

The Arboricity of Graphs with Minimum Genus Embeddings*

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Abstract The arboricity of a graph is the minimum number of forests needed to cover all edges of the graph. Let G be a connected graph embedded in a surface. This paper shows that the arboricity of G is at most $\lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \rceil$ (or $\lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \rceil$) if the orientable genus (or the nonorientable genus) of G is g (or h), and that these bounds are tight. As a conclusion of the results, an upper bound for the outerthickness of a connected graph embedded in an orientable surface (or a non-orientable surface) is obtained. The paper proves that if the orientable genus (or the non-orientable genus) of G is g (≥ 1) (or h), then G can be decomposed into $\lceil \sqrt{3g} \rceil + 3$ (or $\lceil \sqrt{\frac{3}{2}h} \rceil + 3$) forests in which one has maximum degree at most $\lfloor \frac{1}{3} \lceil \sqrt{3g} \rceil \rfloor + 1$ (or $\lfloor \frac{1}{3} \lceil \sqrt{\frac{3}{2}h} \rceil \rfloor + 1$).

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1 Introduction

A surface is a compact connected 2-dimensional manifold without boundary. The orientable surface S_g and the non-orientable surface N_h ($h \geq 1$) can be obtained from the sphere with g handles and h Möbius bands attached, respectively.

The arboricity of a graph G , denoted by $a(G)$, is the minimum number of forests needed to cover all edges in G . For the arboricity of a graph, Nash-Williams [11] established the following famous formula

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil,$$

where H is a subgraph of G and $|V(H)| \geq 2$.

By the above formula, the arboricity of the complete graph K_n is $\lceil \frac{n}{2} \rceil$, and every planar graph has arboricity at most 3. For more about the arboricity of a graph, one can refer to [2, 5, 7–8, 12, 18], etc.

A graph G is embeddable in a surface Σ if it can be drawn in Σ such that no two edges cross each other. An embedding Π of G in Σ is called 2-cell embedding, if any connected component

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of Σ - Π , called a face, is homeomorphic to an open disc. Suppose that G has n vertices and m edges. Then $m \leq 3n + 6g - 6$ (or $m \leq 3n + 3h - 6$) if G is 2-cell embedded in the orientable surface S_g (or the non-orientable surface N_h), by Euler's formula. So $\frac{m}{n-1} \leq \frac{3n+6g-6}{n-1} = 3 + \frac{6g-3}{n-1}$ or $\frac{m}{n-1} \leq \frac{3n+3h-6}{n-1} = 3 + \frac{3h-3}{n-1}$. By Nash-Williams' formula, the arboricity of G is related to g or h . On the other hand, the orientable genus (or the non-orientable genus) of the complete graph K_n is a function of the square of n . Considering that a graph G with n vertices is a subgraph of the complete graph K_n , an upper bound of the arboricity of G seems to be a function related to the square root of the orientable genus (or the non-orientable genus) of G . In the paper we shall explore it.

The orientable genus (or the non-orientable genus) of a connected graph G , denoted by $\gamma(G)$ (or $\bar{\gamma}(G)$), is the smallest number g (or h) such that G can be embedded in the orientable surface S_g (or the non-orientable surface N_h). An embedding of G in the orientable surface S_g (or in the non-orientable surface N_h) is called a minimum genus embedding if $\gamma(G) = g$ (or $\bar{\gamma}(G) = h$). Every minimum genus embedding of a connected graph has a 2-cell embedding (see [13, 19]). In the paper, we prove that for a connected graph G , if the orientable genus (or the non-orientable genus) of G is g (or h), then

$$a(G) \leq \left\lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \right\rceil$$

or

$$a(G) \leq \left\lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \right\rceil,$$

and these bounds are tight.

An outerplanar graph is a planar graph that can be embedded in the plane such that all the vertices are on the boundary of the outer face. The outerthickness of a graph G , denoted by $t_o(G)$, is the minimum integer k such that G is decomposed into k outerplanar subgraphs. Since a forest is an outerplanar graph, it is obvious that $t_o(G) \leq a(G)$.

Xu and Zha [17] showed that $t_o(G) \leq 4 + \lceil \sqrt{3g - \frac{3}{2}} \rceil$ (or $t_o(G) \leq 3 + \lceil \sqrt{\frac{3}{2}(h-1)} \rceil$), if G is a connected graph embedded in the orientable surface $S_g (g \geq 1)$ (or the non-orientable surface N_h). As a conclusion of our results, we have that $t_o(G) \leq \lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \rceil$ (or $t_o(G) \leq \lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \rceil$) if the orientable genus (or non-orientable genus) of G is g (or h). It is obvious that our bounds improve that in [17] if a graph has a minimum genus embedding in the surface $S_g (g \geq 1)$ or in the surface $N_h (h \geq 2)$. For more about the outerthickness of graphs, one can refer to [4, 6, 10, 20], etc.

In 1986, Payan [14] introduced the fractional arboricity of a graph G which is defined as follows:

$$\eta_f(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1},$$

where $|V(H)| \geq 2$. By Nash-Williams' formula, $a(G) = \lceil \eta_f(G) \rceil$ for a graph G . A graph G is said to be $(k, d)^*$ -decomposable if G can be decomposed into $k + 1$ forests in which one

has maximum degree at most d . In 2012, Montassier Ossona de Mendez, Raspaud and Zhu [9] proposed the following conjecture (Nine Dragon Tree Conjecture): If $\eta_f(G) \leq k + \frac{d}{k+d+1}$, where k, d are two positive integers, then G is $(k, d)^*$ -decomposable. Jiang and Yang [8] proved the Nine Dragon Tree Conjecture in 2017.

There are several results on the $(k, d)^*$ -decomposition of planar graphs. Balogh, Kochol, Pluhár and Yu [1] showed that any simple planar graph is $(2, 8)^*$ -decomposable. Gonçalves [7] further proved that every simple planar graph is $(2, 4)^*$ -decomposable. For nonplanar graphs, there are few results on the $(k, d)^*$ -decomposition. In the paper we show that if the orientable genus (or the non-orientable genus) of a connected graph G is $g (\geq 1)$ (or h), then G can be decomposed into $\lceil \sqrt{3g} \rceil + 3$ (or $\lceil \sqrt{\frac{3}{2}h} \rceil + 3$) forests in which one has maximum degree at most $\lfloor \frac{1}{3} \lceil \sqrt{3g} \rceil \rfloor + 1$ (or $\lfloor \frac{1}{3} \lceil \sqrt{\frac{3}{2}h} \rceil \rfloor + 1$). In addition, we prove that any graph embedded in the torus is $(3, 12)^*$ -decomposable.

The paper is arranged as follows. In Sections 2–3, we obtain an upper bound for the arboricity and the fractional arboricity of a connected graph which is minimum genus embedded in the orientable surface and the non-orientable surface, respectively. In Section 4, we explore the arboricity of a connected graph with fixed girth embedded in a surface.

The remainder of the section is devoted to some terminologies of graphs. The other undefined terms can be found in [3, 16].

The order of a graph is the number of vertices of the graph. The girth of a connected graph is the length of a shortest cycle in the graph. The orientable genus and the nonorientable genus of the complete graph K_n have been determined, which gives a proof of the well-known Map Color Theorem (see [15]).

Theorem 1.1 (see [15])

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

and

$$\bar{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, \quad \text{if } n \neq 7 \text{ and } \bar{\gamma}(K_7) = 3.$$

For the orientable surface $S_g (g \geq 1)$, let n be the minimum number satisfying the condition that $\gamma(K_n) \geq g$. If c is such a number that

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil = \frac{(n-3)(n-4) + c}{12},$$

then we call c the orientable deficiency number of complete graphs with orientable genus g . Similarly, if n is the minimum number so that $\bar{\gamma}(K_n) \geq h$, and if c' is such a number that

$$\left\lceil \frac{(n-3)(n-4)}{6} \right\rceil = \frac{(n-3)(n-4) + c'}{6},$$

then we call c' the non-orientable deficiency number of complete graphs with non-orientable genus h .

2 The Orientable Case

The section starts with a technological lemma.

Lemma 2.1 *Let a, b and c be three integers. If $a > b \geq 1$ and $0 \leq c \leq 11$, then*

$$\frac{12a - 6}{5 + \sqrt{48a - (4c - 1)}} > \frac{12b - 6}{5 + \sqrt{48b - (4c - 1)}}. \quad (2.1)$$

Proof If $b = 1$, then

$$\frac{12b - 6}{5 + \sqrt{48b - (4c - 1)}} \leq \frac{6}{5 + \sqrt{5}} < 1.$$

Claim 1 If $a > b = 1$, then the formula (2.1) holds.

In fact, if $a \geq 2$, then

$$\frac{12a - 6}{5 + \sqrt{48a - (4c - 1)}} \geq \frac{12a - 6}{5 + \sqrt{48a + 1}} = \frac{\sqrt{48a + 1} - 5}{4}.$$

Considering that $\sqrt{48a + 1} > 9$ if $a \geq 2$, we have that

$$\frac{12a - 6}{5 + \sqrt{48a - (4c - 1)}} > 1.$$

We now suppose that $a > b \geq 2$.

Claim 2 If $a > b \geq 2$, then $48(a^2b - ab^2) > (4c - 1)(a^2 - b^2)$.

Obviously, the claim is true if $c = 0$. We now suppose that $c \geq 1$. We observe that $ab > a + b$ if $a > b \geq 2$. So $48ab > (4c - 1)(a + b)$. Thus $48ab(a - b) > (4c - 1)(a + b)(a - b)$, i.e., $48(a^2b - ab^2) > (4c - 1)(a^2 - b^2)$.

Claim 3 If $a > b \geq 2$, then $a\sqrt{48b - (4c - 1)} > b\sqrt{48a - (4c - 1)}$.

Otherwise, $a\sqrt{48b - (4c - 1)} \leq b\sqrt{48a - (4c - 1)}$. So $a^2[48b - (4c - 1)] \leq b^2[48a - (4c - 1)]$.

In this case, $48(a^2b - ab^2) \leq (4c - 1)(a^2 - b^2)$, which contradicts Claim 2.

Claim 4 If $a > b \geq 2$, then

$$\frac{12a - 6}{5 + \sqrt{48a - (4c - 1)}} > \frac{12b - 6}{5 + \sqrt{48b - (4c - 1)}}.$$

Otherwise,

$$\frac{12a - 6}{5 + \sqrt{48a - (4c - 1)}} \leq \frac{12b - 6}{5 + \sqrt{48b - (4c - 1)}}.$$

Then

$$(12a - 6)[5 + \sqrt{48b - (4c - 1)}] \leq (12b - 6)[5 + \sqrt{48a - (4c - 1)}],$$

$$(2a - 1)[5 + \sqrt{48b - (4c - 1)}] \leq (2b - 1)[5 + \sqrt{48a - (4c - 1)}].$$

So

$$10(a - b) + 2[a\sqrt{48b - (4c - 1)} - b\sqrt{48a - (4c - 1)}] + [\sqrt{48a - (4c - 1)} - \sqrt{48b - (4c - 1)}] \leq 0. \tag{2.2}$$

On the other hand, $10(a - b) > 0$, $\sqrt{48a - (4c - 1)} - \sqrt{48b - (4c - 1)} > 0$ and $a\sqrt{48b - (4c - 1)} - b\sqrt{48a - (4c - 1)} > 0$ by Claim 3. So the left-hand side of the formula (2.2) should be a positive number, a contradiction. The proof of Lemma 2.1 is completed.

Lemma 2.2 *Let c be the orientable deficiency number of complete graphs with orientable genus $g (\geq 1)$. Let G be a connected graph of order n with m edges. If the orientable genus of G is $g (g \geq 1)$, then*

$$\frac{m}{n - 1} \leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}}. \tag{2.3}$$

Proof Since G has a minimum genus embedding in the orientable surface S_g , we have that $m \leq 3n + 6g - 6$ by Euler’s formula. So

$$\frac{m}{n - 1} \leq \frac{3n + 6g - 6}{n - 1} = 3 + \frac{6g - 3}{n - 1}.$$

Let K_s be the complete graph whose orientable genus is at least g such that s is minimum. Clearly, $s \geq 5$. Since c is the orientable deficiency of complete graphs with orientable genus g , we have

$$\gamma(K_s) = \left\lceil \frac{(s - 3)(s - 4)}{12} \right\rceil = \frac{(s - 3)(s - 4) + c}{12} \geq g.$$

Thus $(s - 3)(s - 4) \geq 12g - c$. In this case,

$$s \geq \frac{7}{2} + \frac{1}{2}\sqrt{48g - (4c - 1)}.$$

Since $\gamma(G) = g$, we claim that $n \geq s$. Otherwise, $n < s$. So $\gamma(K_n) \leq \gamma(K_s)$. By the choice of K_s , we have that $\gamma(K_n) < g$. On the other hand, $\gamma(K_n) \geq \gamma(G) = g$, a contradiction. So

$$n \geq s \geq \frac{7}{2} + \frac{1}{2}\sqrt{48g - (4c - 1)}.$$

Hence

$$\begin{aligned} \frac{m}{n - 1} &\leq 3 + \frac{6g - 3}{n - 1} \leq 3 + \frac{6g - 3}{s - 1} \\ &\leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}}. \end{aligned}$$

Theorem 2.1 *Let c be the orientable deficiency number of complete graphs with orientable genus $g (\geq 1)$. If the orientable genus of a connected graph G is $g (g \geq 1)$, then*

$$\eta_f(G) \leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \tag{2.4}$$

and

$$a(G) \leq \left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil.$$

Proof Let H be an arbitrary subgraph of order at least 2 in G . Suppose that H has s vertices with t edges. We can suppose that H is connected. Otherwise, suppose that H_1, H_2, \dots, H_l are all components of H , where $l \geq 2$. Then $\eta_f(H) = \max\{\eta_f(H_1), \eta_f(H_2), \dots, \eta_f(H_l)\}$. We now select a component, say H_j , such that $\eta_f(H_j) = \eta_f(H)$. Obviously, $\gamma(H_j) \leq g$. Next, H_j is argued in a similar way as the case that H is connected. Suppose that H has a minimum genus embedding in the orientable surface S_k . Clearly, $0 \leq k \leq g$. We consider two cases.

Case 1 $k \geq 1$.

In this case,

$$\frac{t}{s - 1} \leq 3 + \frac{12k - 6}{5 + \sqrt{48k - (4c - 1)}}$$

by Lemma 2.2.

If $k < g$, then

$$\frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} > \frac{12k - 6}{5 + \sqrt{48k - (4c - 1)}}$$

by Lemma 2.1. So

$$\frac{t}{s - 1} \leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}}. \tag{2.5}$$

Case 2 $k = 0$.

In this case, H is a planar graph. Thus

$$\frac{t}{s - 1} \leq a(H) \leq 3.$$

Note that

$$\frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \geq 0$$

if $g \geq 1$. So the formula (2.5) holds.

Considering the arbitrariness of H , we have that

$$\eta_f(G) \leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}}.$$

Applying Nash-Williams' formula, there is the following inequality

$$a(G) \leq \left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil.$$

The theorem below shows that the bound for arboricity in Theorem 2.1 can be reached by the complete graph K_n which is minimum genus embedded in S_g if $n \geq 12$ and $n \not\equiv 2, 6, 8, 10 \pmod{12}$.

Theorem 2.2 *Let $n \geq 12$ be an integer. If $\gamma(K_n) = g$ and if $n \not\equiv 2, 6, 8, 10 \pmod{12}$, then*

$$a(K_n) = \left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil. \tag{2.6}$$

Proof Suppose that the deficiency of complete graphs with orientable genus g is c . Then

$$g = \frac{(n - 3)(n - 4) + c}{12}.$$

So

$$\begin{aligned} \left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil &= \left\lceil 3 + \frac{n^2 - 7n + 6 + c}{5 + \sqrt{4n^2 - 28n + 49}} \right\rceil \\ &= \left\lceil 3 + \frac{(n - 1)(n - 6) + c}{2n - 2} \right\rceil \\ &= \left\lceil \frac{n}{2} + \frac{c}{2n - 2} \right\rceil. \end{aligned}$$

If $n \equiv 0$ or $4 \pmod{12}$, then $c = 0$. In this case

$$\left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

If $n \equiv 1 \pmod{2}$, then

$$0 \leq \frac{c}{2n - 2} \leq \frac{11}{22} = \frac{1}{2}$$

since $n \geq 12$. Thus

$$\left\lceil \frac{n}{2} + \frac{c}{2n - 2} \right\rceil = \frac{n - 1}{2} + \left\lceil \frac{1}{2} + \frac{c}{2n - 2} \right\rceil = \frac{n + 1}{2} = \left\lceil \frac{n}{2} \right\rceil.$$

So

$$\left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil = \left\lceil \frac{n + 1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

On the other hand, $a(K_n) = \left\lceil \frac{n}{2} \right\rceil$. Hence

$$a(K_n) = \left\lceil 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \right\rceil,$$

if $n \not\equiv 2, 6, 8, 10 \pmod{12}$.

The bounds in Theorem 2.1 are functions of g and c . The following theorem gives upper bounds for the arboricity and the fractional arboricity of a connected graph which is minimum genus embedded in the surface $S_g (g \geq 1)$ using functions of g .

Theorem 2.3 *If the orientable genus of a connected graph G is $g (g \geq 1)$, then*

$$\eta_f(G) \leq \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \tag{2.7}$$

and

$$a(G) \leq \left\lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \right\rceil. \tag{2.8}$$

Proof By the formula (2.4), we have that

$$\eta_f(G) \leq 3 + \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}},$$

where c is the orientable deficiency number of complete graphs with orientable genus g . Then $0 \leq c \leq 11$.

Let

$$\alpha(g) = \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \quad \text{and} \quad \beta(g) = \frac{\sqrt{48g + 1}}{4} - \frac{377}{460}.$$

Then

$$\begin{aligned} \alpha(g) &= \frac{12g - 6}{5 + \sqrt{48g - (4c - 1)}} \\ &= \frac{(12g - 6)[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c} \\ &= \frac{(12g - 6 - c)[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c} + \frac{c[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c} \\ &= \frac{\sqrt{48g - (4c - 1)} - 5}{4} + \frac{c[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c}. \end{aligned}$$

Next, we show that

$$\frac{c[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c} \leq \frac{99}{230}, \quad (2.9)$$

if $g \geq 11$. In fact, if $g \geq 11$, then

$$\begin{aligned} \frac{c[\sqrt{48g - (4c - 1)} - 5]}{48g - 24 - 4c} &\leq \frac{11(\sqrt{48g + 1} - 5)}{48g - 68} \\ &= \frac{11(48g - 24)}{(48g - 68)(\sqrt{48g + 1} + 5)} \\ &= \frac{11(12g - 6)}{(12g - 17)(\sqrt{48g + 1} + 5)}. \end{aligned}$$

In addition, if $g \geq 11$, then

$$\begin{aligned} \frac{12g - 6}{(12g - 17)(\sqrt{48g + 1} + 5)} &= \frac{1}{\sqrt{48g + 1} + 5} + \frac{11}{(12g - 17)(\sqrt{48g + 1} + 5)} \\ &\leq \frac{1}{28} + \frac{11}{115 \times 28} = \frac{9}{230}. \end{aligned}$$

So

$$\frac{11(12g - 6)}{(12g - 17)(\sqrt{48g + 1} + 5)} \leq \frac{99}{230}.$$

Thus, the formula (2.9) holds if $g \geq 11$. So

$$\alpha(g) \leq \frac{\sqrt{48g - (4c - 1)} - 5}{4} + \frac{99}{230}$$

$$\begin{aligned} &\leq \frac{\sqrt{48g+1}-5}{4} + \frac{99}{230} \\ &= \frac{\sqrt{48g+1}}{4} - \frac{377}{460} = \beta(g), \end{aligned}$$

if $g \geq 11$.

We now consider the case that $1 \leq g \leq 10$. By the definition of the orientable deficiency number, c can be countered if g is given. Table 1 shows that $\alpha(g) < \beta(g)$ if $1 \leq g \leq 10$.

Table 1 The values of $\alpha(g)$ and $\beta(g)$ where $1 \leq g \leq 10$.

g	1	2	3	4	5	6	7	8
c	10	4	6	6	4	0	6	6
$\alpha(g)$	0.75	< 1.29	< 1.88	< 2.34	2.7	3	< 3.45	3.75
$\beta(g)$	> 0.93	> 1.64	> 2.19	> 2.65	> 3.06	> 3.43	> 3.76	> 4.08

g	9	10
c	10	10
$\alpha(g)$	< 4.11	< 4.39
$\beta(g)$	> 4.38	> 4.66

So

$$\eta_f(G) \leq 3 + \alpha(g) \leq 3 + \beta(g) \leq 3 + \frac{\sqrt{48g+1}}{4} - \frac{377}{460} = \frac{1003}{460} + \sqrt{3g + \frac{1}{16}}.$$

Moreover,

$$a(G) \leq \left\lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \right\rceil.$$

By Theorem 2.3, the arboricity of any connected graph with orientable genus one is at most 4. The orientable genus of K_7 is 1, and by straightforward application of Nash-Williams formula it is clear that $a(K_7) = 4$. So the bound for arboricity in Theorem 2.3 is tight. Since the outerthickness of a graph is no more than its arboricity, we have the following result.

Corollary 2.1 *If the orientable genus of a connected graph G is $g(g \geq 1)$, then*

$$t_o(G) \leq \left\lceil \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \right\rceil.$$

Xu and Zha [17] showed that $t_o(G) \leq 4 + \lceil \sqrt{3g - \frac{3}{2}} \rceil$ if G is a connected graph embedded in the orientable surface $S_g(g \geq 1)$. It is not hard to find that $\frac{1003}{460} + \sqrt{3g + \frac{1}{16}} < 4 + \sqrt{3g - \frac{3}{2}}$ if $g \geq 1$. So the bound in Corollary 2.1 improves the bound in [17] if the orientable genus of a connected graph is $g(g \geq 1)$.

We now discuss the arboricity of a graph embedded in the torus.

Theorem 2.4 *If a connected graph G is embeddable in the torus, then G is $(3, 12)^*$ -decomposable.*

Proof If G is a planar graph, then it is $(2, 4)^*$ -decomposable (see [7]). If G is nonplanar, then $\gamma(G) = 1$. By the formula (2.4), $\eta_f(G) \leq 3 + \frac{12g-6}{5+\sqrt{48g-(4c-1)}}$. According to Table 1 in the proof of Theorem 2.3, we know that $\frac{12g-6}{5+\sqrt{48g-(4c-1)}} = \frac{3}{4}$ if $g = 1$. So $\eta_f(G) \leq 3 + \frac{3}{4} = 3 + \frac{12}{4+12}$. By the Nine Dragon Tree theorem, we have that G is $(3, 12)^*$ -decomposable.

Theorem 2.5 *Let $g \geq 1$ be an integer. If the orientable genus of a connected graph G is g , then G is $(\lceil\sqrt{3g}\rceil + 2, \lfloor\frac{1}{3}\lceil\sqrt{3g}\rceil + 1)^*$ -decomposable.*

Proof Since $g \geq 1$, we have that

$$\sqrt{3g + \frac{1}{16}} < \sqrt{3g} + \frac{1}{48}.$$

By the formula (2.7), we have

$$\begin{aligned} \eta_f(G) &\leq \frac{1003}{460} + \sqrt{3g + \frac{1}{16}} \\ &< \frac{1003}{460} + \sqrt{3g} + \frac{1}{48} \\ &< (\sqrt{3g} + 2) + \frac{1}{4}. \end{aligned}$$

Let $k = \lceil\sqrt{3g}\rceil + 2$ and let

$$\frac{x}{k + x + 1} \leq \frac{1}{4},$$

then

$$x \leq \frac{1}{3}(k + 1).$$

Let $d = \lfloor\frac{1}{3}(k + 1)\rfloor$. Then $d = \lfloor\frac{1}{3}(\lceil\sqrt{3g}\rceil + 3)\rfloor = \lfloor\frac{1}{3}\lceil\sqrt{3g}\rceil\rfloor + 1$. By the Nine Dragon Tree theorem, we have that G is $(\lceil\sqrt{3g}\rceil + 2, \lfloor\frac{1}{3}\lceil\sqrt{3g}\rceil + 1)^*$ -decomposable.

Remark 2.1 In Theorem 2.5, the graph G can be decomposed into $3s + 3$ forests in which one has maximum degree at most $s + 1$, if s is a positive integer and $g = 3s^2$.

3 The Non-orientable Case

Proceeding a similar argument to that in the proof of Lemma 2.1, we have the result below.

Lemma 3.1 *Let a, b and c' be three integers. If $a > b \geq 1$ and $0 \leq c' \leq 5$, then*

$$\frac{6a - 6}{5 + \sqrt{24a - (4c' - 1)}} > \frac{6b - 6}{5 + \sqrt{24b - (4c' - 1)}}.$$

The following result is an analogue of Lemma 2.2.

Lemma 3.2 *Let c' be the non-orientable deficiency number of complete graphs with non-orientable genus h . Let G be a connected graph of order n with m edges. If the non-orientable genus of G is h , then*

$$\frac{m}{n-1} \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}}.$$

Proof We proceed a similar argument to that in the proof of Lemma 2.2. Since G has a minimum genus embedding in the non-orientable surface N_h , G has a 2-cell embedding in N_h . Then $m \leq 3n + 3h - 6$ by Euler's formula. So

$$\frac{m}{n-1} \leq 3 + \frac{3h-3}{n-1}.$$

Let K_s be the complete graph whose non-orientable genus is at least h such that s is minimum. Clearly, $s \geq 5$. Then

$$\overline{\gamma}(K_s) = \left\lceil \frac{(s-3)(s-4)}{6} \right\rceil = \frac{(s-3)(s-4) + c'}{6} \geq h.$$

So

$$s \geq \frac{7}{2} + \frac{1}{2}\sqrt{24h - (4c' - 1)}.$$

Moreover, we have that $n \geq s$ by a similar argument as in the proof of Lemma 2.2. Thus

$$n \geq \frac{7}{2} + \frac{1}{2}\sqrt{24h - (4c' - 1)}.$$

Hence

$$\frac{m}{n-1} \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}}.$$

Theorem 3.1 *Let c' be the non-orientable deficiency number of complete graphs with non-orientable genus h . If the non-orientable genus of a connected graph G is h , then*

$$\eta(G) \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \tag{3.1}$$

and

$$a(G) \leq \left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \right\rceil. \tag{3.2}$$

Proof The proof is similar to that of Theorem 2.1. Let H be an arbitrary subgraph of G which has p vertices with q edges, where $p \geq 2$. As in the proof of Theorem 2.1, we can suppose that H is connected. Suppose that H has a minimum genus embedding in the non-orientable surface N_k . Clearly, $0 \leq k \leq h$. We consider two cases.

Case 1 $k \geq 1$.

Applying Lemma 3.2, we have

$$\frac{q}{p-1} \leq 3 + \frac{6k-6}{5 + \sqrt{24k - (4c' - 1)}}.$$

If $k < h$, then

$$\frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} > \frac{6k-6}{5 + \sqrt{24k - (4c' - 1)}}$$

by Lemma 3.1. So

$$\frac{q}{p-1} \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}}. \tag{3.3}$$

Case 2 $k = 0$.

In this case, H is a planar graph. As in the proof of Theorem 2.1, the formula (3.3) holds. Considering the arbitrariness of H , we have that

$$\eta_f(G) \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}}.$$

So

$$a(G) \leq \lceil \eta_f(G) \rceil = \left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \right\rceil.$$

The theorem below shows that the bound for arboricity in Theorem 3.1 can be reached by the complete graph K_n which is minimum genus embedded in N_h if $n \geq 6$ and $n \not\equiv 2 \pmod{6}$.

Theorem 3.2 *Let $n \geq 8$ be an integer. Let c' be the non-orientable deficiency number of complete graphs with non-orientable genus h . If $\overline{\gamma}(K_n) = h$ and $n \not\equiv 2 \pmod{6}$, then*

$$a(K_n) = \left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \right\rceil. \tag{3.4}$$

Proof Since $n \geq 8$, we have

$$h = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

by Theorem 1.1. Furthermore,

$$h = \frac{(n-3)(n-4) + c'}{6}.$$

Proceeding a similar argument to that in the proof of Theorem 2.2, we have that

$$\left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h + 1 - 4c'}} \right\rceil = \left\lceil \frac{n}{2} + \frac{c'}{2n-2} \right\rceil.$$

If $n \equiv 0$, or $4 \pmod{6}$, then $c' = 0$. In this case,

$$\left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

If $n \equiv 1 \pmod{2}$, then

$$0 \leq \frac{c'}{2n-2} \leq \frac{5}{14} < \frac{1}{2}$$

since $n \geq 8$. Thus

$$\left\lceil \frac{n}{2} + \frac{c'}{2n-2} \right\rceil = \frac{n-1}{2} + \left\lceil \frac{1}{2} + \frac{c'}{2n-2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

So

$$\left\lceil 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

On the other hand, $a(K_n) = \left\lceil \frac{n}{2} \right\rceil$. Hence the theorem holds if $n \not\equiv 2 \pmod{6}$.

The following theorem gives upper bounds for the arboricity and the fractional arboricity of a connected graph embedded in the surface N_h using a function of h .

Theorem 3.3 *If the non-orientable genus of a connected graph G is h , then*

$$\eta_f(G) \leq \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \tag{3.5}$$

and

$$a(G) \leq \left\lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \right\rceil. \tag{3.6}$$

Proof By Theorem 3.1, we have that

$$\eta(G) \leq 3 + \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}},$$

where c' is the non-orientable deficiency number of complete graphs with non-orientable genus h .

Let

$$\alpha'(h) = \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \quad \text{and} \quad \beta'(h) = \frac{\sqrt{24h+1}}{4} - \frac{245}{244}.$$

Then

$$\begin{aligned} \alpha'(h) &= \frac{6h-6}{5 + \sqrt{24h - (4c' - 1)}} \\ &= \frac{(6h-6)[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'} \\ &= \frac{(6h-6-c')[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'} + \frac{c'[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'} \\ &= \frac{\sqrt{24h - (4c' - 1)} - 5}{4} + \frac{c'[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'}. \end{aligned}$$

Next, we show that

$$\frac{c'[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'} \leq \frac{15}{61}, \tag{3.7}$$

if $h \geq 12$. In fact, if $h \geq 12$, then

$$\begin{aligned} \frac{c'[\sqrt{24h - (4c' - 1)} - 5]}{24h - 24 - 4c'} &\leq \frac{5(\sqrt{24h + 1} - 5)}{24h - 44} \\ &= \frac{5(24h - 24)}{(24h - 44)(\sqrt{24h + 1} + 5)} \\ &= \frac{5(6h - 6)}{(6h - 11)(\sqrt{24h + 1} + 5)} \\ &= \frac{5}{\sqrt{24h + 1} + 5} + \frac{25}{(6h - 11)(\sqrt{24h + 1} + 5)} \\ &\leq \frac{5}{22} + \frac{25}{61 \times 22} = \frac{15}{61}. \end{aligned}$$

So the formula (3.7) holds if $h \geq 12$. Thus

$$\begin{aligned} \alpha'(h) = \frac{6h - 6}{5 + \sqrt{24h - (4c' - 1)}} &\leq \frac{\sqrt{24h - (4c' - 1)} - 5}{4} + \frac{15}{61} \\ &\leq \frac{\sqrt{24h + 1} - 5}{4} + \frac{15}{61} \\ &= \frac{\sqrt{24h + 1}}{4} - \frac{245}{244} = \beta'(h), \end{aligned}$$

if $h \geq 12$.

We now consider the case that $1 \leq h \leq 11$. By the definition of the non-orientable deficiency number, c' can be countered if h is given. Table 2 shows that $\alpha'(h) < \beta'(h)$ if $1 \leq h \leq 11$.

Table 2 The values of $\alpha'(h)$ and $\beta'(h)$ where $1 \leq h \leq 11$.

h	1	2	3	4	5	6	7	8
c'	0	0	0	4	0	0	0	4
$\alpha'(h)$	0	0.5	1	< 1.29	1.5	< 1.77	2	< 2.34
$\beta'(h)$	> 0.25	> 0.75	> 1.13	> 1.46	> 1.74	> 2.00	> 2.24	> 2.46
h	9	10	11	So				
c'	4	4	0					
$\alpha'(h)$	< 2.53	2.7	< 2.86					
$\beta'(h)$	> 2.67	> 2.87	> 3.06					

$$\eta_f(G) \leq 3 + \alpha'(h) \leq 3 + \beta'(h) \leq 3 + \frac{\sqrt{24h + 1}}{4} - \frac{245}{244} = \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}}.$$

Furthermore,

$$a(G) \leq \left\lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \right\rceil.$$

By Theorem 3.3, the arboricity of any connected graph with the non-orientable genus 5 is at most 5. The non-orientable genus of K_9 is 5, and by straightforward application of Nash-Williams formula it is clear that $a(K_9) = 5$. So the bound for arboricity in Theorem 3.3 is tight. Since the outerthickness of a graph is no more than its arboricity, we have the following result.

Corollary 3.1 *If the non-orientable genus of a connected graph G is h , then*

$$t_o(G) \leq \left\lceil \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \right\rceil.$$

Xu and Zha [17] showed that $t_o(G) \leq 3 + \lceil \sqrt{\frac{3}{2}(h-1)} \rceil$ if G is a connected graph embedded in the non-orientable surface N_h . It is not hard to find that $\frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} < 3 + \sqrt{\frac{3}{2}(h-1)}$ if $h \geq 2$. So the bound in Corollary 3.1 improves the bound in [17] if the non-orientable genus of a connected graph is $h(h \geq 2)$.

Suppose that G is a graph with the non-orientable genus one. Combining the formula (3.1) and Table 2, we know that $\eta_f(G) \leq 3$. So $a(G) \leq 3$. Thus any graph embedded in the projective plane can be decomposed into three forests. It is known that $\bar{\gamma}(K_5) = \bar{\gamma}(K_6) = 1$. We find that K_5 can be decomposed into three trees in which one is a matching (see Figure 1). However, K_6 does not have such a decomposition. The reason is that K_6 has 15 edges, and that each is a spanning tree if K_6 is decomposed into three forests. So we propose the following problem.

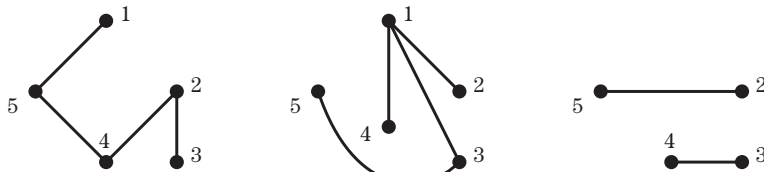


Figure 1 Three forests of K_5 .

Problem 1 What are the conditions for graphs embedded in the projective plane that can be edge partitioned into three forests with one being a matching?

Theorem 3.4 *Let $h \geq 1$ be an integer. If the non-orientable genus of a connected graph G is h , then G is $(\lceil \sqrt{\frac{3}{2}h} \rceil + 2, \lfloor \frac{1}{3} \lceil \sqrt{\frac{3}{2}h} \rceil \rfloor + 1)^*$ -decomposable.*

Proof Since $h \geq 1$, we have that

$$\sqrt{\frac{3}{2}h + \frac{1}{16}} < \sqrt{\frac{3}{2}h} + \frac{1}{5}.$$

By the formula (3.5), we have that

$$\begin{aligned} \eta_f(G) &\leq \frac{487}{244} + \sqrt{\frac{3}{2}h + \frac{1}{16}} \\ &< \left(\sqrt{\frac{3}{2}h} + 2 \right) + \frac{1}{4}. \end{aligned}$$

Let $k = \lceil \sqrt{\frac{3}{2}h} \rceil + 2$ and

$$\frac{x}{k+x+1} \leq \frac{1}{4},$$

then

$$x \leq \frac{1}{3}(k+1).$$

Let $d = \lfloor \frac{1}{3}(k+1) \rfloor$. Then $d = \lfloor \frac{1}{3}(\lceil \sqrt{\frac{3}{2}h} \rceil + 2 + 1) \rfloor = \lfloor \frac{1}{3} \lceil \sqrt{\frac{3}{2}h} \rceil \rfloor + 1$. By the Nine Dragon Tree theorem, we have that G is $(\lceil \sqrt{\frac{3}{2}h} \rceil + 2, \lfloor \frac{1}{3} \lceil \sqrt{\frac{3}{2}h} \rceil \rfloor + 1)^*$ -decomposable.

4 Further Argument

Let G be a connected graph of order n with m edges and girth at least k . It is known by Euler’s formula that

$$m \leq \frac{k}{k-2}(n+2g-2) \quad \text{or} \quad m \leq \frac{k}{k-2}(n+h-2),$$

if G is 2-cell embedded in the orientable surface S_g or in the non-orientable surface N_h . Proceeding parallel arguments to Lemmas 2.1–2.2 and Theorem 2.3, we have that

$$a(G) \leq \left\lceil \frac{k}{k-2} + \frac{k}{k-2} \cdot \frac{4g-2}{5 + \sqrt{48g - (4c-1)}} \right\rceil,$$

if the orientable genus of G is $g (g \geq 1)$, where c is the orientable deficiency number of complete graphs with orientable genus g .

Since $0 \leq c \leq 11$, we have that

$$\frac{4g-2}{5 + \sqrt{48g - (4c-1)}} \leq \frac{4g-2}{5 + \sqrt{48g - 43}}.$$

So

$$a(G) \leq \left\lceil \frac{k}{k-2} + \frac{k}{k-2} \cdot \frac{4g-2}{5 + \sqrt{48g - 43}} \right\rceil. \tag{4.1}$$

Similarly, if the non-orientable genus of G is h , then

$$a(G) \leq \left\lceil \frac{k}{k-2} + \frac{k}{k-2} \cdot \frac{2h-2}{5 + \sqrt{24h - (4c'-1)}} \right\rceil,$$

where c' is the non-orientable deficiency number of complete graphs with non-orientable genus h .

Since $0 \leq c' \leq 5$, we have that

$$a(G) \leq \left\lceil \frac{k}{k-2} + \frac{k}{k-2} \cdot \frac{2h-2}{5 + \sqrt{24h - 19}} \right\rceil. \tag{4.2}$$

Notice that the relation between the order and orientable genus (or non-orientable genus) of the complete graph K_n is utilized in order to obtain the formula (4.1) (or the formula (4.2)). If we want to improve the bound in the formula (4.1) (or the formula (4.2)), one approach is to find a proper lower bound for the order of a connected graph with girth at least k which has a minimum genus embedding in S_g (or N_h). So we propose the following two problems.

Problem 2 For all positive integers $g \geq 1$ and $k \geq 4$, find an integer $f(k, g)$ such that for every connected graph G of order n with girth at least k and $\gamma(G) \geq g$ so that $n \geq f(k, g)$.

Problem 3 For all positive integers $h \geq 1$ and $k \geq 4$, find an integer $f'(k, h)$ such that for every connected graph G of order n with girth at least k and $\bar{\gamma}(G) \geq h$ so that $n \geq f'(k, h)$.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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