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Existence of Solutions to a Generalized Self-dual Chern-Simons Equation on Graphs*

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Abstract Let $G = (V, E)$ be a connected finite graph and Δ be the usual graph Laplacian. In this paper, the authors consider a generalized self-dual Chern-Simons equation on the graph G:

$$\Delta u = -\lambda \mathrm{e}^{F(u)}[\mathrm{e}^{F(u)} - 1]^2 + 4\pi \sum_{j=1}^{M} \delta_{p_j}, \tag{0.1}$$

where

$$F(u) = \begin{cases} \widetilde{F}(u), & u \leq 0, \\ 0, & u > 0, \end{cases}$$

$\widetilde{F}(u)$ satisfies $u = 1 + \widetilde{F}(u) - \mathrm{e}^{\widetilde{F}(u)}$, $\lambda > 0$, M is any fixed positive integer, δ_{p_j} is the Dirac delta mass at the vertex p_j, and p_1, p_2, $\cdots$, p_M are arbitrarily chosen distinct vertices on the graph. They first prove that there is a critical value λ_c such that if $\lambda \geq \lambda_c$, then the generalized self-dual Chern-Simons equation has a solution u_λ. Applying the existence result, they develop a new method to construct a solution of (0.1) which is monotonic with respect to λ when $\lambda \geq \lambda_c$. Then they establish that there exist at least two solutions of the equation for $\lambda > \lambda_c$ via the variational method. Furthermore, they give a fine estimate of the monotone solution, which can be applied to other related problems.

Keywords Chern-Simons equation, Finite graph, Existence, Variational method
2020 MR Subject Classification 35A01, 35A15, 35R02

1 Introduction

In this paper, we study the following generalized self-dual Chern-Simons equation on a connected finite graph G:

$$\Delta u = -\lambda \mathrm{e}^{F(u)}[\mathrm{e}^{F(u)} - 1]^2 + 4\pi \sum_{j=1}^{M} \delta_{p_j}, \tag{1.1}$$

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where Δ is the graph Laplacian,

$$F(u) = \begin{cases} \widetilde{F}(u), & u \leq 0, \\ 0, & u > 0, \end{cases} \tag{1.2}$$

$\widetilde{F}(u)$ satisfies $u = 1 + \widetilde{F}(u) - \mathrm{e}^{\widetilde{F}(u)}$, $p_1, p_2, \cdots, p_M$ are distinct points in the graph G and δ_{p_j} is the Dirac delta mass at the vertex p_j. Note that if $F(u) = u$, then the solutions of the generalized self-dual Chern-Simons model is referred to as vortices. We remark that the notion of vortices plays important roles in many aspects of sciences including superconductivity (see [15]), optics (see [2]), quantum Hall effects (see [19]), for which one can read.

The study of vortices in (2+1)-dimensional Chern-Simons gauge theory has attracted much attention recently. One of the important features of these vortices, which differs from Nielsen-Olesen vortices (see [18]), is that they are magnetically and electrically charged. In the Chern-Simons model, the Yang-Mills (or Maxwell) term does not appear in the action Lagrangian density and only the Chern-Simons term governs electromagnetism. Under the condition that the Higgs potential takes a sextic form, the static equations of motion can be deduced by reducing a system of second-order differential equations to a self-dual system of first-order equations, and then the topological multivortices (see [21, 24]), non-topological multivortices (see [3–6, 20]), and doubly periodic vortices (see [11]) can all be studied rigorously using mathematical methods.

A generalized self-dual Chern-Simons model was later proposed by Hong, Kim and Pac [12] and now plays an important role in various areas of physics, many researchers did a lot of significant work on the existence of non-topological vortices and topological vortices in this Chern-Simons model (see [4, 23, 25]). However, the existence of doubly periodic vortices in this Chern-Simons model had been an open problem, until recently Han solved this problem in [11]. He reduced the generalized self-dual Chern-Simons equation to a quasilinear elliptic equation by appropriate transformations, and established the existence of doubly periodic vortices of the Chern-Simons model by the methods of subsolutions and supersolutions.

In this paper, we investigate existence of solutions to the generalized Chern-Simons equation on a connected finite graph. In recent years, the research on the elliptic equations on graphs has attracted increasing attention from scientists. Grigor'yan, Lin and Yang [9] considered the Kazdan-Warner equation on a finite graph, where the Kazdan-Warner equation was initially studied on a manifold in [16]. In [9], they gave the solvability of the following equation depending on the sign of c:

$$\Delta u = c - h\mathrm{e}^u,$$

where c is a constant and $h : V \to \mathbb{R}$ is a function. Ge and Jiang [8] studied the Kazdan-Warner equation on an infinite graph. They applied a heat flow method which is different from the variational method used in the finite graph case and then gave an existence result for the Kazdan-Warner equation. For more results of solvability of the Kazdan-Warner equation, we refer readers to [7, 10, 17]. The Kazdan-Warner equation is closely related to the mean field equation investigated originally in the prescribed curvature problem in geometry. Huang, Lin and Yau [14] proved the existence of solutions to the following two mean field equations:

$$\Delta u + \mathrm{e}^u = \rho\delta_0 \tag{1.3}$$

and

$$\Delta u = \lambda \mathrm{e}^u(\mathrm{e}^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}$$

on an arbitrary connected finite graph, where $\rho > 0$ and $\lambda > 0$ are constants.

From these results, we study the generalized Chern-Simons equation (1.1) on a finite graph G. Inspired by the idea from [14, 22], we first show that there is a critical value λ_c depending on the graph G such that if $\lambda > \lambda_c$, (1.1) has a solution via the variational method. Moreover, (1.1) also has a solution when $\lambda = \lambda_c$ by the properties of finite graphs. Applying the existence result for $\lambda \geq \lambda_c$, we put forward a new idea to construct a solution of (1.1) which is monotonic with respect to λ. With the aid of the existence results and the Mountain Pass theorem, we show that there exist at least two solutions of (1.1). The crucial point is that we apply the properties of the equation on the finite graph to prove the Palais-Smale condition used in the Mountain Pass theorem which is different from the equations on Euclidean spaces. Furthermore, under the monotonicity property of the solutions, we give a fine estimate of the solutions for almost every $\lambda > \lambda_c$.

The organization of the paper is as follows. In Section 2, we introduce the notations and preliminaries of the paper and then state our main results. In Section 3, we present the existence of the solution to the generalized self-dual Chern-Simons equation. A fine result about the solutions is established in Section 4.

2 Settings and Main Results

Let $G = (V, E)$ be a connected finite graph. For any edge $xy \in E$, we assume that the symmetric weights $\omega_{xy} = \omega_{yx}$ satisfy $\omega_{xy} > 0$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. For any function $u : V \to \mathbb{R}$, the Laplace operator acting on u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{yx}[u(y) - u(x)], \tag{2.1}$$

where $y \sim x$ means $xy \in E$. The associated gradient form stands for

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)) \tag{2.2}$$

for any $u, v : V \to \mathbb{R}$.

For any function $f : V \to \mathbb{R}$, an integral of f over V is defined by

$$\int_V f \mathrm{d}\mu = \sum_{x \in V} \mu(x) f(x), \tag{2.3}$$

and then

$$\frac{1}{2} \int_V |\nabla u|^2 \mathrm{d}\mu := \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} [u(y) - u(x)]^2. \tag{2.4}$$

Define a Sobolev space and a norm by

$$W^{1,2}(V) = \Big\{ u : V \to \mathbb{R} \;\Big|\; \int_V (|\nabla u|^2 + u^2) \mathrm{d}\mu < +\infty \Big\}$$

and

$$\|u\|_{W^{1,2}(V)} = \Big(\int_V (|\nabla u|^2 + u^2) \mathrm{d}\mu \Big)^{\frac{1}{2}},$$

respectively. Since G is a finite graph, the Sobolev space $W^{1,2}(V)$ is the set of all functions on V, which is a finite dimensional linear space. Then the following Sobolev embedding was introduced in [9, Lemma 5].

Lemma 2.1 *Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,2}(V)$ is pre-compact. Namely, if $\{u_j\}$ is bounded in $W^{1,2}(V)$, then there exists some $u \in W^{1,2}(V)$ such that up to a subsequence, $u_j \to u$ in $W^{1,2}(V)$.*

Let $(X, \|\cdot\|)$ be a Banach space, $J : X \to \mathbb{R}$ be a functional. In order to prove the existence of solutions of (1.1), we first need the following Palais-Smale condition.

Definition 2.1 *A functional $J \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition if every sequence $\{u_k\} \subset X$ such that*

(i) *$J(u_k) \to c$ for some constant c, as $k \to +\infty$,*

(ii) *$\|J'(u_k)\| \to 0$ in X, as $k \to +\infty$*

is precompact in X.

With the help of the Palais-Smale condition, Ambrosetti and Rabinowitz established the following Mountain Pass theorem (see [1]).

Lemma 2.2 *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\| > r$ and*

$$b := \inf_{\|u\| = r} J(u) > J(0) \geq J(e).$$

If J satisfies the Palais-Smale condition with $c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$, where

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of J.

Our first main result in this paper can be stated as follows.

Theorem 2.1 *There is a critical value λ_c depending on G satisfying*

$$\lambda_c \geq \frac{27\pi M}{|V|} \tag{2.5}$$

such that

(i) *if $\lambda \geq \lambda_c$,* (1.1) *has a solution u_λ on G, and if $\lambda < \lambda_c$,* (1.1) *has no solution.*

(ii) *If $\lambda \geq \lambda_c$, there exists a solution $\widetilde{u}_\lambda$ of* (1.1) *on G satisfying $\widetilde{u}_{\lambda_1} \geq \widetilde{u}_{\lambda_2}$ when $\lambda_1 > \lambda_2$.*

(iii) *If $\lambda > \lambda_c$,* (1.1) *admits at least two solutions on G.*

Let u_0 be a solution of the Poisson equation

$$\Delta u_0 = -\frac{4\pi M}{|V|} + 4\pi\sum_{j=1}^{M}\delta_{p_j}. \tag{2.6}$$

As is well known, the solution of (2.6) exists if the integral of the right-hand side is equal to 0. Inserting $u = u_0 + v$ into (1.1) yields

$$\Delta v = -\lambda \mathrm{e}^{F(u_0+v)}[\mathrm{e}^{F(u_0+v)} - 1]^2 + \frac{4\pi M}{|V|}. \tag{2.7}$$

By Theorem 2.1, we investigate the existence result of (1.1) and show that $\widetilde{v}_\lambda (= \widetilde{u}_\lambda - u_0)$ is a local minimum of the functional related to (2.7) when $\lambda > \lambda_c$. For the solution $\widetilde{v}_\lambda$, we give a further result as follows.

Theorem 2.2 *The solution $\widetilde{v}_\lambda$ is a strict local minimum of the functional related to* (2.7) *for almost every $\lambda > \lambda_c$.*

Remark 2.1 (1) In [13], Hou and Sun showed the same existence result of (1.1) as (i) in Theorem 2.1 by the method of subsolutions and supersolutions for $\lambda > \lambda_c$. In this paper, we prove the existence result of (1.1) by a different method, i.e., the variational method.

(2) Hou and Sun [13] showed the existence result of (1.1) for $\lambda > \lambda_c$. In this paper, we prove not only the existence of a single solution, but also the existence of multiple solutions. To achieve this, we first put forward a new method to construct a solution $\widetilde{u}_\lambda$ of (1.1) which is monotonic with respect to λ. This implies that $\widetilde{v}_\lambda$ is monotonic with respect to λ. Then we establish that $\widetilde{v}_\lambda$ is a local minimum of the functional related to (2.7) when $\lambda > \lambda_c$ by the

monotonicity of $\widetilde{v}_\lambda$. With the help of this result and the variational method, we show that (1.1) admits at least two solutions on G when $\lambda > \lambda_c$.

(3) Hou and Sun [13] established the existence result of (1.1) at $\lambda = \lambda_c$ by the integral estimation method. In our paper, we develop a new method to prove this result by applying properties of the Chern-Simons equations on the finite graph and the contradiction argument.

3 Existence of Solutions for the Chern-Simons Equations

This section is devoted to the proof of Theorem 2.1. We first establish the existence result of solutions to (1.1) for $\lambda \geq \lambda_c$ by the variational method and the properties of finite graphs. Then we put forward a new idea to construct a solution of (1.1) which is monotonic with respect to λ. This plays a key role in proving the existence of multiple solutions to (1.1).

3.1 Existence of solutions for $\boldsymbol{\lambda > \lambda_c}$

In this subsection, we show the existence result of (1.1) for $\lambda > \lambda_c$. To achieve this purpose, we first prove some basic lemmas.

Definition 3.1 *We say that* $\underline{u}$ *is a subsolution of* (1.1) *if*

$$\Delta \underline{u} \geq -\lambda \mathrm{e}^{F(\underline{u})}[\mathrm{e}^{F(\underline{u})} - 1]^2 + 4\pi \sum_{j=1}^{M} \delta_{p_j} \tag{3.1}$$

for any $x \in V$.

Lemma 3.1 *If* $\underline{u}$ *is a sub-solution to* (1.1)*, then* $\underline{u}$ *is non-positive on* V.

Proof Let

$$\Omega_1 = \{x \in V \mid \underline{u}(x) > 0\}, \quad \Omega_2 = \{x \in V \mid \underline{u}(x) \leq 0\}. \tag{3.2}$$

If Ω_1 is empty, then the conclusion holds. If Ω_1 is non-empty, we first claim that $\Omega_1 \neq V$. Suppose that the claim does not hold, then

$$\Delta \underline{u} \geq 4\pi \sum_{j=1}^{M} \delta_{p_j} \quad \text{in } \Omega_1. \tag{3.3}$$

Summing the two sides of the above equation, one can obtain that the left-hand side of (3.3) is equal to zero and the right-hand side of (3.3) is equal to $4\pi M$. This leads to a contradiction.

Since

$$\Delta \underline{u} \geq -\lambda \mathrm{e}^{F(\underline{u})}[\mathrm{e}^{F(\underline{u})} - 1]^2 + 4\pi \sum_{j=1}^{M} \delta_{p_j} \quad \text{in } \Omega_1,$$

and from (1.2), one has

$$\Delta \underline{u} \geq 4\pi \sum_{j=1}^{M} \delta_{p_j} \geq 0 \quad \text{in } \Omega_1.$$

Therefore, $\underline{u}$ is a subharmonic function in Ω_1. Since Ω_1 and Ω_2 are non-empty, it follows from the maximum principle on graphs that

$$\max_{\Omega_1} \underline{u} \leq 0.$$

This implies that

$$\underline{u}(x) \leq 0 \quad \text{in } \Omega_1,$$

which contradicts the definition of Ω_1. Therefore, $\underline{u}$ is non-positive on V.

Remark 3.1 If u is a solution to (1.1), then u is non-positive on V.

According to (1.1) and Remark 3.1, one can easily see that

$$\Delta u \geq -\frac{4}{27}\lambda + 4\pi \sum_{j=1}^{M} \delta_{p_j}. \tag{3.4}$$

Summing the two sides of the above equation yields

$$\lambda \geq \frac{27\pi M}{|V|}. \tag{3.5}$$

This is a necessary condition for the existence of solutions to (1.1).

In order to prove the existence of solutions to (1.1), with the aid of the decomposition $u = u_0 + v$ and (2.7), we only need to consider the existence result of (2.7).

Lemma 3.2 *If $\underline{v}$ is a subsolution of* (2.7)*, then there exists a solution v^* of* (2.7)*.*

Proof Define the functional $J : W^{1,2}(V) \to \mathbb{R}$ by

$$J(v) = \frac{1}{2}\int_V |\nabla v|^2 \mathrm{d}\mu + \frac{\lambda}{4}\int_V [\mathrm{e}^{F(u_0+v)} - 1]^4 \mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V v \mathrm{d}\mu, \tag{3.6}$$

and let the functional J among all functions v belonging to the set

$$\mathcal{A} := \{v \in W^{1,2}(V) \mid v \geq \underline{v} \text{ a.e. in } V\}. \tag{3.7}$$

Denote

$$J_0 = \inf_{\mathcal{A}} J(v).$$

Then there exists a sequence $\{v_n\} \subset \mathcal{A}$ such that $J(v_n) \to J_0$. We claim that $\{v_n\}$ is bounded in $W^{1,2}(V)$. In fact, we only need to prove that $\{v_n(x)\}$ is bounded for any $x \in V$. Suppose not, then there exists a point $x^k \in V$ and a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that

$$v_{n_k}(x^k) \to +\infty \quad \text{as } k \to +\infty. \tag{3.8}$$

It follows from (3.6) that

$$\begin{aligned} J(v_{n_k}) &= \frac{1}{2}\int_V |\nabla v_{n_k}|^2 \mathrm{d}\mu + \frac{\lambda}{4}\int_V [\mathrm{e}^{F(u_0+v_{n_k})} - 1]^4 \mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V v_{n_k} \mathrm{d}\mu \\ &\geq \frac{4\pi M}{|V|}\int_V v_{n_k} \mathrm{d}\mu \\ &= \frac{4\pi M}{|V|}\int_{V\setminus\{x^k\}} v_{n_k} \mathrm{d}\mu + \frac{4\pi M}{|V|}\mu(x^k) v_{n_k}(x^k) \to +\infty \quad \text{as } k \to +\infty. \end{aligned} \tag{3.9}$$

This is impossible. Thus, $\{v_n(x)\}$ is bounded for any $x \in V$, and therefore $\{v_n\}$ is bounded in $W^{1,2}(V)$. Applying the Sobolev embedding theorem (Lemma 2.1), there exists a subsequence $\{v_{n_l}\} \subset \{v_n\}$ and $v^* \in W^{1,2}(V)$ such that $v_{n_l} \to v^*$ as $l \to +\infty$. One can easily see that $v^* \in \mathcal{A}$. Hence, one has

$$J_0 = J(v^*).$$

In the following, we prove that v^* is a critical point for J in $W^{1,2}(V)$.

For any $\psi \in W^{1,2}(V)$, $\tau > 0$, let

$$v^\tau = v^* + \tau\psi + \psi^\tau \tag{3.10}$$

with $\psi^\tau = \{v^* + \tau\psi - \underline{v}\}_- \geq 0$ a.e. in V. Then, one has

$$v^\tau \geq \underline{v} \quad \text{a.e. in } V. \tag{3.11}$$

This implies that $v^\tau \in \mathcal{A}$. Hence, we have

$$\begin{aligned} 0 &\leq \frac{J(v^\tau) - J(v^*)}{\tau} \\ &= \frac{1}{2\tau}\int_V (|\nabla v^\tau|^2 - |\nabla v^*|^2)\mathrm{d}\mu + \frac{\lambda}{4\tau}\int_V \{[\mathrm{e}^{F(u_0+v^\tau)} - 1]^4 - [\mathrm{e}^{F(u_0+v^*)} - 1]^4\}\mathrm{d}\mu \\ &\quad + \frac{4\pi M}{|V|\tau}\int_V (v^\tau - v^*)\mathrm{d}\mu \\ &= \frac{1}{\tau}\int_V (\nabla v^* \cdot \nabla(\tau\psi + \psi^\tau))\mathrm{d}\mu + \frac{1}{2\tau}\int_V |\nabla(\tau\psi + \psi^\tau)|^2 \mathrm{d}\mu \\ &\quad - \lambda\int_V \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)} - 1]^2 \psi \mathrm{d}\mu + \frac{\lambda}{4\tau}\int_V [\{[\mathrm{e}^{F(u_0+v^\tau)} - 1]^4 - [\mathrm{e}^{F(u_0+v^*)} - 1]^4\} \\ &\quad + 4\tau \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)} - 1]^2 \psi]\mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V \psi \mathrm{d}\mu + \frac{4\pi M}{|V|\tau}\int_V \psi^\tau \mathrm{d}\mu. \end{aligned}$$

Thus,

$$\begin{aligned}
&\int_V \nabla v^* \cdot \nabla\psi \mathrm{d}\mu - \lambda\int_V \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi\mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V \psi\mathrm{d}\mu\\
\geq& -\frac{1}{\tau}\int_V \nabla v^*\cdot\nabla\psi^\tau\mathrm{d}\mu - \frac{1}{2\tau}\int_V |\nabla(\tau\psi+\psi^\tau)|^2\mathrm{d}\mu - \frac{\lambda}{4\tau}\int_V\{[\mathrm{e}^{F(u_0+v^\tau)}-1]^4-[\mathrm{e}^{F(u_0+v^*)}-1]^4\\
&+4\tau\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi + 4\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi^\tau\}\mathrm{d}\mu\\
&+\frac{\lambda}{\tau}\int_V \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi^\tau\mathrm{d}\mu - \frac{4\pi M}{|V|\tau}\int_V\psi^\tau\mathrm{d}\mu.
\end{aligned}$$

By Taylor expansion, one has

$$\begin{aligned}
&-\frac{\lambda}{4\tau}\int_V\{[\mathrm{e}^{F(u_0+v^\tau)}-1]^4-[\mathrm{e}^{F(u_0+v^*)}-1]^4+4\tau\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi\\
&+4\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi^\tau\}\mathrm{d}\mu = O(\tau).
\end{aligned}\tag{3.12}$$

It follows from the fact that $\frac{1}{2\tau}\int_V|\nabla(\tau\psi+\psi^\tau)|^2\mathrm{d}\mu = O(\tau)$ and (3.12) that

$$\begin{aligned}
&\int_V \nabla v^* \cdot \nabla\psi \mathrm{d}\mu - \lambda\int_V \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi\mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V \psi\mathrm{d}\mu\\
\geq& O(\tau) - \frac{1}{\tau}\int_V(\nabla v^*-\nabla\underline{v})\cdot\nabla\psi^\tau\mathrm{d}\mu - \frac{1}{\tau}\int_V\nabla\underline{v}\cdot\nabla\psi^\tau\mathrm{d}\mu\\
&+\frac{\lambda}{\tau}\int_V \mathrm{e}^{F(u_0+\underline{v})}[\mathrm{e}^{F(u_0+\underline{v})}-1]^2\psi^\tau\mathrm{d}\mu\\
&-\frac{4\pi M}{|V|\tau}\int_V\psi^\tau\mathrm{d}\mu + \frac{\lambda}{\tau}\int_V\{\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2-\mathrm{e}^{F(u_0+\underline{v})}[\mathrm{e}^{F(u_0+\underline{v})}-1]^2\}\psi^\tau\mathrm{d}\mu\\
\geq& O(\tau) - \frac{1}{\tau}\int_V(\nabla v^*-\nabla\underline{v})\cdot\nabla\psi^\tau\mathrm{d}\mu\\
&+\frac{\lambda}{\tau}\int_V\{\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2-\mathrm{e}^{F(u_0+\underline{v})}[\mathrm{e}^{F(u_0+\underline{v})}-1]^2\}\psi^\tau\mathrm{d}\mu\\
:=& O(\tau)+I_1+I_2,
\end{aligned}$$

where the last inequality follows from the fact that $\underline{v}$ is the subsolution of (1.1).

Denote

$$V_0=\{x\in V\mid v^*(x)=\underline{v}(x)\}\quad\text{and}\quad V_1=\{x\in V\mid v^*(x)>\underline{v}(x)\}.$$

One can see that $\psi^\tau(x)=0$ for sufficiently small τ when $x\in V_1$. Thus, one has

$$\begin{aligned}
I_1 &= -\frac{1}{\tau}\int_V(\nabla v^*-\nabla\underline{v})\cdot\nabla\psi^\tau\mathrm{d}\mu\\
&= -\frac{1}{\tau}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}[(v^*-\underline{v})(y)-(v^*-\underline{v})(x)](\psi^\tau(y)-\psi^\tau(x))\\
&= -\frac{1}{\tau}\sum_{x\in V_0}\Big\{\Big[\sum_{y\sim x,y\in V_0}+\sum_{y\sim x,y\in V_1}\Big]\omega_{xy}[(v^*-\underline{v})(y)-(v^*-\underline{v})(x)](\psi^\tau(y)-\psi^\tau(x))\Big\}\\
&\quad -\frac{1}{\tau}\sum_{x\in V_1}\Big\{\Big[\sum_{y\sim x,y\in V_0}+\sum_{y\sim x,y\in V_1}\Big]\omega_{xy}[(v^*-\underline{v})(y)-(v^*-\underline{v})(x)](\psi^\tau(y)-\psi^\tau(x))\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\tau}\sum_{x\in V_0}\sum_{y\sim x, y\in V_1}\omega_{xy}[(v^*-\underline{v})(y)-(v^*-\underline{v})(x)](\psi^\tau(y)-\psi^\tau(x))\\
&\quad -\frac{1}{\tau}\sum_{x\in V_1}\sum_{y\sim x, y\in V_0}\omega_{xy}[(v^*-\underline{v})(y)-(v^*-\underline{v})(x)](\psi^\tau(y)-\psi^\tau(x))\\
&\geq 0 \qquad (3.13)
\end{aligned}
$$

for sufficiently small τ.

Similarly, one can obtain that

$$
I_2 = \frac{\lambda}{\tau}\int_V \{\mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2 - \mathrm{e}^{F(u_0+\underline{v})}[\mathrm{e}^{F(u_0+\underline{v})}-1]^2\}\psi^\tau \mathrm{d}\mu = 0. \qquad (3.14)
$$

Therefore,

$$
\int_V \nabla v^*\cdot\nabla\psi \mathrm{d}\mu - \lambda\int_V \mathrm{e}^{F(u_0+v^*)}[\mathrm{e}^{F(u_0+v^*)}-1]^2\psi\mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V\psi\mathrm{d}\mu \geq 0 \quad \text{as } \tau\to 0^+. \qquad (3.15)
$$

This implies that $\langle J'(v^*),\psi\rangle \geq 0$ for any $\psi\in W^{1,2}(V)$. Replacing ψ with $-\psi$ we obtain the reverse inequality, i.e., $\langle J'(v^*),\psi\rangle \leq 0$. Hence, we obtain that v^* is a critical point for J in $W^{1,2}(V)$.

Lemma 3.3 *If $\lambda > 0$ is sufficiently large, then* (1.1) *has a solution on G.*

Proof Choose $\underline{u} = -c$ to be a constant function. Then

$$
\Delta\underline{u} = 0. \qquad (3.16)
$$

If λ is sufficiently large, one has

$$
-\lambda\mathrm{e}^{F(\underline{u})}[\mathrm{e}^{F(\underline{u})}-1]^2 + 4\pi\sum_{j=1}^M \delta_{p_j} \leq 0. \qquad (3.17)
$$

Therefore,

$$
\Delta\underline{u} \geq -\lambda\mathrm{e}^{F(\underline{u})}[\mathrm{e}^{F(\underline{u})}-1]^2 + 4\pi\sum_{j=1}^M \delta_{p_j}. \qquad (3.18)
$$

This implies that $\underline{u}$ is a subsolution of (1.1). It follows from Lemma 3.2 that (1.1) has a solution on G for λ sufficiently large.

This completes the proof of the lemma.

Lemma 3.4 *There is a critical value λ_c depending on G satisfying*

$$
\lambda_c \geq \frac{27\pi M}{|V|} \qquad (3.19)
$$

such that when $\lambda > \lambda_c$, (1.1) *has a solution u_λ.*

Proof Denote

$$\Lambda = \{\lambda > 0 \mid \lambda \text{ is such that (1.1) has a solution}\}. \tag{3.20}$$

We show that Λ is an interval. If $\widehat{\lambda} \in \Lambda$, denote by $\widehat{u} = u_0 + \widehat{v}$ a solution of (1.1) at $\lambda = \widehat{\lambda}$, where $\widehat{v}$ is the corresponding solution of (2.7). If $\lambda \geq \widehat{\lambda}$, one has

$$\begin{aligned}\Delta\widehat{v} &= -\widehat{\lambda}\mathrm{e}^{F(u_0+\widehat{v})}[\mathrm{e}^{F(u_0+\widehat{v})} - 1]^2 + \frac{4\pi M}{|V|} \\ &\geq -\lambda\mathrm{e}^{F(u_0+\widehat{v})}[\mathrm{e}^{F(u_0+\widehat{v})} - 1]^2 + \frac{4\pi M}{|V|}.\end{aligned} \tag{3.21}$$

This implies that $\widehat{v}$ is a subsolution of (2.7) on G for any $\lambda \geq \widehat{\lambda}$. Then by Lemma 3.2, one can obtain the existence of solutions v_λ to (2.7) for any $\lambda \geq \widehat{\lambda}$. Thus, (1.1) has a solution for any $\lambda \geq \widehat{\lambda}$. It implies that

$$[\widehat{\lambda}, +\infty) \subset \Lambda.$$

Set

$$\lambda_c = \inf\{\lambda \mid \lambda \in \Lambda\}. \tag{3.22}$$

It follows from (3.5) that

$$\lambda \geq \frac{27\pi M}{|V|} \quad \text{for any } \lambda > \lambda_c.$$

Taking the limit, one can arrive at

$$\lambda_c \geq \frac{27\pi M}{|V|}.$$

3.2 Existence of solutions for $\lambda = \lambda_c$

This subsection is devoted to showing the existence of solutions to (1.1) on G if $\lambda = \lambda_c$.

Lemma 3.5 *If $\lambda = \lambda_c$, then* (1.1) *has a solution u_λ.*

Proof Choose a sequence $\lambda_n \to \lambda_c$ as $n \to +\infty$. After passing to a subsequence of $\{\lambda_n\}$, we may assume that $u_{\lambda_n}(x)$ converges to a limit point in $[-\infty, 0]$ for any $x \in V$, and we denote this limit by $u(x)$. We prove that $u(x) \in (-\infty, 0]$ for any $x \in V$. Suppose not, then there are two cases:

(i)

$$\lim_{n\to\infty} u_{\lambda_n}(x) = -\infty \quad \text{for any } x \in V. \tag{3.23}$$

(ii) There exists $x \in V$ such that $\lim_{n\to\infty} u_{\lambda_n}(x)$ exists in $(-\infty, 0]$.

In the following, we show that none of these cases happens.

If (i) holds, when $\lambda_n \to \lambda_c$, summing the two sides of (1.1) yields that the left-hand side of (1.1) approaches 0 and the right-hand side remains at least 2π. This leads to a contradiction.

If (ii) holds, we split V into two subsets V_1 and V_2, where

$$\begin{aligned} V_1 &= \Big\{x \in V \ \Big| \ \lim_{n\to\infty} u_{\lambda_n}(x) = -\infty\Big\}, \\ V_2 &= \Big\{x \in V \ \Big| \ \lim_{n\to\infty} u_{\lambda_n}(x) \text{ exists in } (-\infty, 0]\Big\}. \end{aligned} \tag{3.24}$$

If V_1 is empty, then Lemma 3.5 holds. In the following, we consider that both V_1 and V_2 are non-empty sets.

Since G is a connected finite graph, one may choose $x_2 \in V_2$ such that there exists $x_1 \in V_1$ satisfying $x_1 \sim x_2$. Then

$$\begin{aligned} \Delta u_{\lambda_n}(x_2) &= \frac{1}{\mu(x_2)} \sum_{y\sim x_2} \omega_{yx_2}[u_{\lambda_n}(y) - u_{\lambda_n}(x_2)] \\ &= \frac{1}{\mu(x_2)} \sum_{\substack{y\sim x_2 \\ y\in V_1}} \omega_{yx_2}[u_{\lambda_n}(y) - u_{\lambda_n}(x_2)] + \frac{1}{\mu(x_2)} \sum_{\substack{y\sim x_2 \\ y\in V_2}} \omega_{yx_2}[u_{\lambda_n}(y) - u_{\lambda_n}(x_2)] \\ &:= I_1 + I_2. \end{aligned} \tag{3.25}$$

Now we calculate I_1 and I_2, respectively. It follows from (3.24) that

$$I_1 = \frac{1}{\mu(x_2)} \sum_{y\sim x_2, y\in V_1} \omega_{yx_2}[u_{\lambda_n}(y) - u_{\lambda_n}(x_2)] \to -\infty \quad \text{as } n \to \infty. \tag{3.26}$$

For I_2, since $x_2, y \in V_2$, the limit of u_{λ_n} as λ_n tends to λ_c exists. One can obtain that

$$\frac{1}{\mu(x_2)} \sum_{y\sim x_2, y\in V_2} \omega_{yx_2}[u_{\lambda_n}(y) - u_{\lambda_n}(x_2)] \tag{3.27}$$

is bounded. Taking the limit $\lambda_n \to \lambda_c$, one has

$$\Delta u_{\lambda_n}(x_2) \to -\infty. \tag{3.28}$$

However, it follows from (1.1) that $\Delta u_{\lambda_n}(x_2)$ is finite, which is a contradiction to (3.28). This completes the proof of the lemma.

Combining Lemmas 3.4 and 3.5, one can obtain that (1.1) has a solution u_λ on G when $\lambda \geq \lambda_c$. Hence, the statement (i) of Theorem 2.1 holds.

3.3 Monotonicity

In this subsection, we prove that there exists a solution $\widetilde{u}_\lambda$ to (1.1) satisfying the property that $\widetilde{u}_\lambda$ is increasing with respect to λ.

Lemma 3.6 *There is a critical value λ_c depending on G satisfying*

$$\lambda_c \geq \frac{27\pi M}{|V|}$$

such that (1.1) *has a solution $\widetilde{u}_\lambda$ on G for any $\lambda \geq \lambda_c$, and*

$$\widetilde{u}_{\lambda_1} \geq \widetilde{u}_{\lambda_2} \quad \text{when } \lambda_1 > \lambda_2. \tag{3.29}$$

Proof Equation (1.1) has a solution u_λ on G when $\lambda \geq \lambda_c$, and we denote the set of solvability parameters by $\widetilde{\Lambda} := [\lambda_c, +\infty)$. We fix a solution of (1.1) with parameter $\lambda = \lambda_c$, and denote it by u_{λ_c}. For any fixed $m \in \mathbb{N}$, let

$$\widetilde{\Lambda} = \bigcup_{k=0}^{\infty} \widetilde{\Lambda}_{km},$$

where

$$\widetilde{\Lambda}_{km} := \Big[\lambda_c + \frac{k}{2^m}, \lambda_c + \frac{k+1}{2^m}\Big], \quad k \in \mathbb{N}.$$

Note that for any $\lambda > \lambda_c$, u_{λ_c} is a subsolution of (1.1) with parameter λ. It then follows from the argument of Lemma 3.2 that for any $\lambda \in (\lambda_c, \lambda_c + \frac{1}{2^m}]$, there exists a solution $u_{\lambda,m}$ of (1.1) such that

$$u_{\lambda,m} \geq u_{\lambda_c}, \quad \lambda \in \widetilde{\Lambda}_{0m}.$$

Now consider the solution $u_{\lambda_c + \frac{1}{2^m}, m}$. Similarly, since $u_{\lambda_c + \frac{1}{2^m}, m}$ is a subsolution of (1.1) with parameter $\lambda > \lambda_c + \frac{1}{2^m}$, by the argument of Lemma 3.2, one can find a solution $u_{\lambda,m}$ of (1.1) for any $\lambda \in \big(\lambda_c + \frac{1}{2^m}, \lambda_c + \frac{2}{2^m}\big]$ such that

$$u_{\lambda,m} \geq u_{\lambda_c + \frac{1}{2^m}, m}, \quad \lambda \in \widetilde{\Lambda}_{1m}.$$

Repeating the process above yields that for any $\lambda \in [\lambda_c, +\infty)$, there exists a solution $u_{\lambda,m}$ of (1.1) with parameter λ, and if $\lambda \in \widetilde{\Lambda}_{km}$ for some $k \in \{0, 1, 2, \cdots\}$, then

$$u_{\lambda,m} \geq u_{\lambda_c + \frac{k}{2^m}, m}.$$

Here, for convenience, we let $u_{\lambda_c, m} = u_{\lambda_c}$. Then we have constructed an increasing sequence

$$u_{\lambda_c, m} \leq u_{\lambda_c + \frac{1}{2^m}, m} \leq u_{\lambda_c + \frac{2}{2^m}, m} \leq \cdots \tag{3.30}$$

for any $m \in \mathbb{N}$.

Now define

$$Q := \Big\{\lambda_c + \frac{j}{2^m} \Big| m \in \mathbb{N},\ j \in \mathbb{N} \text{ odd}\Big\}. \tag{3.31}$$

We rearrange the elements in the countable set Q and list them as $Q = \{q_l : l \in \mathbb{N}\}$.

For any $q_l \in Q$, we have constructed a sequence of solutions $\{u_{q_l,m}\}_{m\in\mathbb{N}}$ of (1.1) with parameter $\lambda = q_l$. Since the functions $u_{q_l,m}$ $(l, m \in \mathbb{N})$ are defined on the finite connected graph $G = (V, E)$ and

$$u_{\lambda_c} \le u_{q_l,m} \le 0, \quad \forall l, m \in \mathbb{N},$$

there exists a subsequence $\{n_i\} \subset \mathbb{N}$ such that, for any $q_l \in Q$, $\{u_{q_l,n_i}\}$ converges as $i \to \infty$.

Define

$$u_{q_l} := \lim_{i\to\infty} u_{q_l,n_i}$$

and let

$$U = \{u_{q_l} \mid q_l \in Q\}. \tag{3.32}$$

It follows from (3.30) that if $q_{l_1}, q_{l_2} \in Q$, $q_{l_1} < q_{l_2}$ and i is sufficiently large, then

$$u_{q_{l_1},n_i} \le u_{q_{l_2},n_i},$$

and hence by taking limits as $i \to \infty$, we have

$$u_{q_{l_1}} \le u_{q_{l_2}} \quad \text{for } q_{l_1} < q_{l_2}. \tag{3.33}$$

This implies that $\{u_\lambda : \lambda \in Q\}$ is an increasing sequence with respect to the parameter $\lambda \in Q$.

Now for any $\lambda \ge \lambda_c$, there exists an increasing sequence $\{q_l\} \subset Q$ such that $q_l \to \lambda$. The increasing sequence $\{u_{q_l}\}$ then has a limit and we define

$$\widetilde{u}_\lambda := \lim_{l\to\infty} u_{q_l}. \tag{3.34}$$

Then $\widetilde{u}_\lambda$ is a solution of (1.1) for the parameter $\lambda \in \widetilde{\Lambda}$. Let

$$\overline{U} = \{\widetilde{u}_\lambda \mid \lambda \in [\lambda_c, +\infty)\}. \tag{3.35}$$

It follows from (3.33)–(3.34) that if $\lambda_1 > \lambda_2$, then

$$\widetilde{u}_{\lambda_1} \ge \widetilde{u}_{\lambda_2}. \tag{3.36}$$

Hence, $\widetilde{u}_\lambda$ is increasing with respect to $\lambda \in [\lambda_c, \infty)$. This completes the proof of the lemma.

The statement (ii) of Theorem 2.1 then follows from Lemma 3.6.

3.4 Existence of multiple solutions for $\lambda > \lambda_c$

From Lemma 3.6 and the maximum principle, we can obtain that $\widetilde{u}_\lambda > \widetilde{u}_{\lambda_c}$ when $\lambda > \lambda_c$. Thus, $\widetilde{v}_\lambda$ $(= \widetilde{u}_\lambda - u_0)$ is the local minimum point of the functional J for any $\lambda > \lambda_c$. If v_λ obtained by Lemma 3.4 is different from $\widetilde{v}_\lambda$. Then we have already found a second solution. If v_λ is equal to $\widetilde{v}_\lambda$ as the local minimum of the functional J, there exists ρ_λ such that

$$J(v_\lambda) \leq J(v) \quad \text{for any } \|v - v_\lambda\| \leq \rho_\lambda. \tag{3.37}$$

Then we have the following two cases.

Case (i) For any $\rho \in (0, \rho_\lambda)$, one has

$$\inf_{\|v - v_\lambda\| = \rho} J = J(v_\lambda)(:= \eta_\lambda). \tag{3.38}$$

Thus, there is a local minimum v_ρ of J satisfying $\|v_\rho - v_\lambda\| = \rho$ and $J(v_\lambda) = \eta_\lambda$ for any $\rho \in (0, \rho_\lambda)$. Therefore, one can obtain a one-parameter family of solutions to (1.1).

Case (ii) There exists $r_\lambda \in (0, \rho_\lambda)$ such that

$$J(v_\lambda) < \inf_{\|w - v_\lambda\| = r_\lambda} J(w). \tag{3.39}$$

In the following, we prove that there is a second solution of (1.1) on G for $\lambda > \lambda_c$ by the Mountain Pass theorem. We first show that $J(v)$ satisfies the Palais-Smale condition in order to apply the Mountain Pass theorem for the functional J.

Lemma 3.7 *Every sequence $\{v_n\} \subset W^{1,2}(V)$ admits a convergent subsequence if $\{v_n\} \subset W^{1,2}(V)$ satisfies*

(i) $J(v_n) \to c$ *for some constant* c, *as* $n \to +\infty$, (3.40)

(ii) $\|J'(v_n)\| \to 0$ *in* $W^{1,2}(V)$, *as* $n \to +\infty$. (3.41)

Proof It follows from (3.41) that

$$\Big| \int_V \nabla v_n \cdot \nabla \varphi \mathrm{d}\mu - \lambda \int_V \mathrm{e}^{F(u_0+v_n)}[\mathrm{e}^{F(u_0+v_n)} - 1]^2 \varphi \mathrm{d}\mu + \frac{4\pi M}{|V|} \int_V \varphi \mathrm{d}\mu \Big| \leq \varepsilon_n \|\varphi\|, \tag{3.42}$$

where $\varphi \in W^{1,2}(V)$ and $\varepsilon_n \to 0$ as $n \to +\infty$. By taking $\varphi = -1$, one has

$$0 < c \leq \int_V \mathrm{e}^{F(u_0+v_n)}[\mathrm{e}^{F(u_0+v_n)} - 1]^2 \mathrm{d}\mu \leq C, \tag{3.43}$$

where c and C are constants depending only on λ.

We claim that $\{v_n(x)\}$ is bounded for any $x \in V$. Suppose that the claim is not true, there exists a point $x^k \in V$ and a subsequence $\{v_{n_k}(x^k)\} \subset \{v_n\}$ that tends to $+\infty$ or $-\infty$ as $k \to +\infty$. Without loss of generality, we assume that $\{v_{n_k}(x^k)\}$ tends to $+\infty$ as $k \to +\infty$.

In the following, for the subsequence $\{v_{n_k}\}$, we first show that there exists a point $\widehat{x} \in V$ such that $\{v_{n_k}(\widehat{x})\}$ is bounded. If not, one can obtain that $\{v_{n_k}(y)\}$ tends to $+\infty$ or $-\infty$ for any point $y \in V$ as $k \to +\infty$. Therefore, one can derive that

$$\int_V \mathrm{e}^{F((u_0+v_{n_k})(y))}[\mathrm{e}^{F((u_0+v_{n_k})(y))} - 1]^2 \mathrm{d}\mu \to 0 \quad \text{as } k \to +\infty, \tag{3.44}$$

which contradicts (3.43).

Let

$$L_k := \max_V |v_{n_k}(x)| = v_{n_k}(x^k). \tag{3.45}$$

From (3.40), one has

$$\begin{aligned} J(v_{n_k}) &= \frac{1}{2}\int_V |\nabla v_{n_k}|^2 \mathrm{d}\mu + \frac{\lambda}{4}\int_V [\mathrm{e}^{F(u_0+v_{n_k})} - 1]^4 \mathrm{d}\mu + \frac{4\pi M}{|V|}\int_V v_{n_k} \mathrm{d}\mu \\ &= c + o_n(1). \end{aligned} \tag{3.46}$$

Combining (2.4) and (3.46) yields that

$$c \geq \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}[v_{n_k}(y) - v_{n_k}(x)]^2 - 4\pi M L_k. \tag{3.47}$$

It implies that

$$|v_{n_k}(x) - v_{n_k}(y)| \leq C\sqrt{c + 4\pi M L_k}, \quad x \sim y \text{ for any } x, y \in V, \tag{3.48}$$

where C is a constant.

Since the graph G is finite and connected, there is a path $\{x_i\}_{i=0}^l \subset G$ between $\widehat{x}$ and x^k for $l \in \mathbb{N}$, that is

$$\widehat{x} = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_l = x^k. \tag{3.49}$$

Combining (3.48) and (3.49), one can obtain that

$$\begin{aligned} v_{n_k}(\widehat{x}) &\geq v_{n_k}(x^k) - (l-1)C\sqrt{c + 4\pi M L_k} \\ &= L_k - (l-1)C\sqrt{c + 4\pi M L_k} \\ &\to +\infty \quad \text{as } k \to +\infty. \end{aligned} \tag{3.50}$$

This is a contradiction to the conclusion that $\{v_{n_k}(\widehat{x})\}$ is bounded. Therefore, $\{v_n(x)\}$ is bounded for any $x \in V$. This implies that $\{v_n(x)\}$ is bounded in $W^{1,2}(V)$. Applying the Sobolev embedding theorem (Lemma 2.1), there exists a subsequence $v_{n_k}(x)$ converges to some function $v(x)$ in $W^{1,2}(V)$. Hence, the Palais-Smale condition holds.

For $Q > 0$, one has

$$J(v_\lambda - Q) - J(v_\lambda) = \frac{\lambda}{4}\int_V [\mathrm{e}^{F(u_0+v_\lambda \mathrm{d}\mu - Q)} - 1]^4 \mathrm{d}\mu - \frac{\lambda}{4}\int_V [\mathrm{e}^{F(u_0+v_\lambda)} - 1]^4 \mathrm{d}\mu - 4\pi MQ$$
$$\to -\infty \quad \text{as } Q \to +\infty.$$

Thus, one has

$$J(v_\lambda - Q_0) < J(v_\lambda) \quad \text{for some } Q_0 > r_\lambda > 0 \text{ sufficiently large.} \tag{3.51}$$

Let

$$\Gamma := \{\gamma \in C([0,1], W^{1,2}(V)) \mid \gamma(0) = v_\lambda,\ \gamma(1) = v_\lambda - Q_0\}$$

and set

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)).$$

It follows from (3.39) that $c > J(v_\lambda)$. From Lemma 3.7, applying the Mountain Pass theorem (Lemma 2.2), we conclude that c defines a critical value of J. Since $c > J(v_\lambda)$, the corresponding critical point yields a second solution for (2.7). Therefore, there exists a second solution to (1.1). Hence, the statement (iii) of Theorem 2.1 holds.

This completes the proof of Theorem 2.1.

4 Fine Properties of the Solutions

In this section, we prove Theorem 2.2. From the analysis in Section 3, we know that $\widetilde{v}_\lambda$ is a local minimum of the functional $J(v)$ for $\lambda > \lambda_c$. In the following, we show that $\widetilde{v}_\lambda$ is a strict local minimum of the functional J for almost every $\lambda > \lambda_c$.

For convenience, we state Theorem 2.2 again here.

Theorem 4.1 *The solution $\widetilde{v}_\lambda$ is a strict local minimum of the functional related to* (2.7) *for almost every $\lambda > \lambda_c$.*

Proof It follows from Lemma 3.6 that $\widetilde{v}_\lambda$ is increasing with respect to λ for $\lambda > \lambda_c$. This guarantees that

$$\overline{v}_\lambda := \frac{\mathrm{d}\widetilde{v}_\lambda}{\mathrm{d}\lambda} \tag{4.1}$$

is well-defined for almost every $\lambda > \lambda_c$, and therefore $\overline{v}_\lambda \ge 0$ a.e. in V.

Taking the derivative of both sides of (2.7) with respect to λ yields that

$$\Delta\overline{v}_\lambda = -\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)} - 1]^2 - \lambda \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[1 - 3\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}] \cdot \overline{v}_\lambda. \tag{4.2}$$

From the fact that $\widetilde{v}_\lambda$ is the critical point of $J(v)$, one can derive that

$$0=\langle J'(\widetilde{v}_\lambda),\varphi\rangle=\int_V \nabla\widetilde{v}_\lambda\cdot\nabla\varphi \mathrm{d}\mu-\lambda\int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}-1]^2\varphi\mathrm{d}\mu+\frac{4\pi M}{|V|}\int_V\varphi\mathrm{d}\mu \quad (4.3)$$

and

$$\langle J''(\widetilde{v}_\lambda)\psi,\varphi\rangle=\int_V\nabla\psi\cdot\nabla\varphi\mathrm{d}\mu-\lambda\int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[1-3\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}]\varphi\psi\mathrm{d}\mu \quad (4.4)$$

for any $\varphi,\psi\in W^{1,2}(V)$.

From (4.4), one can see that

$$J''(\widetilde{v}_\lambda)h=-\Delta h-\lambda\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[1-3\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}]h \quad \text{for any } h\in W^{1,2}(V). \quad (4.5)$$

Then

$$\langle J''(\widetilde{v}_\lambda)h,g\rangle=\langle J''(\widetilde{v}_\lambda)g,h\rangle \quad \text{for any } h,g\in W^{1,2}(V).$$

This implies that $J''(\widetilde{v}_\lambda)$ is a symmetric and linear operator from $W^{1,2}(V)$ to $W^{1,2}(V)$.

Denote N as the number of vertices in V. Let $\mu_1,\mu_2,\cdots,\mu_N$ be the eigenvalues of $J''(\widetilde{v}_\lambda)$ satisfying $\mu_1\le\mu_2\le\cdots\le\mu_N$ and let $\theta_1,\cdots,\theta_N$ be the corresponding eigenvectors of $J''(\widetilde{v}_\lambda)$, i.e.,

$$J''(\widetilde{v}_\lambda)\theta_i=\mu_i\theta_i,$$

where $\langle\theta_i,\theta_j\rangle=\delta_{ij}$ for any $i,j=1,2,\cdots,N$.

Then the eigenvalue

$$\mu_1=\inf_{h\in W^{1,2}(V)}\frac{\langle J''(\widetilde{v}_\lambda)h,h\rangle}{\langle h,h\rangle}.$$

In the following, we prove that $\mu_1>0$.

Noting that

$$\begin{aligned}\langle J''(\widetilde{v}_\lambda)h,h\rangle&=\int_V|\nabla h|^2\mathrm{d}\mu-\lambda\int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[1-3\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}]h^2\mathrm{d}\mu\\&=\sum_{x\in V}\sum_{y\sim x}\omega_{xy}[h(y)-h(x)]^2-\lambda\int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[1-3\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}]h^2\mathrm{d}\mu.\end{aligned} \quad (4.6)$$

Let $\widetilde{h}=(|h_1|,|h_2|,\cdots,|h_N|)$ for any $h=(h_1,h_2,\cdots,h_N)\in W^{1,2}(V)$, it follows from (4.6) that

$$\langle J''(\widetilde{v}_\lambda)\widetilde{h},\widetilde{h}\rangle\le\langle J''(\widetilde{v}_\lambda)h,h\rangle. \quad (4.7)$$

Thus, one can obtain that all of the components of θ_1 have the same sign. Without loss of generality, we assume that they are all non-negative, i.e.,

$$\theta_{1,x_i}\ge 0 \quad \text{for } 1\le i\le N,\ x_i\in V. \quad (4.8)$$

Taking $\psi = \overline{v}_\lambda$, it follows from (4.4) that

$$\langle J''(\widetilde{v}_\lambda)\overline{v}_\lambda, \varphi\rangle = \int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)} - 1]^2 \varphi \mathrm{d}\mu \tag{4.9}$$

for any $\varphi \in W^{1,2}(V)$. Let $\varphi = \theta_1$, one has

$$\langle J''(u_\lambda)\overline{v}_\lambda, \theta_1\rangle = \int_V \mathrm{e}^{F(u_0+\widetilde{v}_\lambda)}[\mathrm{e}^{F(u_0+\widetilde{v}_\lambda)} - 1]^2 \theta_1 \mathrm{d}\mu > 0. \tag{4.10}$$

Since $\overline{v}_\lambda \in W^{1,2}(V)$, $\overline{v}_\lambda$ can be represented by

$$\overline{v}_\lambda = \sum_{i=1}^{N} a_i \theta_i,$$

where $a_i = \langle \overline{v}_\lambda, \theta_i\rangle$, $1 \le i \le N$. One can see that $a_1 \ge 0$ and

$$0 < \langle J''(\widetilde{v}_\lambda)\overline{v}_\lambda, \theta_1\rangle = a_1\mu_1.$$

Thus, one has

$$\mu_1 > 0. \tag{4.11}$$

Therefore, $\widetilde{v}_\lambda$ is the strict minimum of $J(\widetilde{v}_\lambda)$ for almost every $\lambda > \lambda_c$. This implies that for almost every λ, there exists r_λ such that

$$J(\widetilde{v}_\lambda) < \inf_{\|w-\widetilde{v}_\lambda\|=r_\lambda} J(w). \tag{4.12}$$

This completes the proof of the theorem.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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