Chin. Ann. Math. Ser. B
47(3), 2026, 491–510
DOI: 10.1007/s11401-026-0020-0

Chinese Annals of Mathematics, Series B

General Results on Strong Laws for Weighted Sums Under Sub-linear Expectations with a Statistical Application*

Yi WU[1] Mengmei XI[2] Xuejun WANG[3]

Abstract In this paper, the authors investigate the double-indexed version of the strong law of large numbers under some general conditions in a sub-linear expectation space. The weighted version of the Marcinkiewicz-Zygmund type strong law of large numbers is also established. These results extend or improve some existing ones in the classical probability space or a sub-linear expectation space. As an application, they further study the nonparametric regression model under the sub-linear expectation framework. Some numerical simulations are also presented.

Keywords Strong law of large numbers, Sub-linear expectations, Nonparametric regression, Numerical simulation
2020 MR Subject Classification 60F15, 62G05

1 Introduction

The classical strong laws of large numbers (SLLN for short) are fundamental tools in probability theory. Besides, many well-known estimators in statistics such as the least squares estimators, nonparametric regression function estimators, jackknife estimators, and so on, are the form of weighted sums of random variables. Therefore, it is desirable and of great interest to study the limit behavior of weighted sums of random variables. The central limit theorem and other weak laws for the sums of independent and identically distributed (i.i.d. for short) random variables have been extended to weighted sums and many other dependence cases in the past decades. However, it is not easy to establish the SLLN, especially the Marcinkiewicz-Zygmund type SLLN. Consequently, the study of Marcinkiewicz-Zygmund type SLLN for weighted sums is still a hot topic.

Manuscript received February 1, 2023. Revised August 10, 2023.
[1]School of Big Data and Artificial Intelligence, Chizhou University, Chizhou 247000, Auhui, China. E-mail: wuyi8702@163.com
[2]School of Big Data and Statistics, Anhui University, Hefei 230601, China. E-mail: ximengmei123@163.com
[3]Corresponding author. School of Big Data and Statistics, Anhui University, Hefei 230601, China. E-mail: wxjahdx2000@126.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12471248, 12201079, 12301181, 12361031), the Natural Science Foundation of Anhui Province (No. 2308085MA07), the Provincial Natural Science Research Project of Anhui Colleges (No. KJ2021A1095) and the Excellent Scientific Research and Innovation Team of Anhui Colleges (No. 2022AH010098).

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying

$$\sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} |a_{ni}|^{\alpha} < \infty \tag{1.1}$$

for some $\alpha > 0$. For an i.i.d. sequence $\{X; X_n, n \geq 1\}$ with $EX = 0$, $EX^2 < \infty$ and the weights $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfying (1.1) with $\alpha = 2$, Chow [7] presented the Kolmogorov SLLN, i.e.,

$$n^{-1} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad \text{a.s.}$$

Cuzick [8] extended the result of Chow [7] under the condition $EX = 0$, $E|X|^{\beta} < \infty$ for some $\beta > 0$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Bai and Cheng [1] established the following Marcinkiewicz-Zygmund type SLLN

$$n^{-\frac{1}{p}} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad \text{a.s.} \tag{1.2}$$

under the condition $EX = 0$, $E|X|^{\beta} < \infty$ for some $\beta > 0$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$ with $1 \leq p < 2$. Chen and Gan [3] proved that (1.2) also holds for $0 < p < 1$. Later on, these results above were generalized to many dependent and much more general cases. For more details, we refer the readers to [2, 4, 13, 15, 26, 31] among others.

The results above are all established in the classical probability space, where the probability measures and mathematical expectations are assumed to be additive. However, this additivity assumption is not always plausible because uncertain phenomena cannot be modeled using additive probabilities or additive expectations. Recently, motivated by some problems in statistics, risk measures, mathematical economics and super-hedging in finance, more and more researchers have adopted non-additive probability and non-linear expectation to describe some uncertain phenomena in these fields. Non-additive probabilities and non-additive expectations are useful tools in studying uncertainties in statistics, risk measures, super-hedging in finance and non-linear stochastic calculus. For further details, one can refer to [9, 12, 17–20]. Therefore, statisticians have devoted themselves to investigating the limit theorems for sub-linear expectations in a general function space in recent years. Peng [19–21] introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to sub-additivity and positive homogeneity.

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}| \quad \text{for any } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$. We also let $C_{b,Lip}(\mathbb{R}^n)$ denote the space of bounded Lipschitz functions φ satisfying

$$|\varphi(\boldsymbol{x})| \le C, \ |\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C|\boldsymbol{x} - \boldsymbol{y}| \quad \text{for any } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

for some $C > 0$ depending on φ.

Definition 1.1 *A sub-linear expectation* $\widehat{\mathbb{E}}$ *on* $\mathcal{H}$ *is a functional* $\widehat{\mathbb{E}} : \mathcal{H} \to \overline{\mathbb{R}}$ *satisfying the following properties: For all* $X, Y \in \mathcal{H}$*, we have the following results.*

(a) *Monotonicity: If* $X \ge Y$*, then* $\widehat{\mathbb{E}}[X] \ge \widehat{\mathbb{E}}[Y]$.

(b) *Constant preserving:* $\widehat{\mathbb{E}}[c] = c$.

(c) *Sub-additivity:* $\widehat{\mathbb{E}}[X+Y] \le \widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ *whenever* $\widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ *is not of the form* $+\infty-\infty$ *or* $-\infty+\infty$.

(d) *Positive homogeneity:* $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda > 0$.

Here $\overline{\mathbb{R}} = [-\infty, +\infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space. For a given sub-linear expectation $\widehat{\mathbb{E}}$, denote the conjugate expectation $\widehat{\varepsilon}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\varepsilon}[X] := -\widehat{\mathbb{E}}[-X] \quad \text{for any } X \in \mathcal{H}.$$

From the definition, it is easy to show that $\widehat{\varepsilon}\,[X] \le \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$ and $\widehat{\mathbb{E}}[X - Y] \ge \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Furthermore, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\varepsilon}\,[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

However, it is very hard to study limit theorems unless some additional conditions are assumed. Peng [21] gave a reasonable definition of independence through non-linear expectation. A random vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $\boldsymbol{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$, under $\widehat{\mathbb{E}}$, if for each test function $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$,

$$\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}], \tag{1.3}$$

whenever $\overline{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}[|\overline{\varphi}(\boldsymbol{X})|] < \infty$. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be independent if

$$\widehat{\mathbb{E}}[\varphi(X_1, \cdots, X_n, X_{n+1})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x_1, \cdots, x_n, X_{n+1})]|_{x_1=X_1,\cdots,x_n=X_n}]$$

for all $n \ge 1$ and $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^{n+1})$. Under this framework, many limit theorems have been established successively. We refer the readers to Peng [21–22] for the central limit theorem and weak law of large numbers, Zhang [32] for the small deviation and Chung's law of the iterated logarithm, Zhang [34] for the law of the iterated logarithm, and so on. Because of the significance of the SLLN, this issue has also been generalized to the sub-linear expectation space by some authors. For example, Chen [5] obtained the Kolmogorov SLLN for an i.i.d. sequence under the sub-linear expectations with $\widehat{\mathbb{E}}[|X|^{1+\alpha}] < \infty$ for some $\alpha > 0$; Cheng [6] derived the Kolmogorov SLLN which improves the corresponding one of Chen [5]; Hu et al. [14] also established the Kolmogorov SLLN and presented some applications; Zhang [33] proved the sufficient and necessary conditions for the Kolmogorov SLLN; Wu and Jiang [30] extended the Kolmogorov SLLN to the Marcinkiewicz-Zygmund type SLLN under some general conditions. As far as we know, there is no literature investigating the Marcinkiewicz-Zygmund type SLLN for weighted sums under sub-linear expectations. Inspired by Li et al. [15] and Wu and Jiang [30], we will study the weighted version of Marcinkiewicz-Zygmund type SLLN under sub-linear expectations. Since many basic properties or tools and common methods for classical probability theory are no longer available for sub-linear expectation, the study on limit theorems in sub-additive expectation space is quite different from that in classical probability space.

Zhang [34] extended the concept of independent random variables to negatively dependent random variables in the sub-linear expectation space as follows.

A random vector $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be negatively dependent of another random vector $\boldsymbol{X} = (X_1, \cdots, X_n)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$, if for each pair of test functions $\varphi_1 \in C_{l,\mathrm{Lip}}(\mathbb{R}^m)$ and $\varphi_2 \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$,

$$\widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})] \le \widehat{\mathbb{E}}[\varphi_1(\boldsymbol{X})]\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})], \tag{1.4}$$

whenever $\varphi_1(\boldsymbol{X}) \ge 0$, $\widehat{\mathbb{E}}[\varphi_2(\boldsymbol{Y})] \ge 0$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})\varphi_2(\boldsymbol{Y})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_1(\boldsymbol{X})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_2(\boldsymbol{Y})|] < \infty$, and either φ_1 and φ_2 are coordinatewise nondecreasing or φ_1 and φ_2 are coordinatewise non-increasing.

As pointed out in Zhang [34], if $\boldsymbol{Y}$ is independent of $\boldsymbol{X}$, then $\boldsymbol{Y}$ is negatively dependent of $\boldsymbol{X}$. Moreover, from the definition of negative dependence, it can be seen that the fact that $\boldsymbol{Y}$ is negatively dependent of $\boldsymbol{X}$ does not imply that $\boldsymbol{X}$ is negatively dependent to $\boldsymbol{Y}$.

Peng [21] introduced the concept of identical distribution as follows.

Definition 1.2 *Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two n-dimensional random vectors defined, respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, if*

$$\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)] \quad \text{for any } \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n),$$

whenever the sub-linear expectations are finite.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1, \quad V(A) \le V(B), \quad \forall A \subset B,\ A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \cup B) \le V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

In the sub-linear space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) = \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) = 1 - \mathbb{V}(A^c) \quad \text{for any } A \in \mathcal{F},$$

where A^c is the complement set of A. Thus, we have

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \mathcal{V}(A) := \widehat{\varepsilon}[I_A], \quad \text{if } I_A \in \mathcal{H},$$

and

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\varepsilon}[f] \leq \mathcal{V}(A) \leq \widehat{\varepsilon}[g], \quad \text{if } f \leq I_A \leq g, \ f, g \in \mathcal{H}.$$

It is obvious that $\mathbb{V}$ is sub-additive. Also, the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ can be defined by

$$C_V[X] = \int_0^\infty V(X \geq t)\mathrm{d}t + \int_{-\infty}^0 [V(X \geq t) - 1]\mathrm{d}t$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

In this paper, we investigate the double-indexed version of the strong law of large numbers under some general conditions in a sub-linear expectation space. The weighted version of the Marcinkiewicz-Zygmund type strong law of large numbers is also established. These results extend the corresponding ones of Li et al. [15] from the classical probability space to a sub-linear expectation space and also improve the corresponding ones of Wu and Jiang [30] under sub-linear expectations. As an application, we further study the nonparametric regression model under sub-linear expectation framework. Some numerical simulations are presented to verify the validity of the theoretical results.

The layout of the paper is as follows: The main results are stated in Section 2. The proofs of the main results are provided in Section 3. An application of the main results and some numerical simulations are presented in Section 4. Throughout the paper, the symbol C denotes positive constants which may vary from place to place. The representation A a.s. $\mathcal{V}$ means $\mathcal{V}(A) = 1$. Denote $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, and $\lfloor x \rfloor$ denotes the integer part not exceeding x.

2 Main Results

For $0 < \mu < 1$, let $g(x) \in C_{l,\mathrm{Lip}}(\mathbb{R})$ be even and non-increasing on $[0, \infty)$ such that $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$ and $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x \geq 1$. Then

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1), \quad I(|x| > 1) \leq 1 - g(|x|) \leq I(|x| > \mu). \tag{2.1}$$

For each $1 \leq i \leq n$ and $n \geq 1$, let $X_i(b_n) := X_i g\big(\frac{|X_i|}{b_n}\big)$, where $\{b_n, n \geq 1\}$ is a sequence of positive constants. Now we present our main results as follows.

Theorem 2.1 *Let $0 < p < 2$, $p < \alpha \le \infty$ with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. Let $\{b_n, n \ge 1\}$ be a sequence of positive constants satisfying $0 < b_n/n^{\frac{1}{\beta}} \uparrow$, $\frac{b_{2n}}{b_n} = O(n^{\frac{1}{\alpha}\wedge 1})$, and $a_n := n^{\frac{1}{\alpha}} b_n$. Assume that $\{X; X_n, n \ge 1\}$ is a sequence of identically distributed random variables such that X_i is negatively dependent to $(X_{i+1}, \cdots, X_n)$ for each $1 \le i \le n-1$ and $\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty$, where $\mathbb{V}$ is countably sub-additive. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real constants satisfying* (1.1), *which can be interpreted as $\sup_{n\ge 1} \max_{1\le i\le n} |a_{ni}| < \infty$ if $\alpha = \infty$. Then*

$$\limsup_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni}(X_i - \widehat{\mathbb{E}}[X_i(b_n)]) \le 0 \quad a.s.\ \mathcal{V} \tag{2.2}$$

and

$$\liminf_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni}(X_i - \widehat{\varepsilon}[X_i(b_n)]) \ge 0 \quad a.s.\ \mathcal{V}. \tag{2.3}$$

Remark 2.1 It is easy to see that if $\alpha = \infty$ and $a_{ni} \equiv 1$, Theorem 2.1 reduces to Wu and Jiang [30, Theorem 3.3]. Therefore, our result improves the corresponding one of Wu and Jiang [30] for Marcinkiewicz-Zygmund type SLLN and thus Hu et al. [14], Chen [5], Cheng [6], Zhang [33] and so on for Kolmogorov's SLLN. Moreover, Theorem 2.1 extends the corresponding one of Li et al. [15] from classical probability space to sub-linear expectation space. Furthermore, if $\{b_n\}$ is polynomial increasing, then the restriction $\frac{b_{2n}}{b_n} = O(n^{\frac{1}{\alpha}\wedge 1})$ is satisfied.

From Theorem 2.1, we can conclude the following two corollaries.

Corollary 2.1 *Suppose that the assumptions of Theorem* 2.1 *hold. Assume further that $\frac{a_n}{n} \uparrow \infty$ if $\alpha \ge 1$. Then*

$$\limsup_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni} X_i \le 0 \quad a.s.\ \mathcal{V},$$

$$\liminf_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni} X_i \ge 0 \quad a.s.\ \mathcal{V},$$

and thus

$$a_n^{-1} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad a.s.\ \mathcal{V}.$$

Corollary 2.2 *Suppose that the assumptions of Theorem* 2.1 *hold with $\widehat{\mathbb{E}}[|X|] < \infty$. Assume further that $\frac{n}{a_n} \uparrow$ and $\widehat{\mathbb{E}}$ is countably sub-additive if $\alpha \ge 1$. Then*

$$\limsup_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni}(X_i - \widehat{\mathbb{E}}[X_i]) \le 0 \quad a.s.\ \mathcal{V}$$

and

$$\liminf_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} a_{ni}(X_i - \widehat{\varepsilon}[X_i]) \ge 0 \quad a.s.\ \mathcal{V}.$$

Moreover, if $\widehat{\mathbb{E}}[X] = \widehat{\varepsilon}[X]$, *then*

$$a_n^{-1}\sum_{i=1}^{n} a_{ni}(X_i - \widehat{\mathbb{E}}[X_i]) \to 0 \quad a.s.\ \mathcal{V}.$$

Let $b_n = n^{\frac{1}{\beta}}$. We derive the following Marcinkiewicz-Zygmund type SLLN for weighted sums of negatively dependent random variables.

Theorem 2.2 *Let* $0 < p < 2$, $p < \alpha \le \infty$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. *Assume that* $\{X; X_n, n \ge 1\}$ *is a sequence of identically distributed random variables such that* X_i *is negatively dependent to* $(X_{i+1}, \cdots, X_n)$ *for each* $1 \le i \le n-1$ *and* $C_{\mathbb{V}}[|X|^\beta] < \infty$, *where* $\mathbb{V}$ *is countably sub-additive. Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of real constants satisfying* (1.1), *which can be interpreted as* $\sup_{n\ge1}\max_{1\le i\le n}|a_{ni}| < \infty$ *if* $\alpha = \infty$. *If* $0 < p < 1$, *then*

$$n^{-\frac{1}{p}}\sum_{i=1}^{n} a_{ni}X_i \to 0 \quad a.s.\ \mathcal{V}.$$

If $1 \le p < 2$, *assume further that* $\widehat{\mathbb{E}}$ *is countably sub-additive, then*

$$\limsup_{n\to\infty} n^{-\frac{1}{p}}\sum_{i=1}^{n} a_{ni}(X_i - \widehat{\mathbb{E}}[X_i]) \le 0 \quad a.s.\ \mathcal{V}$$

and

$$\liminf_{n\to\infty} n^{-\frac{1}{p}}\sum_{i=1}^{n} a_{ni}(X_i - \widehat{\varepsilon}[X_i]) \ge 0 \quad a.s.\ \mathcal{V}.$$

Remark 2.2 For $1 \le p < 2$, since $\widehat{\mathbb{E}}$ is countably sub-additive, by Zhang [33, Lemma 3.9] we have $\widehat{\mathbb{E}}[|X|^\beta] \le C_{\mathbb{V}}[|X|^\beta] < \infty$, which together with the Jensen's inequality and $\beta > p$ gives $\widehat{\mathbb{E}}[|X|] < \infty$.

3 Proofs of the Main Results

To prove the main results of this paper, the following lemmas are needed. The first two lemmas can be seen in Zhang [34] and Zhang [33], respectively.

Lemma 3.1 (*Borel-Cantelli Lemma*) *Let* $\{A_n, n \ge 1\}$ *be a sequence of events in* $\mathcal{F}$. *Suppose that* V *is a countably sub-additive capacity. If* $\sum_{n=1}^{\infty} V(A_n) < \infty$, *then* $V(A_n, \text{i.o.}) = 0$, *where* $\{A_n, i.o.\} = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m$.

Lemma 3.2 *Suppose that* X_i *is negatively dependent to* $(X_{i+1}, \cdots, X_n)$ *for each* $1 \le i \le n-1$ *and* $\widehat{\mathbb{E}}[X_i] \le 0$, $1 \le i \le n$. *Then for* $1 \le p \le 2$,

$$\widehat{\mathbb{E}}\Big[\Big|\max_{1\le k\le n}\sum_{i=1}^{k} X_i\Big|^p\Big] \le 2^{2-p}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|X_i|^p].$$

Lemma 3.3 *Let $\{b_n, n \geq 1\}$ be a sequence of positive constants satisfying $0 < b_n/n^{\frac{1}{\beta}} \uparrow$ and X be a random variable under sub-linear expectations. Then for any $c > 0$,*

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty \quad \textit{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{V}(|X| > cb_n) < \infty.$$

Proof The proof is inspired by Sung [26]. It follows from $0 < b_n/n^{\frac{1}{\beta}} \uparrow$ that $\{b_n, n \geq 1\}$ is strictly increasing. Hence, we have $b_{2n} < b_{2n+1}$ for each $n \geq 1$ and thus

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) \leq \mathbb{V}(|X| > b_1) + 2\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_{2n}). \tag{3.1}$$

By virtue of $b_n/n^{\frac{1}{\beta}} \leq b_{2n}/(2n)^{\frac{1}{\beta}}$, we have $2^{\frac{1}{\beta}} b_n \leq b_{2n}$ and hence

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) \geq \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{\frac{1}{\beta}} b_n) \geq \sum_{n=1}^{\infty} \mathbb{V}(|X| > b_{2n}). \tag{3.2}$$

A combination of (3.1)–(3.2) gives

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{\frac{1}{\beta}} b_n) < \infty.$$

Replacing b_n with $2^{\frac{1}{\beta}} b_n$ in (3.1)–(3.2), we have

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{\frac{2}{\beta}} b_n) < \infty.$$

Repeating the foregoing procedure for $k-2$ times more, we obtain that

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{\frac{k}{\beta}} b_n) < \infty. \tag{3.3}$$

Replacing b_n with $2^{-\frac{k}{\beta}} b_n$ in (3.3), we further have

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{-\frac{k}{\beta}} b_n) < \infty. \tag{3.4}$$

Therefore, if $c > 1$, we have that $c \leq 2^{\frac{k}{\beta}}$ for some $k \geq 1$ and thus

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) \geq \sum_{n=1}^{\infty} \mathbb{V}(|X| > cb_n) \geq \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{\frac{k}{\beta}} b_n),$$

which together with (3.3) yields the desired result. If $0 < c < 1$, the assertion also follows from (3.4) and

$$\sum_{n=1}^{\infty} \mathbb{V}(|X| > b_n) \leq \sum_{n=1}^{\infty} \mathbb{V}(|X| > cb_n) \leq \sum_{n=1}^{\infty} \mathbb{V}(|X| > 2^{-\frac{k}{\beta}} b_n)$$

for some $k \geq 1$ such that $c \geq 2^{-\frac{k}{\beta}}$. The proof is completed.

With the three lemmas above, we can now give the proofs of the main results in Section 2.

Proof of Theorem 2.1 Note from Lemma 3.3 that $\sum_{n=1}^{\infty}\mathbb{V}(|X|>b_n)<\infty$ is equivalent to $\sum_{n=1}^{\infty}\mathbb{V}(|X|>cb_n)<\infty$ for any $c>0$. Therefore, we have

$$\infty>\sum_{n=1}^{\infty}\mathbb{V}(|X|>cb_n)=\sum_{k=1}^{\infty}\sum_{2^{k-1}\le n<2^k}\mathbb{V}(|X|>cb_n)\ge\frac{1}{2}\sum_{k=1}^{\infty}2^k\mathbb{V}(|X|>cb_{2^k}). \tag{3.5}$$

For each n, there exists a positive integer k such that $2^{k-1}\le n<2^k$. Hence, we can obtain

$$\begin{aligned}&\frac{\sum_{i=1}^{n}a_{ni}(X_i-\widehat{\mathbb{E}}[X_i(b_n)])}{a_n}\\&=\frac{\sum_{i=1}^{n}a_{ni}(X_i-\widehat{\mathbb{E}}[X_i(b_{2^k})])+\sum_{i=1}^{n}a_{ni}(\widehat{\mathbb{E}}[X_i(b_{2^k})]-\widehat{\mathbb{E}}[X_i(b_n)])}{a_n}\\&\le\frac{\Big|\max_{1\le n<2^k}\sum_{i=1}^{n}a_{ni}(X_i-\widehat{\mathbb{E}}[X_i(b_{2^k})])\Big|}{a_{2^{k-1}}}+\frac{\max_{2^{k-1}\le n<2^k}\sum_{i=1}^{n}|a_{ni}||\widehat{\mathbb{E}}[X(b_{2^k})]-\widehat{\mathbb{E}}[X(b_n)]|}{a_{2^{k-1}}}\\&=:I_1(k)+I_2(k).\end{aligned}$$

Consequently, in order to prove (2.2), it suffices to show that

$$\limsup_{k\to\infty}I_1(k)\le 0\quad\text{a.s. }\mathcal{V},\qquad\lim_{k\to\infty}I_2(k)=0.$$

Noting that $g(x)$ is non-increasing for $x\ge 0$, we obtain that

$$\begin{aligned}|X(b_{2^k})-X(b_n)|&\le|X|\Big|g\Big(\frac{|X|}{b_{2^k}}\Big)-g\Big(\frac{|X|}{b_{2^{k-1}}}\Big)\Big|\\&\le|X|I(\mu b_{2^{k-1}}<|X|\le b_{2^k})\\&\le b_{2^k}I(|X|>\mu b_{2^{k-1}}).\end{aligned}$$

In the sequel, we may assume without loss of generality that $a_{ni}=0$ if $i>n$. It follows from (1.1) and Hölder's inequality that $\sum_{i=1}^{n}|a_{ni}|^s=O(n)$ for any $0<s\le\alpha$. Hence, we obtain from (3.5) that if $\alpha\ge 1$,

$$\begin{aligned}I_2(k)&\le C\frac{2^k\widehat{\mathbb{E}}\Big[|X|\Big|g\Big(\frac{|X|}{b_{2^k}}\Big)-g\Big(\frac{|X|}{b_{2^{k-1}}}\Big)\Big|\Big]}{a_{2^{k-1}}}\\&\le C\frac{b_{2^k}2^k\mathbb{V}(|X|>\mu b_{2^{k-1}})}{a_{2^{k-1}}}\\&\le C\sup_{n\ge1}\frac{b_{2n}}{a_n}2^k\mathbb{V}(|X|>\mu b_{2^{k-1}})\to 0\quad\text{ as }k\to\infty;\end{aligned}$$

if $0<\alpha<1$, noting that

$$\sum_{i=1}^{n}|a_{ni}|\le\Big(\sum_{i=1}^{n}|a_{ni}|^{\alpha}\Big)^{\frac{1}{\alpha}}=O(n^{\frac{1}{\alpha}}),\tag{3.6}$$

we can also obtain that

$$\begin{aligned}I_2(k)&\le C\frac{b_{2^k}2^{\frac{k}{\alpha}}\mathbb{V}(|X|>\mu b_{2^{k-1}})}{a_{2^{k-1}}}\\&\le C\frac{b_{2^k}2^{-k}}{b_{2^{k-1}}}2^k\mathbb{V}(|X|>\mu b_{2^{k-1}})\to0\quad\text{as }k\to\infty.\end{aligned}$$

Next we turn to deal with $I_1(k)$. It is easy to obtain that

$$\begin{aligned}&\sum_{k=1}^{\infty}\mathbb{V}\Big(\Big|\max_{1\le n\le2^k}\sum_{i=1}^{n}a_{ni}(X_i-\widehat{\mathbb{E}}[X_i(b_{2^k})])\Big|\ge\epsilon a_{2^{k-1}}\Big)\\&\le\sum_{k=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{2^k}|a_{ni}X_i|\Big(1-g\Big(\frac{|X_i|}{b_{2^k}}\Big)\Big)>0\Big)\\&\quad+\sum_{k=1}^{\infty}\mathbb{V}\Big(\Big|\max_{1\le n\le2^k}\sum_{i=1}^{n}a_{ni}(X_i(b_{2^k})-\widehat{\mathbb{E}}[X_i(b_{2^k})])\Big|\ge\epsilon a_{2^{k-1}}\Big)\\&=:J_1+J_2.\end{aligned}$$

For J_1, it follows from (2.1) and (3.5) that

$$\begin{aligned}J_1&\le\sum_{k=1}^{\infty}\mathbb{V}\Big(\sum_{i=1}^{2^k}|a_{ni}X_i|I(|X_i|>\mu b_{2^k})>0\Big)\\&\le\sum_{k=1}^{\infty}\mathbb{V}\Big(\bigcup_{i=1}^{2^k}(|X_i|>\mu b_{2^k})\Big)\le\sum_{k=1}^{\infty}\sum_{i=1}^{2^k}\mathbb{V}(|X_i|>\mu b_{2^k})\\&\le\sum_{k=1}^{\infty}\sum_{i=1}^{2^k}\widehat{\mathbb{E}}\Big[1-g\Big(\frac{|X_i|}{\mu b_{2^k}}\Big)\Big]=\sum_{k=1}^{\infty}2^k\widehat{\mathbb{E}}\Big[1-g\Big(\frac{|X|}{\mu b_{2^k}}\Big)\Big]\\&\le\sum_{k=1}^{\infty}2^k\mathbb{V}(|X|>\mu^2b_{2^k})<\infty.\end{aligned}$$

Define $Y_i=X_i(b_{2^k})-\widehat{\mathbb{E}}[X_i(b_{2^k})]$ for each $i\ge1$. Let $g_j(x)\in C_{l,\mathrm{Lip}}(\mathbb{R})$, $j\ge1$ such that $0\le g_j(x)\le1$ for all x and $g_j\big(\frac{x}{b_{2^j}}\big)=1$ if $b_{2^{j-1}}<x\le b_{2^j}$, $g_j\big(\frac{x}{b_{2^j}}\big)=0$ if $x\le\mu b_{2^{j-1}}$ or $x>(1+\mu)b_{2^j}$. Then we have

$$|X|^qg\Big(\frac{|X|}{b_{2^k}}\Big)\le b_1^q+\sum_{j=1}^{k}|X|^qg_j\Big(\frac{|X|}{b_{2^j}}\Big)\quad\text{for any }q>0\tag{3.7}$$

and

$$g_j\Big(\frac{|X|}{b_{2^j}}\Big)\le I(\mu b_{2^{j-1}}<|X|\le(1+\mu)b_{2^j}).\tag{3.8}$$

It follows from $g(\cdot) \in C_{l,\mathrm{Lip}}(\mathbb{R})$ that $\{Y_n, n \geq 1\}$ is still a sequence of identically distributed random variables such that Y_i is negatively dependent to $(Y_{i+1}, \cdots, Y_n)$ for each $1 \leq i \leq n-1$ with $\widehat{\mathbb{E}}[Y_i] = 0$, $1 \leq i \leq n$. If $\alpha \geq 1$, let $q = 2 \wedge \alpha$; we then derive from Markov's inequality, Jensen's inequality, Lemma 3.2, (2.1), (3.5), (3.7)–(3.8) and $b_n/n^{\frac{1}{\beta}} \uparrow$ that

$$\begin{aligned}
J_2 &\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-q} \widehat{\mathbb{E}} \Big| \max_{1 \leq n \leq 2^k} \sum_{i=1}^{n} a_{ni}(X_i(b_{2^k}) - \widehat{\mathbb{E}}[X_i(b_{2^k})]) \Big|^q \\
&\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-q} \sum_{i=1}^{2^k} |a_{ni}|^q \widehat{\mathbb{E}}\Big[|X|^q g\Big(\frac{|X|}{b_{2^k}}\Big)\Big] \\
&\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-q} 2^k \Big\{ b_1^q + \sum_{j=1}^{k} \widehat{\mathbb{E}}\Big[|X|^q g_j\Big(\frac{|X|}{b_{2^j}}\Big)\Big]\Big\} \\
&\leq C \sum_{k=1}^{\infty} 2^{k(1-\frac{q}{p})} + C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-q} 2^k \sum_{j=1}^{k} (1+\mu)^q b_{2^j}^q \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&\leq C \sum_{j=1}^{\infty} b_{2^j}^q \mathbb{V}(|X| > \mu b_{2^{j-1}}) \sum_{k=j}^{\infty} \Big(\frac{2^{\frac{k-1}{\beta}}}{b_{2^{k-1}}} \Big)^q 2^{k(1-\frac{q}{p})} \\
&\leq C \sum_{j=1}^{\infty} \Big(\frac{b_{2^j}}{b_{2^{j-1}}} \Big)^q 2^{j(1-\frac{q}{\alpha})} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&\leq C \sum_{j=1}^{\infty} 2^{j-1} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&< \infty.
\end{aligned}$$

If $0 < \alpha < 1$, we derive further from (3.5)–(3.6) that

$$\begin{aligned}
J_2 &\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-1} \widehat{\mathbb{E}} \Big[\Big| \max_{1 \leq n \leq 2^k} \sum_{i=1}^{n} a_{ni}(X_i(b_{2^k}) - \widehat{\mathbb{E}}[X_i(b_{2^k})]) \Big| \Big] \\
&\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-1} \sum_{i=1}^{2^k} |a_{ni}| \widehat{\mathbb{E}}\Big[|X| g\Big(\frac{|X|}{b_{2^k}}\Big)\Big] \\
&\leq C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-1} 2^{\frac{k}{\alpha}} \Big\{ b_1 + \sum_{j=1}^{k} \widehat{\mathbb{E}}\Big[|X| g_j\Big(\frac{|X|}{b_{2^j}}\Big)\Big]\Big\} \\
&\leq C \sum_{k=1}^{\infty} 2^{k(\frac{1}{\alpha}-\frac{1}{p})} + C \sum_{k=1}^{\infty} a_{2^{k-1}}^{-1} 2^{\frac{k}{\alpha}} \sum_{j=1}^{k} (1+\mu) b_{2^j} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&\leq C \sum_{j=1}^{\infty} b_{2^j} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \sum_{k=j}^{\infty} \frac{2^{\frac{k-1}{\beta}}}{b_{2^{k-1}}} 2^{k(\frac{1}{\alpha}-\frac{1}{p})} \\
&\leq C \sum_{j=1}^{\infty} \frac{b_{2^j}}{b_{2^{j-1}}} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&\leq C \sum_{j=1}^{\infty} 2^{j-1} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \\
&< \infty.
\end{aligned}$$

Therefore, we have proved that

$$\sum_{k=1}^{\infty}\mathbb{V}\Big(\Big|\max_{1\le n\le 2^k}\sum_{i=1}^{n}a_{ni}(X_i-\widehat{\mathbb{E}}[X_i(b_{2^k})])\Big|\ge \epsilon a_{2^{k-1}}\Big)<\infty,$$

which together with Lemma 3.1 yields $\limsup_{k\to\infty} I_1(k)\le 0$ *a.s.* $\mathcal{V}$ and thus (2.2) follows immediately. Replacing X_i by $-X_i$ for each $i\ge 1$ in (2.2), we can obtain (2.3). This completes the proof of the theorem.

Proof of Corollary 2.1 Note that

$$a_n^{-1}\Big|\sum_{i=1}^{n}a_{ni}\widehat{\mathbb{E}}[X_i(b_n)]\Big|\le a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}[|X(b_n)|]$$

and

$$a_n^{-1}\Big|\sum_{i=1}^{n}a_{ni}\widehat{\varepsilon}[X_i(b_n)]\Big|\le a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\varepsilon}[|X(b_n)|]\le a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}[|X(b_n)|].$$

Therefore, it suffices to prove

$$a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}[|X(b_n)|]\to 0\qquad \text{as } n\to\infty.$$

For $2^{k-1}\le n<2^k$, it follows from $g(x)\downarrow$ on $[0,\infty)$, (3.7)–(3.8) that if $\alpha\ge 1$,

$$\begin{aligned}a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}[|X(b_n)|]&\le (a_{2^{k-1}})^{-1}2^k\widehat{\mathbb{E}}\Big[|X|g\Big(\frac{|X|}{b_{2^k}}\Big)\Big]\\&\le 2b_1 2^{k-1}(a_{2^{k-1}})^{-1}+(1+\mu)2^k(a_{2^{k-1}})^{-1}\sum_{j=1}^{k}b_{2^j}\mathbb{V}(|X|>\mu b_{2^{j-1}}).\end{aligned}$$

Since $\frac{a_n}{n}\uparrow\infty$, we can obtain that $2^{k-1}(a_{2^{k-1}})^{-1}\to 0$ as $k\to\infty$. Noting from (3.5) that

$$\sum_{j=1}^{\infty}2^j(a_{2^{j-1}})^{-1}b_{2^j}\mathbb{V}(|X|>\mu b_{2^{j-1}})\le C\sum_{j=1}^{\infty}2^{j-1}\mathbb{V}(|X|>\mu b_{2^{j-1}})<\infty,$$

thus, we conclude by Kronecker's lemma that

$$2^k(a_{2^{k-1}})^{-1}\sum_{j=1}^{k}b_{2^j}\mathbb{V}(|X|>\mu b_{2^{j-1}})\to 0\qquad\text{as } k\to\infty.$$

If $0<\alpha<1$, we obtain further from (3.6)–(3.7) that

$$a_n^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}[|X(b_n)|]\le 2^{\frac{1}{\alpha}}b_1 2^{\frac{k-1}{\alpha}}(a_{2^{k-1}})^{-1}+(1+\mu)2^{\frac{k}{\alpha}}(a_{2^{k-1}})^{-1}\sum_{j=1}^{k}b_{2^j}\mathbb{V}(|X|>\mu b_{2^{j-1}}).$$

Obviously,

$$2^{\frac{k-1}{\alpha}}(a_{2^{k-1}})^{-1}\to 0$$

as $k \to \infty$. On the other hand, similar to the proof of the case $\alpha \geq 1$, by

$$\sum_{j=1}^{\infty} 2^{\frac{j}{\alpha}} (a_{2^{j-1}})^{-1} b_{2^j} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \leq C \sum_{j=1}^{\infty} 2^{j-1} \mathbb{V}(|X| > \mu b_{2^{j-1}}) < \infty$$

and Kronecker's lemma, we have

$$2^{\frac{k}{\alpha}} (a_{2^{k-1}})^{-1} \sum_{j=1}^{k} b_{2^j} \mathbb{V}(|X| > \mu b_{2^{j-1}}) \to 0 \quad \text{as } k \to \infty.$$

The proof of the corollary is completed.

Proof of Corollary 2.2 For any $n \geq 1$, there always exists a positive integer k such that $2^{k-1} \leq n < 2^k$. Therefore, similar to the proof of Corollary 2.1, it is enough to show

$$\begin{aligned} a_n^{-1} \sum_{i=1}^{n} |a_{ni}| \widehat{\mathbb{E}}\Big[|X|\Big(1 - g\Big(\frac{|X|}{b_n}\Big)\Big)\Big] &\leq (a_{2^{k-1}})^{-1} \sum_{i=1}^{2^k} |a_{ni}| \widehat{\mathbb{E}}\Big[|X|\Big(1 - g\Big(\frac{|X|}{b_{2^{k-1}}}\Big)\Big)\Big] \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

It follows from the definitions of $g(\cdot)$ and $g_j(\cdot)$ that

$$1 - g\Big(\frac{|X|}{b_{2^{k-1}}}\Big) \leq \sum_{j=k}^{\infty} g_j\Big(\frac{|X|}{\mu b_{2^j}}\Big). \tag{3.9}$$

Hence, if $\alpha \geq 1$, we derive from $\frac{n}{a_n} \uparrow$, (2.1), (3.5), (3.8)–(3.9), and the countable sub-additivity of $\widehat{\mathbb{E}}$ that

$$\begin{aligned} &(a_{2^{k-1}})^{-1} \sum_{i=1}^{2^k} |a_{ni}| \widehat{\mathbb{E}}\Big[|X|\Big(1 - g\Big(\frac{|X|}{b_{2^{k-1}}}\Big)\Big)\Big] \\ &\leq C 2^{k-1} (a_{2^{k-1}})^{-1} \sum_{j=k}^{\infty} \widehat{\mathbb{E}}\Big[|X| g_j\Big(\frac{|X|}{\mu b_{2^j}}\Big)\Big] \\ &\leq C 2^{k-1} (a_{2^{k-1}})^{-1} \sum_{j=k}^{\infty} b_{2^j} \mathbb{V}(|X| > \mu^2 b_{2^{j-1}}) \\ &\leq C \sum_{j=k}^{\infty} b_{2^j} (a_{2^{j-1}})^{-1} 2^{j-1} \mathbb{V}(|X| > \mu^2 b_{2^{j-1}}) \\ &\leq C \sum_{j=k}^{\infty} 2^{j-1} \mathbb{V}(|X| > \mu^2 b_{2^{j-1}}) \to 0 \quad \text{as k} \to \infty. \end{aligned}$$

If $0 < \alpha < 1$, we obtain from (3.6) and $\widehat{\mathbb{E}}[|X|] < \infty$ that

$$(a_{2^{k-1}})^{-1} \sum_{i=1}^{2^k} |a_{ni}| \widehat{\mathbb{E}}\Big[|X|\Big(1 - g\Big(\frac{|X|}{b_{2^{k-1}}}\Big)\Big)\Big] \leq C 2^{\frac{k}{\alpha}} (a_{2^{k-1}})^{-1} \widehat{\mathbb{E}}[|X|] \to 0 \quad \text{as } k \to \infty.$$

Therefore, the proof of the corollary is completed.

4 Application in Nonparametric Regression Model

In this section, we will present an application of the main results established in Section 2 to nonparametric regression models based on negatively dependent errors.

Consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \xi_i, \quad i = 1, 2, \cdots, n, \ n \geq 1, \tag{4.1}$$

where x_{ni} are known fixed design points from A, $A \subset \mathbb{R}^m$ is a given compact set for some $m \geq 1$, $f(\cdot)$ is an unknown regression function defined on A and ξ_i are random errors. As an estimator of $f(\cdot)$, we consider the following weighted regression estimator:

$$f_n(x) = \sum_{i=1}^{n} \omega_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^m, \tag{4.2}$$

where $\omega_{ni}(x) = \omega_{ni}(x; x_{n1}, x_{n2}, \cdots, x_{nn})$, $i = 1, 2, \cdots, n$ are the weight functions.

The above weighted estimator (4.2) was first proposed by Stone [25] and adapted by Georgiev [11] to the fixed design case and then constantly studied by many authors. One can refer to Fan [10], Liang and Jing [16], Roussas [23], Roussas et al. [24], Tran et al. [28], Thanh and Yin [27], Wang et al. [29], among others.

4.1 Limit property for the weighted estimator

In this subsection, we will study the limit property of the estimator (4.2) in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Let $c(f)$ denote the set of continuity points of the function f on A. Let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^m$.

Now we present the limit property for the estimator (4.2) in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. For any fixed design point $x \in A$, the following assumptions on the nonnegative weight functions $\omega_{ni}(x)$ will be used:

(A_1) $\sum_{i=1}^{n} \omega_{ni}(x) \to 1$ as $n \to \infty$;

(A_2) $\sum_{i=1}^{n} \omega_{ni}(x) \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$;

(A_3) $\mathbb{V}$ and $\widehat{\mathbb{E}}$ are countably sub-additive.

It is known that the design assumptions (A_1)–(A_2) are satisfied for the nearest neighbor weights, Gasser-Müller weights

$$\omega_{ni}(x) = \frac{K\Big(\dfrac{x - x_{ni}}{h_n}\Big)}{\sum_{j=1}^{n} K\Big(\dfrac{x - x_{nj}}{h_n}\Big)}$$

under suitable conditions, and so on. Assumption (A_3) is a technical requirement which can also be easily satisfied. Based on the assumptions above, we present the following result for the nonparametric regression estimator $f_n(x)$.

Theorem 4.1 *Let $1 \le p < 2$, $p < \alpha \le \infty$ and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. Let $\{\xi, \xi_n, n \ge 1\}$ be a sequence of identically distributed random errors such that ξ_i is negatively dependent on $(\xi_{i+1}, \cdots, \xi_n)$ for each $1 \le i \le n-1$ with $C_{\mathbb{V}}[|X|^\beta] < \infty$ in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. Suppose that (A_1)–(A_3) hold and*

$$\sum_{i=1}^n [\omega_{ni}(x)]^\alpha = O(n^{1-\frac{\alpha}{p}}), \tag{4.3}$$

which can be interpreted as $\max\limits_{1 \le i \le n} \omega_{ni}(x) = O(n^{-\frac{1}{p}})$ if $\alpha = \infty$. Then for all $x \in c(f)$,

$$\widehat{\varepsilon}[\xi] \le \liminf_{n\to\infty}(f_n(x) - f(x)) \le \limsup_{n\to\infty}(f_n(x) - f(x)) \le \widehat{\mathbb{E}}[\xi] \quad a.s.\ \mathcal{V}, \tag{4.4}$$

i.e.,

$$\mathcal{V}\Big(\widehat{\varepsilon}[\xi] \le \liminf_{n\to\infty}(f_n(x) - f(x)) \le \limsup_{n\to\infty}(f_n(x) - f(x)) \le \widehat{\mathbb{E}}[\xi]\Big) = 1.$$

Moreover, if $\widehat{\varepsilon}[\xi] = \widehat{\mathbb{E}}[\xi] = 0$, then

$$f_n(x) \to f(x) \quad a.s.\ \mathcal{V}.,\ n \to \infty.$$

Remark 4.1 We point out that condition (4.3) can be easily satisfied. For example, if $\max\limits_{1\le i\le n} \omega_{ni}(x) = O(n^{-\frac{1}{p}})$, then it is easy to check that (4.3) holds.

Remark 4.2 In Theorem 4.1, we assume that $\mathbb{V}$ is countably sub-additive to obtain the results on strong consistency with respect to $\mathcal{V}$. If $\mathbb{V}$ is not countably sub-additive, we can define an outer capacity $\mathbb{V}^*$ as in [35] by

$$\mathbb{V}^*(A) = \inf\Big\{\sum_{n=1}^\infty \mathbb{V}(A_n) : A \in \bigcup_{n=1}^\infty A_n\Big\}, \quad \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), \quad A \in \mathcal{F}.$$

Then $\mathbb{V}^*(A)$ is a countably sub-additive capacity with $\mathbb{V}^*(A) \le \mathbb{V}(A)$.

Proof of Theorem 4.1 In order to prove (4.4), it suffices to show

$$\sum_{i=1}^n \omega_{ni}(x) f(x_{ni}) - f(x) \to 0 \quad \text{ as } n \to \infty, \tag{4.5}$$

$$\limsup_{n\to\infty} \sum_{i=1}^n \omega_{ni}(x)(\xi_i - \widehat{\mathbb{E}}[\xi]) = \limsup_{n\to\infty}\Big[f_n(x) - \sum_{i=1}^n \omega_{ni}(x) f(x_{ni})\Big] - \widehat{\mathbb{E}}[\xi] \le 0 \quad a.s.\ \mathcal{V} \tag{4.6}$$

and

$$\liminf_{n\to\infty} \sum_{i=1}^n \omega_{ni}(x)(\xi_i - \widehat{\varepsilon}[\xi]) = \liminf_{n\to\infty}\Big[f_n(x) - \sum_{i=1}^n \omega_{ni}(x) f(x_{ni})\Big] - \widehat{\varepsilon}[\xi] \ge 0 \quad \text{a.s. } \mathcal{V}. \tag{4.7}$$

Actually, for $a > 0$ and $x \in c(f)$, from (4.1)–(4.2), we obtain that

$$\Big|\sum_{i=1}^n \omega_{ni}(x) f(x_{ni}) - f(x)\Big| \le \sum_{i=1}^n \omega_{ni}(x) \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| \le a)$$

$$+\sum_{i=1}^{n}\omega_{ni}(x)\cdot|f(x_{ni})-f(x)|I(\|x_{ni}-x\|>a)$$
$$+|f(x)|\cdot\Big|\sum_{i=1}^{n}\omega_{ni}(x)-1\Big|. \tag{4.8}$$

It follows from $x\in c(f)$ that for all $\varepsilon>0$, there exists a constant $\delta>0$ such that for all x' satisfying $\|x'-x\|<\delta$, we have $|f(x')-f(x)|<\epsilon$. Setting $a\in(0,\delta)$ in (4.8), we obtain that

$$\begin{aligned}\Big|\sum_{i=1}^{n}\omega_{ni}(x)f(x_{ni})-f(x)\Big| &\le \epsilon\sum_{i=1}^{n}\omega_{ni}(x)\\ &\quad+\sum_{i=1}^{n}\omega_{ni}(x)\cdot|f(x_{ni})-f(x)|I(\|x_{ni}-x\|>a)\\ &\quad+|f(x)|\cdot\Big|\sum_{i=1}^{n}\omega_{ni}(x)-1\Big|.\end{aligned}$$

From assumptions (A_1)–(A_2) and the arbitrariness of ϵ, for all $x\in c(f)$, we derive that (4.5) holds. The conclusions (4.6)–(4.7) follow immediately from (A_3) and Theorem 2.2 by letting $a_{ni}=n^{\frac{1}{p}}\omega_{ni}(x)$ and $X_i=\xi_i$. The proof is completed.

4.2 Numerical simulation

In this subsection, we will use Monte Carlo method and R software to study the numerical performance of the nearest neighbor estimator $f_n(x)$ with negatively dependent errors. First let us recall the concept of the nearest neighbor weight function estimator as follows.

Put $A=[0,1]$ and let $x_{ni}=\frac{i}{n}$, $i=1,2,\cdots,n$. For any $x\in A$, we rewrite

$$|x_{n1}-x|,|x_{n2}-x|,\cdots,|x_{nn}-x|$$

as follows:

$$|x_{n,R_1(x)}-x|\le|x_{n,R_2(x)}-x|\le\cdots\le|x_{n,R_n(x)}-x|,$$

if $|x_{ni}-x|=|x_{nj}-x|$, then $|x_{ni}-x|$ is permuted before $|x_{nj}-x|$ when $x_{ni}<x_{nj}$.

Let $1\le k_n\le n$, the nearest neighbor weight function is defined as follows:

$$\widetilde{\omega}_{ni}(x)=\begin{cases}\dfrac{1}{k_n}, & \text{if } |x_{ni}-x|\le|x_{n,R_{k_n}(x)}-x|,\\ 0, & \text{otherwise.}\end{cases}$$

Reindex the sequence $\{\xi_n\}$ as $\xi_{ij}=\xi_{m(i-1)+j}$, where $1\le i\le k, 1\le j\le m$. For simplicity, let $m=5$ and $\xi_{ij}\overset{\text{i.i.d.}}{\sim}N(\mu_j,4)$ for each $1\le i\le k$ with $\mu_j=-0.5+0.25(j-1)$. It is easy to see that $\{\xi;\xi_n,n\ge1\}$ is a sequence of negatively dependent random variables under sub-linear expectations with $\widehat{\mathbb{E}}[\xi]=\sup_{1\le j\le m}E[\xi_{ij}]=0.5$ and $\widehat{\varepsilon}[\xi]=\inf_{1\le j\le m}E[\xi_{ij}]=-0.5$. Choose $f(x)=x^2-2x$ and $k_n=\lfloor n^{0.8}\rfloor$. Then for $k=100,200,500$, we replicate the experiment for 1000 times to obtain the differences $f_n(x)-f(x)$ with $x=0.3,0.5,0.8$, respectively. The results are presented in Figures 1–9 as follows.

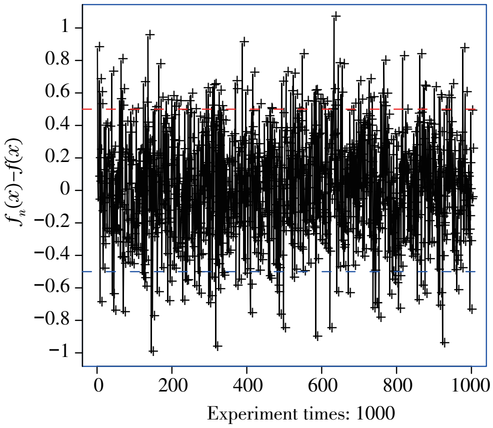

Figure 1 $f_n(x) - f(x)$ with $x = 0.3$, $n = 100$, and $f(x) = x^2 - 2x$.

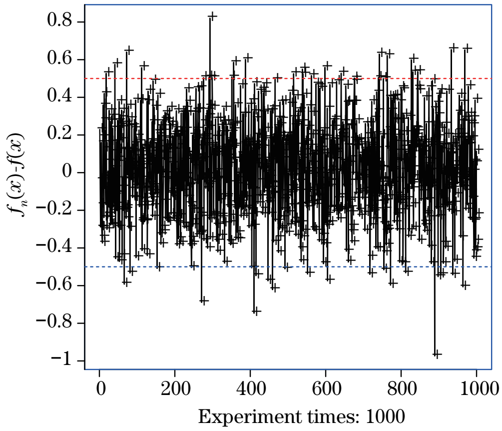

Figure 2 $f_n(x) - f(x)$ with $x = 0.3$, $n = 200$, and $f(x) = x^2 - 2x$.

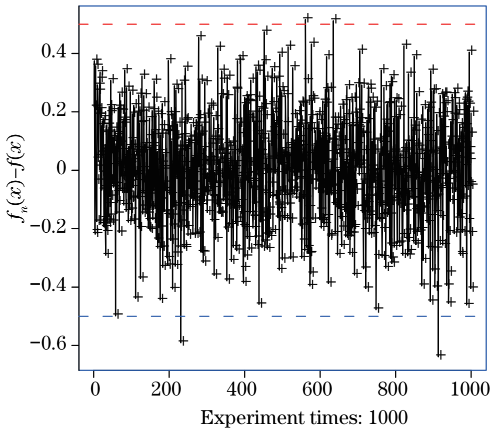

Figure 3 $f_n(x) - f(x)$ with $x = 0.3$, $n = 500$, and $f(x) = x^2 - 2x$.

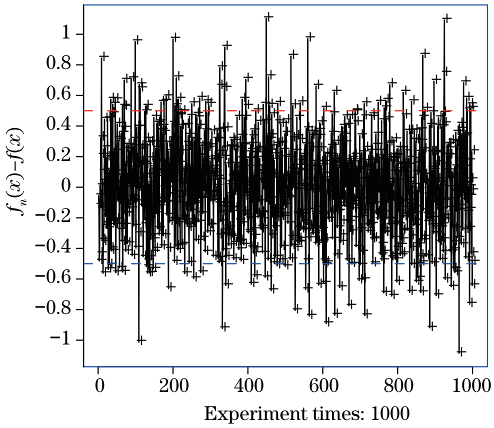

Figure 4 $f_n(x) - f(x)$ with $x = 0.5$, $n = 100$, and $f(x) = x^2 - 2x$.

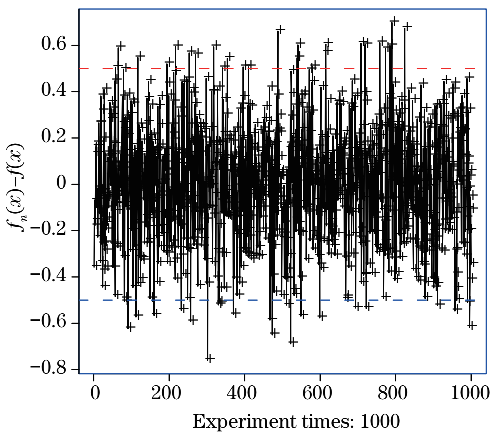

Figure 5 $f_n(x) - f(x)$ with $x = 0.5$, $n = 200$, and $f(x) = x^2 - 2x$.

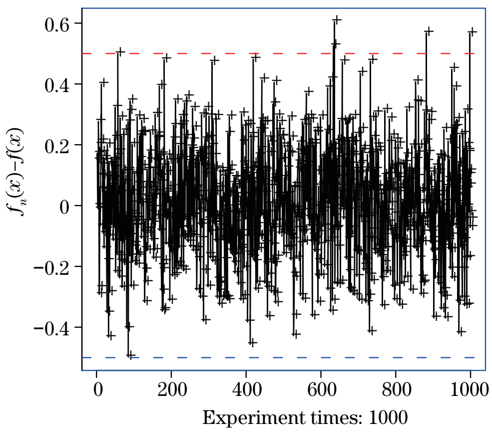

Figure 6 $f_n(x) - f(x)$ with $x = 0.5$, $n = 500$, and $f(x) = x^2 - 2x$.

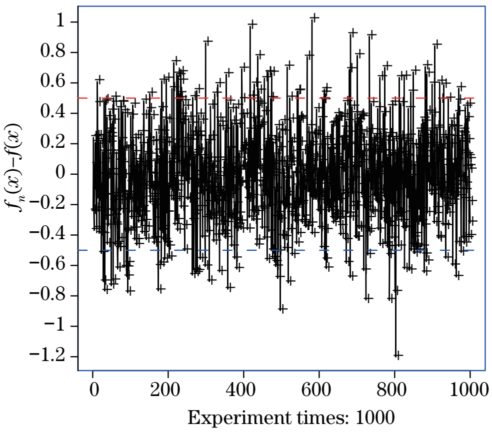

Figure 7 $f_n(x) - f(x)$ with $x = 0.8$, $n = 100$, and $f(x) = x^2 - 2x$.

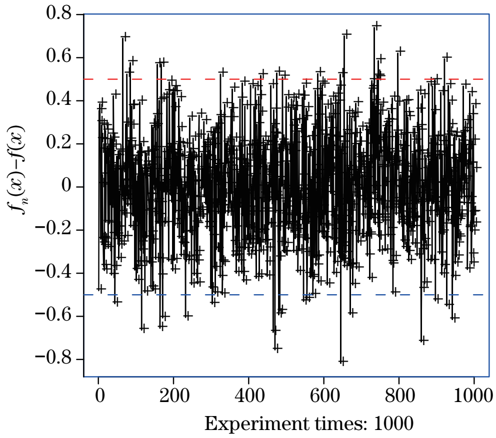

Figure 8 $f_n(x) - f(x)$ with $x = 0.8$, $n = 200$, and $f(x) = x^2 - 2x$.

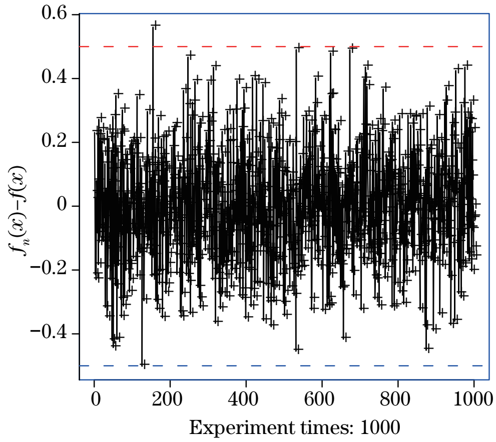

Figure 9 $f_n(x) - f(x)$ with $x = 0.8, n = 500$, and $f(x) = x^2 - 2x$.

It reveals in the figures above that as n increases, the values of $f_n(x) - f(x)$ converge uniformly to the interval $[\widehat{\varepsilon}[\xi], \widehat{\mathbb{E}}[\xi]] = [-0.5, 0.5]$, no matter $x = 0.3, 0.5$ or 0.8. These results basically agree with the theoretical results.

5 Conclusion

It is well known that many interesting results about probability limit theory and statistical large sample properties were all established in the classical probability space. Recently, motivated by some problems in statistics, measures of risk, mathematical economics and super-hedging in finance, more and more researches adopted non-additive probability and non-linear expectation to describe some uncertain phenomena in these fields. Therefore, statisticians were devoted to investigating the limit theorems for the sub-linear expectations in a general function space in recent years.

In this work, we investigate the double-indexed version of strong law of large numbers under some general conditions in sub-linear expectation space. The weighted version of Marcinkiewicz-Zygmund type strong law of large numbers are also established. These results extend or improve some existing ones in classical probability space or sub-linear expectation space. As an application, we further study the nonparametric regression model under sub-linear expectation framework. Some numerical simulations are also presented.

Acknowledgements The authors are most grateful to the editors and anonymous referees for carefully reading the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Bai, Z. D. and Cheng, P. E., Marcinkiewicz strong laws for linear statistics, *Statist. Probab. Lett.*, **46**, 2000, 105–112.

[2] Çağin, T. and Oliveira, P. E., A note on weighted sums of associated random variables, *Acta Math. Hung.*, **143**(1), 2014, 96–106.

[3] Chen, P. Y. and Gan, S. X., Limiting behavior of weighted sums of i.i.d. random variables, *Statist. Probab. Lett.*, **77**, 2007, 1589–1599.

[4] Chen, P. Y., Yi, J. and Sung, S. H., An extension of the Baum-Katz theorem to i.i.d. random variables with general moment conditions, *J. Inequal. Appl.*, **2015**, 2015, 9 pp, https:// doi.org/10.1186/s13660-015-0939-2.

[5] Chen, Z. J., Strong laws of large numbers for sub-linear expectations, *Sci. China Math.*, **59**(5), 2016, 945–954.

[6] Cheng, H., A strong law of large numbers for sub-linear expectation under a general moment condition, *Statist. Probab. Lett.*, **119**, 2016, 248–258.

[7] Chow, Y. S., Some convergence theorems for independent random variables, *Ann. Math. Statist.*, **37**(6), 1966, 1482–1493.

[8] Cuzick, J., A strong law for weighted sums of i.i.d. random variables, *J. Theor. Probab.*, **8**, 1995, 625–641.

[9] Denis, L. and Martini, C., A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, *Ann. Appl. Probab.*, **16**(2), 2006, 827–852.

[10] Fan, Y., Consistent nonparametric multiple regression for dependent heterogeneous processes: The fixed design case, *J. Multivariate Anal.*, **33**, 1990, 72–88.

[11] Georgiev, A. A., Local properties of function fitting estimates with applications to system identification, Grossmann, W., et al. (eds.), Mathematical Statistics and Applications, Vol. B, Proceedings 4th Pannonian Symposium on Mathematical Statistics, Bad Tatzmannsdorf, Austria, Reidel, Dordrecht, 1985, 141–151.

[12] Gilboa, I., Expected utility theory with purely subjective non-additive probabilities, *J. Math. Econ.*, **16**(1), 1987, 65–88.

[13] Hu, D., Chen, P. Y. and Sung, S. H., Strong laws for weighted sums of ψ-mixing random variables and applications in errors-in-variables regression models, *TEST*, **26**, 2017, 600–617.

[14] Hu, F., Chen, Z. J. and Wu, P. Y., A general strong law of large numbers for non-additive probabilities and its applications, *Statistics*, **50**(4), 2016, 733–749.

[15] Li, W., Chen, P. Y. and Hu, T. C., Strong laws for weighted sums of ρ^*-mixing random variables under general moment conditions, *RACSAM*, **113**, 2019, 1765–1772.

[16] Liang, H. Y. and Jing, B. Y., Asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences, *J. Multivariate Anal.*, **95**, 2005, 227–245.

[17] Marinacci, M., Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econ. Theory*, **84**(2), 1999, 145–195.

[18] Peng, S. G., Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer type, *Probab. Theory Rel.*, **113**, 1999, 473–499.

[19] Peng, S. G., G-expectation, G-Brownian motion and related stochastic calculus of Itô type, *Stoch. Anal. Appl.*, **2**, 2007, 541–567.

[20] Peng, S. G., Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stoch. Proc. Appl.*, **118**(12), 2008, 2223–2253.

[21] Peng, S. G., A new central limit theorem under sublinear expectations, 2008, arXiv: 0803.2656.

[22] Peng, S. G., Nonlinear Expectations and Stochastic Calculus under Uncertainty, Springer-Verlag, Berlin, Heidelberg, 2019.

[23] Roussas, G. G., Consistent regression estimation with fixed design points under dependence conditions, *Statist. Probab. Lett.*, **8**, 1989, 41–50.

[24] Roussas, G. G., Tran, L. T. and Ioannides, D. A., Fixed design regression for time series: Asymptotic normality, *J. Multivariate Anal.*, **40**, 1992, 262–291.

[25] Stone, C. J., Consistent nonparametric regression, *Ann. Statist.*, **5**, 1977, 595–645.

[26] Sung, S. H., On the strong law of large numbers for pairwise i.i.d. random variables with general moment conditions, *Statist. Probab. Lett.*, **83**, 2013, 1963–1968.

[27] Thanh, L. V. and Yin, G., Weighted sums of strongly mixing random variables with an application to nonparametric regression, *Statist. Probab. Lett.*, **105**, 2015, 195–202.

[28] Tran, L., Roussas, G., Yakowitz, S. and Truong Van, B., Fixed-design regression for linear time series, *Ann. Statist.*, **24**, 1996, 975–991.

[29] Wang, X. J., Xu, C., Hu, T. C., et al., On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models, *TEST*, **23**, 2014, 607–629.

[30] Wu, Q. Y. and Jiang, Y. Y., Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations, *J. Math. Anal. Appl.*, **460**, 2018, 252–270.

[31] Wu, Y., Wang, X. J., Hu, S. H. and Yang L. Q., Weighted version of strong law of large numbers for a class of random variables and its applications, *TEST*, **27**, 2018, 379–406.

[32] Zhang, L. X., Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm, *Commun. Math. Stat.*, **3**(2), 2015, 187–214.

[33] Zhang, L. X., Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.*, **59**(4), 2016, 751–768.

[34] Zhang, L. X., Exponential inequalities under sub-linear expectations with applications to laws of the iterated logarithm, *Sci. China Math.*, **59**(12), 2016, 2503–2526.

[35] Zhang, L. X., Strong limit theorems for extended independent random variables and extended negatively dependent random variables under sub-linear expectations, *Acta Math. Sci. Ser. B* (*Engl.*), **42**(2), 2022, 467–490.