Chin. Ann. Math. Ser. B
47(3), 2026, 475–490
DOI: 10.1007/s11401-026-0021-z

Chinese Annals of Mathematics, Series B

On the Univalence and Quasiconformal Extensions Criterion for Harmonic Mappings Associated with Pre-Schwarzian Derivative*

Xiaoyuan WANG[1] Jinhua FAN[2] Zhenyong HU[3] Zhigang WANG[4]

Abstract As a generalization of Ahlfors's results for analytic functions, by using the pre-Schwarzian derivative of harmonic mappings, the authors obtain a criterion of univalence and quasiconformal extension for harmonic functions. As applications, they give a lower bound of the inner radius of univalency by means of pre-Schwarzian derivative of harmonic mappings for a planar domain.

Keywords Harmonic mapping, Quasiconformal extension, Pre-Schwarzian derivative, Inner radius of univalency
2020 MR Subject Classification 30C55, 31C45, 30C62, 31A05

1 Introduction and Statements of the Main Results

1.1 Univalence and quasiconformal extension criteria for analytic functions

Let D be a domain in the extended complex plane $\overline{\mathbb{C}}$ and η_D be its Poincaré density. For a locally univalent analytic function ϕ in D, the pre-Schwarzian derivative $P\phi$ and Schwarzian derivative $S\phi$ are defined by

$$P\phi = \frac{\phi''}{\phi'}, \quad S\phi = (P\phi)' - \frac{1}{2}(P\phi)^2,$$

respectively. The pre-Schwarzian norm $\|P\phi\|$ and Schwarzian norm $\|S\phi\|$ are

$$\|P\phi\|_D = \sup_{z\in D} |P\phi|\eta_D^{-1}, \quad \|S\phi\|_D = \sup_{z\in D} |S\phi|\eta_D^{-2}.$$

Manuscript received February 22, 2023. Revised December 7, 2023.

[1] School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China. E-mail: mewangxiaoyuan@163.com

[2] Corresponding author. School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China. E-mail: jinhuafan@hotmail.com

[3] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466000, Henan, China. E-mail: huzhenyongad@163.com

[4] School of Mathematics and Statistics, Hunan First Normal University, Changsha 410205, China. E-mail: wangmath@163.com

[*] This work was supported by the National Natural Science Foundation of China (No. 12471074), the China Scholarship Council (No. 202306840137), the Natural Science Foundation of Henan Province (No. 252300420942), the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX23_0410), the Key Project of Education Department of Hunan Province (No. 25A0668) and the Natural Science Foundation of Changsha (No. kq2502003).

When $D=\triangle:=\{z:|z|<1\}$, for simplicity, we denote $\|S\phi\|_D$ and $\|P\phi\|_D$ by $\|S\phi\|$ and $\|P\phi\|$, respectively.

The quantities $\|S\phi\|$ and $\|P\phi\|$ are widely used to characterize the univalence and quasiconformal extensions of ϕ. As far back as 1949, Nehari [30] proved that

$$\|S\phi\|\le 2 \tag{1.1}$$

implies that ϕ is univalent. Hille [19] proved that the bound 2 in (1.1) is the best possible. Later, Ahlfors and Weill [3] showed that, for any $t\in(0,1)$, if

$$\|S\phi\|\le 2t, \tag{1.2}$$

then ϕ is not only univalent but can also be quasiconformally extended to $\overline{\mathbb{C}}$. In 1972, instead of Schwarzian derivative by pre-Schwarzian derivative, Becker [5] gave parallel results as follows. If

$$\|P\phi\|\le 1, \tag{1.3}$$

then ϕ is univalent; furthermore, there exists a quasiconormal extension of ϕ to $\overline{\mathbb{C}}$ if

$$\|P\phi\|\le k<1. \tag{1.4}$$

It should be noted that the constant 1 in (1.3) is sharp (see [6]).

In 1974, Ahlfors [2] gave a more widely criterion to justify the univalence and quasiconformal extensions.

Theorem A *Let ϕ be a locally univalent analytic function in $\triangle$. Let σ be a continuous function in $\triangle$, which satisfies the following conditions*:

(i) σ_z *and* $\sigma_{\overline{z}}$ *exist in* $\triangle$ *a.e.*;

(ii) $\frac{1}{\sigma}=0$ *on* $\partial\triangle$;

(iii) $\frac{\sigma_{\overline{z}}}{\sigma^2}\neq 0$ *in* $\overline{\triangle}$.

Then the inequality

$$|\sigma P\phi+\sigma^2-\sigma_z|\le|\sigma_{\overline{z}}|,\quad z\in\triangle \tag{1.5}$$

is sufficient to imply that ϕ is univalent in $\triangle$. Moreover, the inequality

$$|\sigma P\phi+\sigma^2-\sigma_z|\le k|\sigma_{\overline{z}}|,\quad 0\le k<1 \tag{1.6}$$

is sufficient to imply that ϕ has an explicit homeomorphic extension

$$\Phi(z)=\begin{cases}\widetilde{\phi}(z), & |z|\le 1,\\ \phi\Big(\dfrac{1}{\overline{z}}\Big)+u\Big(\dfrac{1}{\overline{z}}\Big), & |z|>1,\end{cases}$$

where $u(z)=\frac{\phi'(z)}{\sigma(z)}$ for $z\in\triangle\backslash\{0\}$. Also, the mapping Φ is a K-quasiconformal mapping in $\overline{\mathbb{C}}$, where $K=\frac{1+k}{1-k}$.

Remark 1.1 In fact, by taking suitable functions σ in Theorem A, we can get different criteria for univalence and quasiconformal extensions. For example, (1.1) and (1.3) come from (1.5) by taking

$$\sigma(z)=\frac{\overline{z}}{1-|z|^2}-\frac{1}{2}P\phi,\qquad \sigma(z)=\frac{\overline{z}}{1-|z|^2},$$

respectively.

1.2 Univalence and quasiconformal extension criteria for harmonic mappings

As natural generalizations of analytic mappings, planar harmonic mappings are widely studied by many mathematicians as early as 1980s (see [12]). A complex function f in $\triangle$ is harmonic if

$$\Delta f=4f_{z\overline{z}}=0.$$

It is well-known that f has a canonical representation $f=h+\overline{g}$, where h and g are analytic in $\triangle$ and $g(0)=0$ (see [13]). Lewy [27] proved that a harmonic mapping f is locally univalent if and only if its Jacobian

$$J_f=|h'|^2-|g'|^2\neq 0.$$

If $J_f>0$ ($J_f<0$), then f is sense-preserving (sense-reserving). We call $\omega=\frac{g'}{h'}$ the second complex dilatation of f. Since $P\phi$ plays an important role in the study of locally univalent analytic functions, many mathematicians try to introduce a similar concept for harmonic functions (see [18, 28–29, 31]). For harmonic function f, Hernández and Martín [18] introduced the pre-Schwarzian derivative P_f as

$$P_f=\frac{\partial}{\partial z}\log|J_f|=Ph-\frac{\overline{\omega}\omega'}{1-|\omega|^2}.\tag{1.7}$$

In fact, it was proved that P_f also has similar properties as $P\phi$ for locally univalent analytic functions (see [14, 16, 18, 34]). Especially, by using P_f, Hernández and Martín [16, 18] extended Becker's results (1.3) and (1.4) to the case of harmonic mappings.

Theorem B *Let $f=h+\overline{g}$ be a sense-preserving harmonic mapping in $\triangle$ and the second complex dilatation ω with $\|\omega\|_\infty=\sup\limits_{z\in\triangle}|\omega|<1$. If f satisfies*

$$(1-|z|^2)|P_f|+\frac{|\omega'(z)(1-|z|^2)|}{1-|\omega(z)|^2}\leq 1,\quad z\in\triangle,\tag{1.8}$$

then f is univalent in $\triangle$. The constant 1 is sharp.

Moreover, if f satisfies the condition

$$(1-|z|^2)|P_f|+\frac{|\omega'(z)(1-|z|^2)|}{1-|\omega(z)|^2}\leq k<1,\quad z\in\triangle,\tag{1.9}$$

then f has a continuous and injective extension $\widetilde{f}$ to $\overline{\triangle}$.

Furthermore, if

$$k<\frac{1-\|\omega\|_\infty}{1+\|\omega\|_\infty},\tag{1.10}$$

then the mapping

$$F(z)=\begin{cases}\widetilde{f}(z), & |z|\le 1,\\ f\Big(\dfrac{1}{\overline{z}}\Big)+U\Big(\dfrac{1}{\overline{z}}\Big), & |z|>1\end{cases}\tag{1.11}$$

is an explicit K-quasiconformal mapping of $\overline{\mathbb{C}}$ onto itself, where

$$U(z)=\frac{h'(z)}{\overline{z}}(1-|z|^2)+\frac{\overline{g'(z)}}{z}(1-|z|^2),\quad z\in\triangle\backslash\{0\}$$

and

$$K=\frac{1+k+\|\omega\|_\infty(1-k)}{1-k-\|\omega\|_\infty(1+k)}.$$

Finally, under the condition (1.9) *with $\|\omega\|_\infty<1$, $\widetilde{f}(\partial\triangle)$ is a quasicircle and f can be extended to a quasiconformal mapping of $\overline{\mathbb{C}}$.*

As mentioned in Remark 1.1, Theorem A is more general than Becker's result (1.3) and Nehari's result (1.1), so it is natural to ask whether there exists a general result analogous to Theorem A for harmonic mappings. From this idea, we prove the following theorem.

Theorem 1.1 *Let $f=h+\overline{g}$ be a sense-preserving harmonic mapping in $\triangle$, and let ω be the second complex dilatation of f with $\|\omega\|_\infty<1$. Let σ be a continuous function in $\triangle$, which satisfies the following conditions*:

(i) *σ_z and $\sigma_{\overline{z}}$ exist in $\triangle$ a.e.*;

(ii) *$\frac{1}{\sigma}=0$ on $\partial\triangle$*;

(iii) *$\frac{\sigma_{\overline{z}}}{\sigma^2}\neq 0$ in $\overline{\triangle}$.*

If f satisfies the condition

$$|\sigma P_f+\sigma^2-\sigma_z|+\frac{|\sigma\omega'|}{1-|\omega|^2}\le|\sigma_{\overline{z}}|,\quad z\in\triangle,\tag{1.12}$$

then $f_\lambda=h+\lambda\overline{g}$ ($|\lambda|\le 1$) is univalent in $\triangle$. Moreover, if

$$|\sigma P_f+\sigma^2-\sigma_z|+\frac{|\sigma\omega'|}{1-|\omega|^2}\le k|\sigma_{\overline{z}}|,\quad 0\le k<1,\ z\in\triangle,\tag{1.13}$$

then the harmonic mapping $f_\lambda=h+\lambda\overline{g}$ ($|\lambda|\le 1$) has a continuous and injective extension $\widetilde{f_\lambda}$ to $\overline{\triangle}$, and the function

$$F_\lambda(z)=\begin{cases}\widetilde{f_\lambda}(z), & |z|\le 1,\\ f_\lambda\Big(\dfrac{1}{\overline{z}}\Big)+U_\lambda\Big(\dfrac{1}{\overline{z}}\Big), & |z|>1\end{cases}\tag{1.14}$$

is a homeomorphic extension of $\overline{\mathbb{C}}$ onto itself, where

$$U_\lambda(z)=\frac{h'(z)}{\sigma(z)}+\lambda\frac{\overline{g'(z)}}{\overline{\sigma(z)}},\quad z\in\triangle\backslash\{0\}.$$

Furthermore, if k satisfies the condition (1.10)*, then the family of mappings $F_\lambda(z)$ is a K-quasiconformal homeomorphisms of $\overline{\mathbb{C}}$, where K is given by*

$$K=\frac{1+k+|\lambda|\|\omega\|_\infty(1-k)}{1-k-|\lambda|\|\omega\|_\infty(1+k)}.\tag{1.15}$$

Without the restriction of k in (1.10), then (1.14) may not be a quasiconformal mapping of $\overline{\mathbb{C}}$. However, we can also present that $f_\lambda = h + \lambda\overline{g}$ ($|\lambda| \le 1$) has a quasiconformal extension to $\overline{\mathbb{C}}$ as follows.

Theorem 1.2 *Let* $f = h+\overline{g}$ *be a sense-preserving harmonic mapping in* $\triangle$ *with* $\|\omega\|_\infty < 1$. *If* f *satisfies the condition* (1.13), *then* $\widetilde{f}_\lambda(\partial\triangle)(|\lambda| \le 1)$ *is a quasicircle and* f_λ *can be extended to a quasiconformal mapping of* $\overline{\mathbb{C}}$.

Remark 1.2 It is easy to see that Theorem A can be deduced from Theorems 1.1–1.2 by letting $g = 0$. Furthermore, many criteria for univalence and quasiconformal extension of harmonic functions can be deduced from Theorems 1.1–1.2. For example,

- By setting $\sigma(z) = \frac{\overline{z}}{1-|z|^2}$ and $\lambda = 1$ in Theorems 1.1–1.2, we get the results of Theorem 1.2.
- By taking $\sigma(z) = \frac{(c+1)\overline{z}}{1-|z|^2}$ in (1.12), we get [8, Proposition 1], which is a special harmonic case of Theorem A for univalence.
- By putting $\sigma(z) = \frac{(c+1)\overline{z}}{1-|z|^2}$ in (1.13) and Theorem 1.2, we get the results of [20, Theorems 1 and 2] and [7, Theorem 2.2], respectively, which are special harmonic case of Theorem A for quasiconformal extensions.
- By setting $\sigma(z) = \frac{\overline{z}}{1-|z|^2}$ in Theorem 1.1, we get the result of [9, Theorem 1.1], which is a special harmonic case of Theorem A for quasiconformal extensions.

1.3 Inner radius of univalency and quasidisk

For a simply connected domain D in $\overline{\mathbb{C}}$, the inner radius of univalency of D by the Schwarzian derivative and the pre-Schwarzian derivative are defined as

$$\rho(D) = \sup\{a : \|S\phi\|_D \le a \Rightarrow \phi \text{ univalent in } D\}$$

and

$$\rho_1(D) = \sup\{a : \|P\phi\|_D \le a \Rightarrow \phi \text{ univalent in } D\},$$

respectively. The inner radius of univalency has a significant meaning to judge the quasidisk. It was shown by Gehring [15] that $\rho(D) > 0$ if and only if D is a quasidisk. Instead of $\rho(D)$, Astala and Gehring [4] proved that $\rho_1(D) > 0$ if and only if D is a quasidisk. Recently, Efraimidis proved similar results as Gehring's by using the inner radius of univalency of D by the Schwarzian derivative of harmonic mappings, we refer to [14] for details.

For a locally injective sense-preserving harmonic mapping f in a domain D, we define

$$\|P_f\|_D = \sup_{z\in D}\Big\{\Big(|P_f| + \frac{|\omega'|}{1-|\omega|^2}\Big)\frac{1}{\eta_D(z)}\Big\}. \tag{1.16}$$

For the domain D, we define the inner radius of univalency by the pre-Schwarzian derivative for harmonic mappings as

$$\widehat{\rho}_1(D) = \sup\Big\{a : \|P_f\|_D \le a \Rightarrow f \text{ univalent in } D\Big\}.$$

Following these definitions, Hu et al. [21] proved that $\widehat{\rho}_1(D) > 0$ if and only if D is a quasidisk.

For the importance of the inner radius of univalency, it is interesting to estimate $\rho(D)$ and $\rho_1(D)$. However, except for some special domains D (see [22–23]), it is difficult to find the exact values of $\rho(D)$ and $\rho_1(D)$. Ahlfors [1], Lehto [24] and Sugawa [33] gave an estimate of $\rho(D)$ as follows.

Theorem C *Let D be a quasidisk and ∂D admit a quasiconformal reflection λ. Then*

$$\rho(D) \geq \inf_{z\in D} \frac{|\overline{\partial}\lambda| - |\partial\lambda|}{|\lambda - z|^2 \eta_D(z)}.$$

Remark 1.3 When λ is continuously differentiable, the idea to get Theorem C belongs to Ahlfors while the representation form of Theorem C is due to Lehto. Sugawa obtained Theorem C by removing the assumption of the continuous differentiability of λ.

Considering $\rho_1(D)$, by using Theorem A, Cheng and Chen [10] gave the estimate of $\rho_1(D)$ as follows.

Theorem D *Let D be a quasidisk and λ be the quasiconformal reflection across ∂D. Then*

$$\rho_1(D) \geq \inf_{z\in D} \frac{|\overline{\partial}\lambda| - |\partial\lambda|}{|\lambda - z| \eta_D(z)}.$$

In this paper, as an application of Theorem 1.1, we give a lower bound of $\widehat{\rho}_1(D)$ as follows.

Theorem 1.3 *Let D be a quasidisk and λ be a quasiconformal reflection across ∂D. Then*

$$\widehat{\rho}_1(D) \geq \inf_{z\in D} \frac{|\overline{\partial}\lambda| - |\partial\lambda|}{|\lambda - z| \eta_D(z)}. \tag{1.17}$$

2 Proof of Theorem 1.1

In this section, following the ideas developed by Hernández and Martín [16, 18], we give the proof of Theorem 1.1. The following lemmas play an important role in the proof of Theorem 1.1.

Lemma 2.1 (see [18]) *Let $f = h + \overline{g}$ be a locally univalent sense-preserving harmonic mapping. For $a \in \triangle$, let*

$$f_{[a]} = f + a\overline{f} = h_a + \overline{g_a},$$

where $h_a = h + ag$ and $g_a = g + \overline{a}h$. Then $P_{f_{[a]}} = P_f$ and the second complex dilatation ω_a of $f_{[a]}$ is

$$\omega_a = \varphi_a \circ \omega,$$

where φ_a is the automorphism of $\triangle$ defined by

$$\varphi_a(z) = \frac{\overline{a} + z}{1 + az}, \quad z \in \triangle.$$

It follows that

$$\frac{|z\omega_a'|}{1 - |\omega_a|^2} = \frac{|z\omega'|}{1 - |\omega|^2}.$$

By Hurwitz's theorem, the following property was shown in [16, p. 622].

Lemma 2.2 (see [16]) *Let $f = h + \overline{g}$ be a sense-preserving harmonic function in $\triangle$. If $h + ag$ is univalent for all $a \in \triangle$, then $h + ag$ is univalent for all $a \in \overline{\triangle}$.*

Lemma 2.3 (see [17]) *Let $f = h + \overline{g}$ be a sense-preserving harmonic function in $\triangle$. If $h + ag$ is univalent for all $a \in \partial\triangle$, then $f = h + \overline{g}$ is univalent in $\triangle$.*

Lemma 2.4 (see [25, Chapter I, Lemma 6.1]) *Let C be a quasicircle and f a homeomorphism of the plane which is K-quasiconformal in the complement of C. Then f is a K-quasiconformal mapping of the plane.*

Proof of Theorem 1.1 We divide the proof of Theorem 1.1 into the following three parts.

Part 1 We will prove that (1.12) implies that f_λ is univalent in $\triangle$ for all $|\lambda| \leq 1$.

If $f = h + \overline{g}$ satisfies the condition (1.12), we know that

$$\begin{aligned}\Big|\sigma\frac{h''}{h'} + \sigma^2 - \sigma_z\Big| &= \Big|\sigma\Big(\frac{h''}{h'} - \frac{\overline{\omega}\omega'}{1-|\omega|^2}\Big) + \sigma^2 - \sigma_z + \frac{\sigma\overline{\omega}\omega'}{1-|\omega|^2}\Big| \\ &\leq |\sigma P_f + \sigma^2 - \sigma_z| + \Big|\frac{\sigma\overline{\omega}\omega'}{1-|\omega|^2}\Big| \\ &\leq |\sigma P_f + \sigma^2 - \sigma_z| + \Big|\frac{\sigma\omega'}{1-|\omega|^2}\Big| \\ &\leq |\sigma_{\overline{z}}|.\end{aligned}$$

Thus, from Theorem A, we see that h is univalent in $\triangle$. By Lemma 2.1, we have $P_{f_{[\lambda]}} = P_f$ and $\frac{|\omega_\lambda'|}{1-|\omega_\lambda|^2} = \frac{|\omega'|}{1-|\omega|^2}$, so $f_{[\lambda]} = h + \lambda g + \overline{g + \overline{\lambda} h}$ also satisfies the hypothesis of Theorem 1.1. Then $h_\lambda = h + \lambda g$ is univalent for all $|\lambda| < 1$. By Lemma 2.2, the functions $h + \lambda g$ are univalent for all $|\lambda| \leq 1$. Then $f_\lambda = h + \lambda\overline{g}$ are univalent for all $|\lambda| \leq 1$ follows from Lemma 2.3.

Part 2 Under the condition (1.13), we will show that (1.14) is a homeomorphic extension of f_λ to $\overline{\mathbb{C}}$. We divide the proof of this part into two steps.

Step 1 When $\lambda = 1$, we shall prove the harmonic mapping $f_1 := f = h + \overline{g}$ has a continuous and injective extension $\widetilde{f}_1$ to $\overline{\triangle}$, and the function F_1 in (1.14) is a homeomorphism extension of $\overline{\mathbb{C}}$. We divide this to four substeps.

Substep 1 We shall prove that for all $a \in \overline{\triangle}$, the functions $h_a = h + ag$ have a continuous and injective extension to $\overline{\mathbb{C}}$. It follows from (1.13) and Lemma 2.1 that for all $z \in \triangle$,

$$\begin{aligned}&\Big|\sigma\frac{h_a''(z)}{h_a'(z)} + \sigma^2 - \sigma_z\Big| \\ &\leq \Big|\sigma\Big(\frac{h''(z) + ag''(z)}{h'(z) + ag'(z)} - \frac{\overline{\omega_a(z)}\omega_a'(z)}{1-|\omega_a(z)|^2}\Big) + \sigma^2 - \sigma_z\Big| + \Big|\frac{\sigma\overline{\omega_a(z)}\omega_a'(z)}{1-|\omega_a(z)|^2}\Big| \\ &\leq \Big|\sigma\Big(\frac{h''(z) + ag''(z)}{h'(z) + ag'(z)} - \frac{\overline{\omega_a(z)}\omega_a'(z)}{1-|\omega_a(z)|^2}\Big) + \sigma^2 - \sigma_z\Big| + \frac{|\sigma\omega_a'(z)|}{1-|\omega_a(z)|^2} \\ &= |\sigma P_{f_{[a]}}(z) + \sigma^2 - \sigma_z| + \Big|\frac{\sigma\omega_a'(z)}{1-|\omega_a(z)|^2}\Big| \\ &\leq k|\sigma_{\overline{z}}|. \end{aligned} \tag{2.1}$$

By Theorem A, it implies that for each $a \in \triangle$, the function h_a is univalent and can be extended to a continuous and injective mapping $\widetilde{h}_a$ in $\overline{\triangle}$ and the function

$$H_a(z) = \begin{cases} \widetilde{h}_a(z), & |z| \le 1, \\ h_a\Big(\dfrac{1}{\overline{z}}\Big) + u_a\Big(\dfrac{1}{\overline{z}}\Big), & |z| > 1 \end{cases} \tag{2.2}$$

is a K-quasiconformal extension of $\overline{\mathbb{C}}$ onto itself with

$$K = \frac{1+k}{1-k},$$

where

$$u_a(z) = \frac{h_a'(z)}{\sigma(z)}, \quad z \in \triangle\backslash\{0\}.$$

Hence, H_a is continuous and univalent in $\overline{\mathbb{C}}$.

Substep 2 We will construct an explicit function for a continuous and injective extension of f to $\overline{\mathbb{C}}$. We assume that $H = H_0$ and H_1 are the corresponding extensions of $h_0 = h$ and $h_1 = h + g$, respectively. By virtue of (2.2), we define the function

$$G(z) = H_1(z) - H(z)$$

to obtain a continuous extension G of g to $\overline{\mathbb{C}}$ as

$$G(z) = \begin{cases} \widetilde{g}(z), & |z| \le 1, \\ g\Big(\dfrac{1}{\overline{z}}\Big) + v\Big(\dfrac{1}{\overline{z}}\Big), & |z| > 1, \end{cases}$$

where

$$v(z) = \frac{g'(z)}{\sigma(z)}, \quad z \in \triangle\backslash\{0\}.$$

Since H_a is univalent in $\overline{\mathbb{C}}$ for all $a \in \overline{\triangle}$, so is

$$H_a(z) = H(z) + aG(z). \tag{2.3}$$

Indeed, it is obvious that (2.3) holds when $|z| \le 1$. For $|z| > 1$,

$$\begin{aligned} H(z) + aG(z) &= h_0\Big(\frac{1}{\overline{z}}\Big) + u_0\Big(\frac{1}{\overline{z}}\Big) + ag\Big(\frac{1}{\overline{z}}\Big) + av\Big(\frac{1}{\overline{z}}\Big) \\ &= h\Big(\frac{1}{\overline{z}}\Big) + ag\Big(\frac{1}{\overline{z}}\Big) + \frac{h'\big(\frac{1}{\overline{z}}\big)}{\sigma\big(\frac{1}{\overline{z}}\big)} + a\frac{g'\big(\frac{1}{\overline{z}}\big)}{\sigma\big(\frac{1}{\overline{z}}\big)} \\ &= H_a(z). \end{aligned}$$

Next, we construct an explicit candidate for a continuous and injective extension of f to $\overline{\mathbb{C}}$ defined by

$$F_1(z) = H(z) + \overline{G(z)} = \begin{cases} f_1(z), & |z| < 1, \\ \widetilde{h}(z) + \overline{\widetilde{g}}(z), & |z| = 1, \\ f_1\Big(\dfrac{1}{\overline{z}}\Big) + U_1\Big(\dfrac{1}{\overline{z}}\Big), & |z| > 1, \end{cases}$$

where U_1 is given by

$$U_1(z)=\frac{h'(z)}{\sigma(z)}+\frac{\overline{g'(z)}}{\overline{\sigma(z)}},\quad z\in\triangle\backslash\{0\}.$$

Substep 3 We will prove the univalence of $F_1(z)$. The univalence of $F_1(z)$ can be obtained from the univalence of $H(z)$ in $\overline{\mathbb{C}}$, which we can infer [16–17].

Substep 4 We will show that F_1 is a homeomorphism extension to $\overline{\mathbb{C}}$. Here, we adopt the method due to Ahlfors discussed in Chiang [11, p. 9]. We take into account f_1 to be locally univalent in $\triangle$. On one hand, it is easy to see that the Jacobian J_{f_1} of f_1 is non-vanishing in $\triangle$. On the other hand, let $Q(z)=f_1(z)+U_1(z)$ be sense-preserving and sufficiently smooth in $\triangle$. Note that F_1 is a K-quasiconformal mapping in $|z|\geq 1$ if and only if Q is a sense-reserving K-quasiconformal mapping in $|z|\leq 1$. Then we can derive the Jacobian of Q, i.e., $J_Q\neq 0$ in $\overline{\triangle}$ from the weaker condition $|Q_z(z)|<|Q_{\overline{z}}(z)|$. Hence, F_1 is locally homeomorphic everywhere in $\overline{\mathbb{C}}$ by the Inverse Function Theorem (see [32, p. 221]) and so homeomorphic in $\overline{\mathbb{C}}$.

Step 2 When $|\lambda|\leq 1$, the harmonic mapping $f_\lambda=h+\lambda\overline{g}$ $(|\lambda|\leq 1)$ has a continuous and injective extension $\widetilde{f}_\lambda$ to $\overline{\triangle}$, and the function F_λ in (1.14) is also a homeomorphism extension of $\overline{\mathbb{C}}$. For $f_\lambda=h+\lambda\overline{g}$, the second complex dilatation ω_λ of f_λ is $\omega_\lambda=\overline{\lambda}\omega$. By observing that

$$|\lambda|+|\omega|\leq 1+|\lambda||\omega|$$

following (1.13) we obtain

$$\begin{aligned}
&|\sigma P_{f_\lambda}+\sigma^2-\sigma_z|+\frac{|\sigma\omega'_\lambda|}{1-|\omega_\lambda|^2}\\
&=\left|\sigma\Big(\frac{h''}{h'}-\frac{\overline{\omega_\lambda}\omega'_\lambda}{1-|\omega_\lambda|^2}\Big)+\sigma^2-\sigma_z\right|+\frac{|\sigma\omega'_\lambda|}{1-|\omega_\lambda|^2}\\
&\leq\left|\sigma\Big(\frac{h''}{h'}-\frac{\overline{\omega}\omega'}{1-|\omega|^2}\Big)+\sigma^2-\sigma_z\right|+\left|\sigma\Big(\frac{\overline{\omega}\omega'}{1-|\omega|^2}-\frac{\overline{\omega_\lambda}\omega'_\lambda}{1-|\omega_\lambda|^2}\Big)\right|+\frac{|\sigma\omega'_\lambda|}{1-|\omega_\lambda|^2}\\
&=|\sigma P_f+\sigma^2-\sigma_z|+\frac{|\sigma||\omega||\omega'|(1-|\lambda|^2)}{(1-|\omega|^2)(1-|\lambda|^2|\omega|^2)}+\frac{|\sigma||\lambda||\omega'|}{1-|\lambda|^2|\omega|^2}\\
&=|\sigma P_f+\sigma^2-\sigma_z|+\frac{|\sigma||\omega'|}{1-|\omega|^2}\frac{|\lambda|+|\omega|}{1+|\lambda||\omega|}\\
&\leq|\sigma P_f+\sigma^2-\sigma_z|+\frac{|\sigma\omega'|}{1-|\omega|^2}.\\
&\leq k|\sigma_{\overline{z}}|.
\end{aligned}\tag{2.4}$$

By Step 1, we can obtain this conclusion.

Part 3 We will show that F_λ is a concrete K-quasiconformal mapping.

Firstly, we shall estimate the maximal complex dilatation of the mapping F_λ. We divide it into the following two cases, since it is a removable set in $\overline{\mathbb{C}}$ for quasiconformality when $|z|=1$ (see [26, p. 45, Theorem 8.3]).

Case 1 If $|z|<1$, then

$$|\mu_{F_\lambda}|=|\lambda||\omega(z)|\leq|\lambda|\|\omega\|_\infty<1$$

for $z \in \triangle$.

Case 2 If $|z| > 1$, we make a reciprocal transformation $w = \frac{1}{\overline{z}}$ for some $|z| < 1$, then

$$
\begin{aligned}
|\mu_{F_\lambda}(w)| &= \Big|\frac{h'(z) + (U_\lambda)_z(z)}{\lambda\overline{g'(z)} + (U_\lambda)_{\overline{z}}(z)}\Big| \\
&= \left|\frac{h' + \dfrac{h''\sigma - h'\sigma_z}{\sigma^2} - \lambda\dfrac{\overline{g'}\,\overline{\sigma_{\overline{z}}}}{\overline{\sigma}^2}}{\lambda\overline{g'} + \lambda\dfrac{\overline{g''}\,\overline{\sigma} - \overline{g'}\,\overline{\sigma_z}}{\overline{\sigma}^2} - \dfrac{h'\sigma_{\overline{z}}}{\sigma^2}}\right| \\
&= \left|\frac{\dfrac{\sigma^2}{\sigma_{\overline{z}}} + \dfrac{\sigma}{\sigma_{\overline{z}}}\dfrac{h''}{h'} - \dfrac{\sigma_z}{\sigma_{\overline{z}}} - \lambda\overline{\omega}\dfrac{\sigma^2}{\overline{\sigma}^2}\dfrac{\overline{\sigma_{\overline{z}}}}{\sigma_{\overline{z}}}\dfrac{\overline{h'}}{h'}}{\overline{\lambda}\dfrac{\sigma^2}{\sigma_{\overline{z}}}\omega + \overline{\lambda}\dfrac{\sigma}{\sigma_{\overline{z}}}\dfrac{g''}{h'} - \overline{\lambda}\dfrac{\sigma_z}{\sigma_{\overline{z}}}\omega - \dfrac{\sigma^2}{\overline{\sigma}^2}\dfrac{\overline{\sigma_{\overline{z}}}}{\sigma_{\overline{z}}}\dfrac{\overline{h'}}{h'}}\right| \\
&\le \frac{\Big|\dfrac{\sigma^2}{\sigma_{\overline{z}}} + \dfrac{\sigma}{\sigma_{\overline{z}}}\dfrac{h''}{h'} - \dfrac{\sigma_z}{\sigma_{\overline{z}}}\Big| + |\lambda|\|\omega\|_\infty}{1 - |\lambda|\Big|\dfrac{\sigma^2}{\sigma_{\overline{z}}}\omega - \dfrac{\sigma_z}{\sigma_{\overline{z}}}\omega + \dfrac{\sigma}{\sigma_{\overline{z}}}\dfrac{g''}{h'}\Big|}. \qquad (2.5)
\end{aligned}
$$

Since f satisfies the condition (1.13), by the triangle inequality, we get

$$
\begin{aligned}
\Big|\frac{\sigma^2}{\sigma_{\overline{z}}} + \frac{\sigma}{\sigma_{\overline{z}}}\frac{h''}{h'} - \frac{\sigma_z}{\sigma_{\overline{z}}}\Big| &= \Big|\frac{\sigma}{\sigma_{\overline{z}}}\Big(\frac{h''}{h'} - \frac{\overline{\omega}\omega'}{1-|\omega|^2}\Big) + \frac{\sigma^2}{\sigma_{\overline{z}}} - \frac{\sigma_z}{\sigma_{\overline{z}}} + \frac{\sigma}{\sigma_{\overline{z}}}\frac{\overline{\omega}\omega'}{(1-|\omega|^2)}\Big| \\
&\le \Big|\frac{\sigma}{\sigma_{\overline{z}}}\Big(\frac{h''}{h'} - \frac{\overline{\omega}\omega'}{1-|\omega|^2}\Big) + \frac{\sigma^2}{\sigma_{\overline{z}}} - \frac{\sigma_z}{\sigma_{\overline{z}}}\Big| + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\overline{\omega}\omega'}{(1-|\omega|^2)}\Big| \\
&\le k - \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{1-|\omega|^2}\Big| + \|\omega\|_\infty\Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big| \\
&= k - (1 - \|\omega\|_\infty)\Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big|. \qquad (2.6)
\end{aligned}
$$

On the other hand, we have

$$g'' = (\omega h')' = \omega' h' + \omega h''.$$

Therefore, for all $z \in \triangle$, we obtain

$$
\begin{aligned}
\Big|\frac{\sigma^2}{\sigma_{\overline{z}}}\omega - \frac{\sigma_z}{\sigma_{\overline{z}}}\omega + \frac{\sigma}{\sigma_{\overline{z}}}\frac{g''}{h'}\Big| &= \Big|\frac{\sigma^2}{\sigma_{\overline{z}}}\omega - \frac{\sigma_z}{\sigma_{\overline{z}}}\omega + \frac{\sigma}{\sigma_{\overline{z}}}\frac{h''}{h'}\omega + \frac{\sigma}{\sigma_{\overline{z}}}\omega'\Big| \\
&= \Big|\frac{\sigma\omega}{\sigma_{\overline{z}}}\Big(\frac{h''}{h'} - \frac{\overline{\omega}\omega'}{1-|\omega|^2}\Big) + \frac{\sigma^2}{\sigma_{\overline{z}}}\omega - \frac{\sigma_z}{\sigma_{\overline{z}}}\omega + \frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big| \\
&\le \|\omega\|_\infty\Big|\frac{\sigma}{\sigma_{\overline{z}}}P_f + \frac{\sigma^2}{\sigma_{\overline{z}}} - \frac{\sigma_z}{\sigma_{\overline{z}}}\Big| + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big| \\
&\le k\|\omega\|_\infty - \|\omega\|_\infty\Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big| + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big| \\
&= k\|\omega\|_\infty + (1 - \|\omega\|_\infty)\Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}\Big|. \qquad (2.7)
\end{aligned}
$$

By substituting (2.6)–(2.7) into (2.5), we know

$$
\begin{aligned}
|\mu_{F_\lambda}(w)| &\le \frac{k-(1-\|\omega\|_\infty)|\omega^*(z)|+|\lambda|\|\omega\|_\infty}{1-(k|\lambda|\|\omega\|_\infty+(1-\|\omega\|_\infty)|\omega^*(z)|)} \\
&= \frac{k+|\lambda|\|\omega\|_\infty-(1-\|\omega\|_\infty)|\omega^*(z)|}{1-k|\lambda|\|\omega\|_\infty-(1-\|\omega\|_\infty)|\omega^*(z)|},
\end{aligned}
\tag{2.8}
$$

where

$$\omega^*(z)=\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{(1-|\omega|^2)}.$$

Since f satisfies the condition (1.13), we deduce that $|\omega^*(z)|\le k$.

Define the function $\rho(x):[0,k]\to\mathbb{R}$ by

$$\rho(x)=\frac{k+|\lambda|\|\omega\|_\infty-(1-\|\omega\|_\infty)x}{1-k|\lambda|\|\omega\|_\infty-(1-\|\omega\|_\infty)x}.$$

According to the condition (1.10) holds, we see that $\rho'(x)<0$ for all $x\in[0,k]$. Thus

$$\rho(x)\le\rho(0)=\frac{k+|\lambda|\|\omega\|_\infty}{1-k|\lambda|\|\omega\|_\infty}=\frac{K-1}{K+1}.$$

We then find from (2.8) that

$$|\mu_{F_\lambda}(w)|\le\frac{K-1}{K+1}=:k_1$$

holds for all $|w|>1$. Since the assumption (1.10) shows that

$$\|\omega_\lambda\|_\infty=|\lambda|\|\omega\|_\infty\le\|\omega\|_\infty<\frac{1-k}{1+k}$$

holds for all $\lambda\in\overline{\triangle}$, which implies that $k_1<1$. Therefore, $F_\lambda(w)$ is a quasiconformal mapping in $\overline{\mathbb{C}}\backslash\overline{\triangle}$.

Secondly, by Lemma 2.4, the mapping F_λ defined by (1.14) is K-quasiconformal whenever (2.8) holds with

$$|\mu_{F_\lambda}(z)|\le\frac{K-1}{K+1}$$

for all $|z|\ne1$ in $\overline{\mathbb{C}}$. Moreover, by noting that

$$|\lambda|\|\omega\|_\infty\le k_1<1,$$

we deduce that F_λ is a K-quasiconformal mapping of f_λ in $\overline{\mathbb{C}}$, where K is given by (1.15).

3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we will need the following lemmas.

Lemma 3.1 (see [26, Chapter II, Theorem 5.3]) *The limit function ϖ of a sequence ϖ_n of K-quasiconformal mappings convergent in D is either a constant, a mapping of D onto two points, or a K-quasiconformal mapping of D.*

Lemma 3.2 (see [9]) *Let $\varepsilon \in \triangle$ and $T(z) = \frac{z+|\varepsilon|}{1+|\varepsilon|z}$. Then $T(z)$ is a Möbius transformation of the unit disk $\triangle$ onto itself and*

$$\frac{||\varepsilon| - |z||}{1 - |\varepsilon||z|} \le |T(z)| \le \frac{|\varepsilon| + |z|}{1 + |\varepsilon||z|}.$$

Lemma 3.3 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in $\triangle$ with complex dilatation $\omega \not\equiv 0$. Assume that $\|\omega\|_\infty < 1$ and that f satisfies (1.13). Then, the analytic functions $h_a = h + ag$ are univalent in $\triangle$ for all $0 \le |a| < \delta$, where*

$$1 < \delta = \frac{1 + k\|\omega\|_\infty}{k + \|\omega\|_\infty} \le \frac{1}{\|\omega\|_\infty}.$$

Moreover, h_a has a continuous and injective extension $\widetilde{h}_a$ to $\overline{\triangle}$.

Proof In the proof of Theorem 1.1, we have found that h_a has a continuous and injective extension $\widetilde{h}_a$ to $\overline{\triangle}$ for all $|a| \le 1$.

Next, we shall prove that h_a has a continuous and injective extension $\widetilde{h}_a$ to $\overline{\triangle}$ for $1 < |a| < \delta$. By noting that

$$\frac{h_a''}{h_a'} = \frac{h''}{h'} + \frac{a\omega'}{1 + a\omega}, \tag{3.1}$$

in view of (3.1) and the formula (1.7) for pre-Schwarzian derivative of f, we get

$$\begin{aligned}\Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{h_a''}{h_a'} + \frac{\sigma^2}{\sigma_{\overline{z}}} - \frac{\sigma_z}{\sigma_{\overline{z}}}\Big| &\le \Big|\frac{\sigma}{\sigma_{\overline{z}}}P_f + \frac{\sigma^2}{\sigma_{\overline{z}}} - \frac{\sigma_z}{\sigma_{\overline{z}}}\Big| + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{1-|\omega|^2}\frac{\overline{\omega}+a}{1+a\omega}\Big| \\ &\le k - \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{1-|\omega|^2}\Big| + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{1-|\omega|^2}\Big|\Big|\frac{\overline{\omega}+a}{1+a\omega}\Big| \\ &= k + \Big|\frac{\sigma}{\sigma_{\overline{z}}}\frac{\omega'}{1-|\omega|^2}\Big|\Big(\Big|\frac{\overline{\omega}+a}{1+a\omega}\Big| - 1\Big) \\ &\le k\Big|\frac{\overline{\omega}+a}{1+a\omega}\Big|. \end{aligned} \tag{3.2}$$

From Lemma 3.2, we know that

$$\sup_{z\in\triangle}\Big|\frac{\omega + \overline{a}}{1 + a\omega}\Big| \le \frac{|a| - \|\omega\|_\infty}{1 - \|\omega\|_\infty|a|} < \frac{1}{k} \tag{3.3}$$

with

$$1 < |a| < \delta = \frac{1 + k\|\omega\|_\infty}{k + \|\omega\|_\infty} \le \frac{1}{\|\omega\|_\infty}.$$

Combining (3.2)–(3.3) with Theorem A, we conclude that h_a is univalent in $\triangle$ and h_a has a continuous and injective extension $\widetilde{h}_a$ to $\overline{\triangle}$ for all $0 \le |a| < \delta$, where

$$1 < \delta = \frac{1 + k\|\omega\|_\infty}{k + \|\omega\|_\infty} \le \frac{1}{\|\omega\|_\infty}.$$

We thus complete the proof of Lemma 3.3.

By using the similar method as in [16] (also see [14, 20]), we can prove the following lemma. For completeness, we give the proof.

Lemma 3.4 *Let $f = h + \overline{g}$ satisfy the hypothesis of Lemma 3.3. Assume in addition that both h and g are analytic functions in $\overline{\triangle}$. Then*

$$\sup_{\alpha,\beta\in\overline{\triangle},\alpha\neq\beta}\Big|\frac{\lambda(g(\alpha)-g(\beta))}{h(\alpha)-h(\beta)}\Big| = \sup_{\alpha,\beta\in\overline{\triangle},\alpha\neq\beta}\Big|\frac{\lambda\overline{(g(\alpha)-g(\beta))}}{h(\alpha)-h(\beta)}\Big| \leq \frac{1}{\delta} = \frac{k+\|\omega\|_\infty}{1+k\|\omega\|_\infty} < 1 \tag{3.4}$$

for $|\lambda| \leq 1$.

Proof We fix $\beta \in \overline{\triangle}$ and define the generalized dilatation

$$\psi_\beta(\alpha) = \begin{cases} \dfrac{\lambda(g(\alpha)-g(\beta))}{h(\alpha)-h(\beta)}, & \alpha \neq \beta, \\ \omega(\alpha), & \alpha = \beta, \end{cases}$$

where $\alpha \in \overline{\triangle}$, $|\lambda| \leq 1$ and ω denote the second complex dilatation of f. Evidently, $\psi_\beta(\alpha)$ is continuous for $|\alpha| \leq 1$, so that there exists an $\alpha_0 \in \overline{\triangle}$ such that $\sup\limits_{\alpha\in\overline{\triangle}} |\varphi_\beta(\alpha)| = |\varphi_\beta(\alpha_0)|$.

We consider two cases that $\alpha_0 = \beta$ and $\alpha_0 \neq \beta$. If $\alpha_0 = \beta$, then

$$\sup_{\alpha\in\overline{\triangle}} |\varphi_\beta(\alpha)| \leq \|\omega\|_\infty.$$

By $\delta \leq \frac{1}{\|\omega\|_\infty}$, we get

$$\sup_{\alpha\in\overline{\triangle}} |\varphi_\beta(\alpha)| \leq \frac{1}{\delta}.$$

If $\alpha_0 \neq \beta$, we will use the law of disproof to get contradicts with Lemma 3.3. Assume that $\sup\limits_{\alpha\in\overline{\triangle}} |\varphi_\beta(\alpha)| > \frac{1}{\delta}$. Then there exist $0 < \varepsilon < \delta$ and $|\mu| = 1$ such that

$$\frac{\lambda(g(\alpha_1)-g(\beta))}{h(\alpha_1)-h(\beta)} = \frac{\mu}{\delta-\varepsilon}, \tag{3.5}$$

where $\alpha_1 \in \overline{\triangle}$ and $|\lambda| \leq 1$. (3.5) shows that $h - \overline{\mu}(\delta-\varepsilon)\lambda g$ is not univalent in $\overline{\triangle}$, which contradicts with Lemma 3.3.

Proof of Theorem 1.2 For the harmonic mappings f_λ, we have

$$(f_\lambda)_r(z) = f_\lambda(rz) = h(rz) + \lambda\overline{g(rz)}, \quad 0 < r < 1.$$

Each of these functions $(f_\lambda)_r$ satisfies

$$|\lambda||\omega_r| = |\lambda||\omega(rz)|,$$

so that

$$|\lambda|\|\omega_r\|_\infty = \|\omega_r\|_\infty \leq \|\omega\|_\infty < 1.$$

If we prove that the mapping $(f_\lambda)_r$ can be extended to a K-quasiconformal mapping in $\overline{\mathbb{C}}$, where K does not depend on r, then by Lemma 3.1, we conclude that f_λ can be extended to a K-quasiconformal mapping in $\overline{\mathbb{C}}$.

We note that if f satisfies (1.13), then

$$\begin{aligned}
&\sup_{z\in\triangle}\Big\{|\sigma P_{(f_\lambda)_r}+\sigma^2-\sigma_z|+\frac{|\sigma\omega'|}{1-|\omega|^2}\Big\}\\
&\le r\sup_{z\in\triangle}|\sigma P_{f_\lambda}+\sigma^2-\sigma_z|+r\sup_{z\in\triangle}\frac{|\sigma\omega'|}{1-|\omega|^2}\\
&\le r\sup_{z\in\triangle}|\sigma P_f+\sigma^2-\sigma_z|+r\sup_{z\in\triangle}\frac{|\sigma\omega'|}{1-|\omega|^2}\\
&\le rk|\sigma_{\overline{z}}|\\
&\le k|\sigma_{\overline{z}}|.
\end{aligned}$$

Therefore, $(f_\lambda)_r$ satisfies the condition (1.13). By Lemmas 3.3–3.4, and use the similar method as in [20, Theorem 3], we get the assertion $(f_\lambda)_r$ admits a K-quasiconformal reflection. By allowing $r\to 1$, we deduce that f_λ has a quasiconformal extension to $\overline{\mathbb{C}}$. The proof of Theorem 1.2 is thus completed.

4 Proof of Theorem 1.3

In this section, we will give the proof of Theorem 1.3 following the idea of Cheng and Chen [10].

Proof of Theorem 1.3 According to the Riemann Mapping Theorem, we let g be a conformal mapping from $\triangle$ onto domain D and $w=g(z)(z\in\triangle)$. Set $\tau=\frac{\lambda o g}{g}$ and $\sigma=-\frac{g'}{g}\frac{1}{1-\tau}$, then

$$\sigma_z=-\frac{g''}{g}\frac{1}{1-\tau}+\frac{(g')^2}{g^2}\frac{1}{1-\tau}-\frac{g'}{g}\frac{\partial_z\tau}{(1-\tau)^2} \tag{4.1}$$

and

$$\sigma_{\overline{z}}=-\frac{g'}{g}\frac{\partial_{\overline{z}}\tau}{(1-\tau)^2}. \tag{4.2}$$

Substituting (4.1)–(4.2) into (1.13), we get

$$\begin{aligned}
&\Big|-\frac{g'}{g}\frac{1}{1-\tau}P_f+\frac{(g')^2}{g^2}\frac{1}{(1-\tau)^2}+\frac{g''}{g}\frac{1}{1-\tau}-\frac{(g')^2}{g^2}\frac{1}{1-\tau}+\frac{g'}{g}\frac{\partial_z\tau}{(1-\tau)^2}\Big|\\
&+\Big|\frac{g'}{g}\frac{1}{1-\tau}\frac{\omega'}{1-|\omega|^2}\Big|\le k\Big|\frac{g'}{g}\frac{\partial_{\overline{z}}\tau}{(1-\tau)^2}\Big|,
\end{aligned}$$

i.e.,

$$\Big|-(1-\tau)(P_f-Pg)+\frac{g'}{g}\tau+\partial_z\tau\Big|+\Big|\frac{(1-\tau)\omega'}{1-|\omega|^2}\Big|\le k|\partial_{\overline{z}}\tau|. \tag{4.3}$$

For any locally univalent and sense-preserving harmonic mapping f in $\triangle$, (4.3) implies that f has a quasiconformal extension. We can see from (4.3) that

$$\Big|\frac{1}{\eta_\triangle}(P_f-Pg)-\frac{\partial_z(g\tau)}{g(1-\tau)\eta_\triangle}\Big|+\Big|\frac{1}{\eta_\triangle}\frac{\omega'}{1-|\omega|^2}\Big|\le k\frac{|\partial_{\overline{z}}(g\tau)|}{|g(1-\tau)\eta_\triangle|}. \tag{4.4}$$

On the other hand, if f satisfies the inequality

$$\begin{aligned}\left|\frac{1}{\eta_\triangle}(P_f - Pg)\right| + \left|\frac{1}{\eta_\triangle}\frac{\omega'}{(1-|\omega|^2)}\right| &\le \inf_{z\in\triangle} \frac{k|\partial_{\overline{z}}(\lambda\circ g(z))| - |\partial_z(\lambda\circ g(z))|}{|\lambda\circ g(z) - g(z)|\eta_\triangle(z)}\\ &= \inf_{w\in D}\frac{k|\partial_{\overline{w}}(\lambda(w))|-|\partial_w(\lambda(w))|}{|\lambda(w)-w|\eta_D(w)},\end{aligned} \tag{4.5}$$

then f must satisfy the inequality (4.4), which implies that f is univalent.

Note that $\eta_\triangle = \eta_D(g)|g'(z)|$, then

$$\|P_{f\circ g^{-1}}\|_D = \|P_f - Pg\| = \sup_{z\in\triangle}\Big\{\Big(|P_f - Pg| + \frac{|\omega'|}{1-|\omega|^2}\Big)\eta_\triangle^{-1}\Big\}. \tag{4.6}$$

It follows from (4.5)–(4.6) that if f satisfies the inequality

$$\|P_{f\circ g^{-1}}\|_D \le \inf_{w\in D}\frac{k|\partial_{\overline{w}}\lambda(w)|-|\partial_w\lambda(w)|}{|\lambda(w)-w|\eta_D(w)},$$

then f is univalent and hence $f\circ g^{-1}$ is univalent on D. Therefore,

$$\widehat{\rho}_1(D) \ge \sup_{0\le k<1}\Big\{\inf_{w\in D}\frac{k|\partial_{\overline{w}}\lambda(w)|-|\partial_w\lambda(w)|}{|\lambda(w)-w|\eta_D(w)}\Big\} = \inf_{w\in D}\frac{|\partial_{\overline{w}}\lambda(w)|-|\partial_w\lambda(w)|}{|\lambda(w)-w|\eta_D(w)}.$$

Acknowledgement The authors would like to express gratitude to the referees for their careful reading and valuable suggestions to improve the paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Ahlfors, L., Quasiconformal reflections, *Acta Math.,* **109**, 1963, 291–301.

[2] Ahlfors, L., Sufficient conditions for quasiconformal extension, Discontinuous groups and Riemann surfaces, *Ann. of Math. Studies*, **79**, 1974, 23–29.

[3] Ahlfors, L. and Weill, G., A uniqueness theorem for Beltrami equation, *Proc. Amer. Math. Soc.*, **13**, 1962, 975–978.

[4] Astala, K. and Gehring, F., Injectivity, the BMO norm and the universal Teichmüller space, *J. Analyse Math.*, **46**, 1986, 16–57.

[5] Becker, J., Löwnersche differentialgleichung und quasikonform fortsetzbare schlichet functionen, *J. Reine Angew. Math.*, **255**, 1972, 23–43 (in German).

[6] Becker, J. and Pommerenke, C., Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math.*, **354**, 1984, 74–94 (in German).

[7] Bhowmik, B. and Satpati, G., Quasiconformal extension of integral transforms of analytic and harmonic mappings, *Complex Var. Elliptic Equ.*, **68**(10), 2023, 1734–1750.

[8] Bravo, V., Hernández, R. and Venegas, O., On the univalence of certain integral for harmonic mappings, *J. Math. Anal. Appl.*, **455**(1), 2017, 381–388.

[9] Chen, X. and Que, Y., Quasiconformal extensions of harmonic mappings with a complex parameter, *J. Aust. Math. Soc.*, **102**(3), 2017, 307–315.

[10] Cheng, T. and Chen, J., On the inner radius of univalency by pre-Schwarzian derivative, *Sci. China Ser. A.*, **50**(7), 2007, 987–996.

[11] Chiang, Y. M., Schwarzian derivative and second order differential equations, Ph. D. thesis, Univ. of London, London, 1991.

[12] Clunie, J. and Sheil-Small, T., Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **9**, 1984, 3–25.

[13] Duren, P., Harmonic Mappings in the Plane, Cambridge University Press, Cambridge, 2004.

[14] Efraimidis, I., Criteria for univalence and quasiconformal extension for harmonic mappings on planar domains, *Ann. Fenn. Math.*, **46**(2), 2021, 1123–1134.

[15] Gehring, F. W., Univalent functions and the Schwarzian derivative, *Comment. Math. Helv.*, **52**(4), 1977, 561–572.

[16] Hernández, R. and Martín, M. J., Quasiconformal extension of harmonic mappings in the plane, *Ann. Acad. Sci. Fenn. Math.,* **38**(2), 2013, 617–630.

[17] Hernández, R. and Martín, M. J., Stable geometric properties of analytic and harmonic functions, *Math. Proc. Cambridge Philos. Soc.*, **155**(2), 2013, 343–359.

[18] Hernández, R. and Martín, M. J., Pre-Schwarzian and Schwarzian derivatives of harmonic mappings, *J. Geom. Anal.*, **25**(1), 2015, 64–91.

[19] Hille, E., Remarks on a paper by Zeev Nehari, *Bull. Amer. Math. Soc.*, **55**, 1949, 552–553.

[20] Hu, Z. and Fan, J., Criteria for univalency and quasiconformal extension for harmonic mappings, *Kodai Math. J.*, **44**(2), 2021, 273–289.

[21] Hu, Z., Fan, J. and Wang, X., Quasiconformal extensions and inner radius of univalence by pre-Schwarzian derivatives of analytic and harmonic mappings, *J. Math. Phys. Anal. Geo.*, **19**(4), 2024, 781–798.

[22] Lehtinen, M., Estimates of the inner radius of univalency of domains bounded by conic sections, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **10**, 1985, 349–353.

[23] Lehtinen, M., Angles and the inner radius of univalency, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **11**(2), 1986, 161–165.

[24] Lehto, O., Remark on Nehari's theorem about the Schwarzian derivative and Schlicht function, *J. Analyse Math.*, **36**, 1979, 184–190.

[25] Lehto, O., Univalent Functions and Teichmüller Spaces, Springer-Verlag, New York, 1987.

[26] Lehto, O. and Virtanen, K. I., Quasiconformal Mappings in the Plane, Springer-Verlag, Berlin, New York-Heidelberg, 1973.

[27] Lewy, H., On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, **42**(10), 1936, 689–692.

[28] Liu, G. and Ponnusamy, S., Uniformly locally univalent harmonic mappings associated with the pre-Schwarzian norm, *Indag. Math.* (*N.S.*), **29**(2), 2018, 752–778.

[29] Liu, G. and Ponnusamy, S., Harmonic pre-Schwarzian and its applications, *Bull. Sci. Math.*, **152**, 2019, 150–168.

[30] Nehari, Z., The Schwarzian derivative and Schlicht functions, *Bull. Amer. Math. Soc.*, **55**, 1949, 545–551.

[31] Nie, L. and Yang, Z., The Schwarzian derivative of harmonic mappings in the plane, *Chinese Ann. Math. Ser. B*, **41**(2), 2020, 193–208.

[32] Rudin, W., Principles of Mathematical Analysis, 3rd., McGraw-Hill, Book Co., New York-Auckland-Dusseldorf, 1976.

[33] Sugawa, T., A remark on the Ahlfors-Lehto univalence criterion, *Ann. Acad. Sci. Fenn. A I Math.*, **27**(1), 2002, 151–161.

[34] Tang, S., Strongly quasiconformal extension of harmonic mappings, *Comput. Methods Funct. Theory*, **21**(2), 2021, 297–305.