Chin. Ann. Math. Ser. B
47(3), 2026, 529–554
DOI: 10.1007/s11401-026-0008-9

Chinese Annals of Mathematics, Series B

Finite Non-abelian Groups Whose Non-abelian Subgroups Have Minimum Centralizers*

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Abstract A finite non-abelian group G is called an $\mathcal{MC}$-group if all non-abelian subgroups H of G have minimum centralizers (i.e., $C_G(H) = Z(G)$). In this paper, the authors give some characterizations of $\mathcal{MC}$-groups, and it is proved that $\mathcal{MC}$-groups are just the finite groups with modular centralizer lattice of length 2 depicted by Schmidt, which leads to a classification of $\mathcal{MC}$-groups. However, Schmidt's depiction said nothing for $\mathcal{MC}$-p-groups. They give a characterization of $\mathcal{MC}$-p-groups. In particular, they characterize special $\mathcal{MC}$-p-groups by means of the commutator matrices, and provide a method to determine or classify special $\mathcal{MC}$-p-groups. As applications, some examples are given, and special $\mathcal{MC}$-p-groups with an abelian maximal subgroup are classified up to isoclinism.

Keywords Centralizers of groups, Finite p-groups, Special p-groups, Isoclinism, Commutator matrix
2020 MR Subject Classification 20D15, 20D30, 05C25

1 Introduction

It is well-known that the centralizers of subgroups play an important role in studying the structure of finite groups. Some classical results were obtained. Burnside proved that a finite group G is p-nilpotent if G has a Sylow p-subgroup P such that $C_G(P) = N_G(P)$ (see [7, Chapter 10, Section 1]). Zassenhaus [16] proved that a finite group G is abelian if $C_G(X) = N_G(X)$ for every abelian subgroup X of G. Suzuki [2, Proposition 1.8] proved that if a finite non-abelian p-group G contains a self-centralizing subgroup of order p^2, then G is of maximal class. The influence of centralizers of subgroups on the structure of groups was explored. Schmidt [8] characterized finite groups with modular centralizer lattice of length 2 and revisited the results in his book [9, Chapter 9]. Recently, An et al. [1] gave a classification of finite 2-groups whose non-abelian subgroups are self-centralizing. Wang et al. [12] classified the p-groups G in which $C_G(H)/H$ is cyclic for every non-cyclic and non-central abelian subgroup H of G. Zhang et al. [17] classified the p-groups G in which $C_G(H)/H$ is cyclic for every non-central cyclic subgroup H of G.

Manuscript received December 17, 2022. Revised February 28, 2024.
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*This work was supported by the National Natural Science Foundation of China (Nos. 12571018, 12171213, 11771258).

For a finite group G, it is clear that

$$Z(G) \leq C_G(H)$$

holds for any subgroup H of G. A natural question is: What can be said about finite non-abelian groups G with

$$Z(G) = C_G(H)$$

for each non-abelian subgroup H of G? In other words, what can be said about finite non-abelian groups whose non-abelian subgroups have minimum centralizers? For convenience, we call such groups $\mathcal{MC}$-groups.

In this paper, we will study $\mathcal{MC}$-groups. It turns out that $\mathcal{MC}$-groups are just the finite groups whose centralizer lattice is modular of length 2, which were depicted by Schmidt [9]. However, Schmidt's depiction said nothing about $\mathcal{MC}$-p-groups. We will explore $\mathcal{MC}$-p-groups and give a characterization of such groups. In particular, we will systematically study special $\mathcal{MC}$-p-groups, give a characterization of special $\mathcal{MC}$-p-groups, and provide a method to determine or classify special $\mathcal{MC}$-p-groups.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3, some characterizations of $\mathcal{MC}$-groups are given, and it is shown that $\mathcal{MC}$-groups are just the finite groups whose centralizer lattice is modular of length 2, which leads to a classification of $\mathcal{MC}$-groups. A characterization of $\mathcal{MC}$-p-groups is also given in this section. In Section 4, special $\mathcal{MC}$-p-groups are characterized by means of the commutator matrices, and a method to determine or classify special $\mathcal{MC}$-p-groups is provided, and some examples are given.

2 Preliminaries

In this section, we list some notations and known results which will be used in this paper. The readers may refer to [5, 15] for any undefined notation and terminology related to groups and graphs, respectively.

Throughout this paper, we denote by $\mathbb{F}_p$, $\mathbb{F}_p^n$, and $\mathbb{F}_p^{m\times n}$ the field with p elements, the column vector space of dimension n over $\mathbb{F}_p$, and the set of all $m \times n$ matrices over $\mathbb{F}_p$, respectively. Vectors are denoted by boldface italic lower-case letters, e.g., $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, in particular, $\mathbf{0}$ denotes the zero vector. Matrices are denoted by boldface italic capital letters, e.g., $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, in particular, $\boldsymbol{E}$ denotes the identity matrix and $\boldsymbol{O}$ denotes the zero matrix. For any natural numbers i, $\boldsymbol{e}_i$ is the ith elementary unit column vector. We use $\mathrm{rank}(\boldsymbol{A})$, $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{A}^{-1}$ to denote the rank, the transpose and the inverse matrix (if exists) of $\boldsymbol{A}$, respectively. If $\boldsymbol{A}^{-1}$ exists, then $\boldsymbol{A}$ is said to be non-singular. An $n \times n$ matrix $\boldsymbol{K} = (k_{ij})$ is said to be skew-symmetric if $k_{ij} = -k_{ji}$ for $i \neq j$, $1 \leq i, j \leq n$, and $k_{ii} = 0$ for $i = 1, 2, \cdots, n$.

Let G be a finite group. As usual, we use $\exp(G)$, $c(G)$, and $d(G)$ to denote the exponent, the nilpotency class, and the minimal cardinality of a generating set of G, respectively. For any positive integer s, we define

$$\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1\rangle, \quad \mho_s(G) = \langle a^{p^s} \mid a \in G\rangle.$$

Let A and B be two subgroups of G. We use $A * B$ to denote the central product of A and B. For a nilpotent group G, let

$$G = K_1(G) > G' = K_2(G) > K_3(G) > \cdots > K_{c+1}(G) = 1$$

be the lower central series of G, where $c = c(G)$ and $K_{i+1}(G) = [K_i(G), G]$ for $1 \leq i \leq c$, and let

$$1 = Z_0(G) < Z(G) = Z_1(G) < Z_2(G) < \cdots < Z_c(G) = G$$

be the upper central series of G, where $c = c(G)$ and $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ for $0 \leq i \leq c-1$.

A finite group G is called a metabelian group if G' is abelian. A finite non-abelian group G is said to be special if $G' = \Phi(G) = Z(G)$.

A finite non-abelian group G is called an $\mathcal{MC}$-group if all non-abelian subgroups of G have minimum centralizers, that is $C_G(H) = Z(G)$ for all non-abelian subgroups H of G.

Lemma 2.1 (see [11, Theorem 3.1]) *Let $\boldsymbol{K}$ be an $n \times n$ skew-symmetric matrix over $\mathbb{F}_p$. Then* $\operatorname{rank}(\boldsymbol{K})$ *is necessarily even. Furthermore, if* $\operatorname{rank}(\boldsymbol{K}) = 2\nu (\leq n)$, *then K is cogredient to*

$$\begin{pmatrix} \boldsymbol{J}_{2\nu} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}_{n\times n}, \quad \textit{where } \boldsymbol{J}_{2\nu} = \begin{pmatrix} \boldsymbol{J} & & & \\ & \boldsymbol{J} & & \\ & & \ddots & \\ & & & \boldsymbol{J} \end{pmatrix}_{2\nu\times 2\nu}, \quad \boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Schmidt studied the centralizer lattices of groups and characterized the groups whose centralizer lattice is modular of length 2.

Lemma 2.2 (see [9, p. 519, (17)]) *Let G be a finite group. Then the centralizer lattice of G is modular of length* 2 *if and only if G is non-abelian and every proper centralizer of G is abelian.*

Lemma 2.3 (see [9, Theorem 9.3.12] or [8, SATZ 5.9, SATZ 5.12]) *Let G be a finite group. Then the centralizer lattice of G is modular of length* 2 *if and only if G is a group of one of the following types*:

(1) *$G = A \times P$, where A is abelian and P is a p-group with modular centralizer lattice of length* 2.

(2) *G is non-abelian and has an abelian normal subgroup N of prime index.*

(3) *$G/Z(G)$ is a Fröbenius group with Fröbenius kernel $F/Z(G)$ and Fröbenius complement $K/Z(G)$, where F and K are abelian.*

(4) *$G/Z(G)$ is a Fröbenius group with Fröbenius kernel $F/Z(G)$ and Fröbenius complement $K/Z(G)$, where K is abelian, $Z(F) = Z(G)$, $F/Z(G)$ is a p-group, p a prime, and F is a group with modular centralizer lattice of length* 2.

(5) *$G/Z(G) \cong S_4$ and V is not abelian for any subgroup $V \leq G$ such that $V/Z(G)$ is the Klein four-group.*

(6) *$G/Z(G) \cong PSL(2, p^n)$ or $PGL(2, p^n)$, $G' \cong SL(2, p^n)$, p a prime, $p^n > 3$.*

(7) $G/Z(G) \cong PSL(2,9)$ *or* $PGL(2,9)$, G' *is isomorphic to the representation group of* $PSL(2,9)$ *in the sense of Schur.*

We need the following results about finite p-groups in the sequel.

Lemma 2.4 (see [2, Lemma 4.2]) *Let* G *be a finite* p*-group with* $|G'| = p$. *Then*

$$G = (A_1 * A_2 * \cdots * A_s)Z(G),$$

where A_i *is minimal non-abelian for* $1 \le i \le s$.

Lemma 2.5 (see [10] or [2, Lemma 1.1]) *Let* G *be a finite non-abelian* p*-group with an abelian maximal subgroup. Then* $|G| = p|G'||Z(G)|$.

Lemma 2.6 (see [3, Theorem 255.1]) *Let* G *be a finite non-abelian* p*-group with* $Z(A) \le Z(B)$ *for any non-abelian subgroups* A *and* B *satisfying* $A \le B \le G$. *Then either* G *has an abelian maximal subgroup or* $\mho_1(G) \le Z(G)$.

Hall [4] introduced the concept of isoclinism, which will be used in this paper.

Definition 2.1 (see [4]) *Two groups* G *and* H *are said to be isoclinic if there exist isomorphisms*

$$\alpha : G/Z(G) \longrightarrow H/Z(H), \quad \beta : G' \longrightarrow H'$$

such that $[a,b]^\beta = [a',b']$ *for* $a, b \in G$, *where* $Z(H)a' = (Z(G)a)^\alpha$ *and* $Z(H)b' = (Z(G)b)^\alpha$.

It is easy to see that the isoclinic relation is an equivalence relation. The pair (α, β) of isomorphisms is called an isoclinism, and the equivalence classes are called isoclinism families. A group in an isoclinism family that has minimal order is called a stem group.

Lemma 2.7 (see [4]) *Let* G *be a group of the isoclinism family. Then* G *is a stem group if and only if* $Z(G) \le G'$.

Lemma 2.8 (see [4]) *Let* G *be a group and* $N \trianglelefteq G$. *Then* G/N *and* $G/(N \cap G')$ *are isoclinic. In particular,* G *and* G/N *are isoclinic if and only if* $N \cap G' = 1$.

Lemma 2.9 (see [2, §29, Exercise 5]) *Let* G *be a group and* $U \le G$. *Then* U *and* $UZ(G)$ *are isoclinic. Moreover,* U *and* G *are isoclinic if and only if* $UZ(G) = G$.

As a direct consequence of Lemma 2.9, we have the following result.

Corollary 2.1 *Let* G *be a group and* H *be an abelian group. Then* G *and* $G * H$ *are isoclinic.*

Proof Let $L = G * H$. Then $G \le L$ and $L = GZ(L)$. It follows from Lemma 2.9 that G and L are isoclinic.

3 $\mathcal{MC}$-Groups

In this section, we give some characterizations of a finite group to be an $\mathcal{MC}$-group. It is shown that $\mathcal{MC}$-groups are just groups whose centralizer lattice is modular of length 2, which leads to a classification of $\mathcal{MC}$-groups. We also give a characterization of $\mathcal{MC}$-p-groups.

We start with some characterizations of $\mathcal{MC}$-groups.

Theorem 3.1 *Let G be a finite non-abelian group. Then the following statements are equivalent*:

(1) *G is an $\mathcal{MC}$-group.*

(2) *$C_G(H) = Z(G)$ for any two-generator non-abelian subgroup H of G.*

(3) *$C_G(H) = Z(G)$ for any minimal non-abelian subgroup H of G.*

(4) *$Z(H) \leq Z(G)$ for any non-abelian subgroup H of G.*

(5) *$C_G(x)$ is abelian for each $x \in G \setminus Z(G)$.*

(6) *$C_G(H)$ is abelian for any subgroup H satisfying $H \nleq Z(G)$.*

(7) *$Z(A) \leq Z(B)$ for any non-abelian subgroups A and B with $A \leq B \leq G$.*

(8) *$A \cap B = Z(G)$ for any two distinct maximal abelian subgroups A, B of G.*

Proof (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Assume H is a minimal non-abelian subgroup of G. Then $d(H) = 2$. Thus $C_G(H) = Z(G)$.

(3) $\Rightarrow$ (4) Assume H is a non-abelian subgroup of G. Let K be a minimal non-abelian subgroup of H. Then $Z(H) \leq C_G(H) \leq C_G(K) = Z(G)$.

(4) $\Rightarrow$ (5) Otherwise, there exists $x \in G \setminus Z(G)$ such that $C_G(x)$ is not abelian. Thus $x \in Z(C_G(x)) \leq Z(G)$. This contradicts $x \notin Z(G)$.

(5) $\Rightarrow$ (6) Assume H is a subgroup of G satisfying $H \nleq Z(G)$. Take $x \in H \setminus Z(G)$. Then $C_G(H) \leq C_G(x)$. Thus $C_G(H)$ is abelian.

(6) $\Rightarrow$ (7) Assume A and B are two non-abelian subgroups of G and $A \leq B$. Since $A \leq C_G(Z(A))$, $C_G(Z(A))$ is non-abelian. Thus $Z(A) \leq Z(G)$. Therefore $Z(A) \leq Z(G) \cap B \leq Z(B)$.

(7) $\Rightarrow$ (1) Assume H is a non-abelian subgroup of G. Obviously, $Z(G) \leq C_G(H)$ and $x \in Z(\langle x, H\rangle) \leq Z(G)$ for all $x \in C_G(H)$. Thus $C_G(H) \leq Z(G)$. Therefore $C_G(H) = Z(G)$.

(5) $\Rightarrow$ (8) Assume A and B are two distinct maximal abelian subgroups of G. Clearly, $Z(G) \leq A \cap B$. If $Z(G) < A \cap B$, then there exists $x \in (A \cap B) \setminus Z(G)$. Thus $C_G(x)$ is abelian. Obviously, $A, B \leq C_G(x)$. Thus $A = C_G(x) = B$. This contradicts $A \neq B$. Therefore $Z(G) = A \cap B$.

(8) $\Rightarrow$ (5) Otherwise, there exists $x \in G \setminus Z(G)$ such that $C_G(x)$ is not abelian. Take $y, z \in C_G(x)$ such that $[y, z] \neq 1$. Let A be a maximal abelian subgroup containing y, x and B be a maximal abelian subgroup containing z, x. Then $A \neq B$. But $x \in (A \cap B) \setminus Z(G)$. This is a contradiction. Therefore $C_G(x)$ is abelian for each $x \in G \setminus Z(G)$.

From Theorem 3.1 (7), we have the following result.

Corollary 3.1 *Let G be an $\mathcal{MC}$-group. Then every non-abelian subgroup of G is also a $\mathcal{MC}$-group.*

It is easy to see that Theorem 3.1 (6) is equivalent to that every proper centralizer of G is abelian. Now by Lemma 2.2 we get a characterization of $\mathcal{MC}$-groups.

Theorem 3.2 *Let G be a finite non-abelian group. Then G is an $\mathcal{MC}$-group if and only if the centralizer lattice of G is modular of length* 2, *and this holds if and only if G is one of the groups listed in Lemma* 2.3.

$\mathcal{MC}$-groups can be characterized by means of graph theory. For each finite non-abelian group G, let $\Gamma(G) = (V, E)$ be a simple undirected graph with

$$V = \{Z(G)x \mid x \in G \setminus Z(G)\} \quad \text{and} \quad E = \{\{Z(G)x, Z(G)y\} \mid [x, y] = 1, xy^{-1} \notin Z(G)\}.$$

By Theorem 3.1 (5), G is an $\mathcal{MC}$-group if and only if $C_G(x)$ is abelian for each $x \in G \setminus Z(G)$. Obviously, $C_G(x)$ is abelian for each $x \in G \setminus Z(G)$ if and only if the induced subgraph of the neighborhood of each vertex of $\Gamma(G)$ is a complete graph. Thus we immediately get the following result.

Proposition 3.1 *Let G be a finite non-abelian group. Then G is an $\mathcal{MC}$-group if and only if each connected component of $\Gamma(G)$ is a complete graph.*

The next lemma follows directly from Definition 2.1.

Lemma 3.1 *Let G and H be two finite non-abelian groups. If G and H are isoclinic, then $\Gamma(G) \cong \Gamma(H)$.*

By Proposition 3.1 and Lemma 3.1, we get the following result.

Theorem 3.3 *Let G and H be two finite non-abelian groups. If G and H are isoclinic, then G is an $\mathcal{MC}$-group if and only if H is an $\mathcal{MC}$-group.*

Corollary 3.2 *Let G be a finite non-abelian group and H a finite group. Then the central product $G * H$ is an $\mathcal{MC}$-group if and only if G is an $\mathcal{MC}$-group and H is abelian.*

Proof ($\Leftarrow$) By Corollary 2.1, G and $G * H$ are isoclinic. Since G is an $\mathcal{MC}$-group, $G * H$ is also an $\mathcal{MC}$-group by Theorem 3.3.

($\Rightarrow$) By Corollary 3.1, G is an $\mathcal{MC}$-group. Let $K = G * H$. Since G is non-abelian, there exists an element $y \in G \setminus Z(K)$. Obviously, $H \le C_K(y)$. By Theorem 3.1 (5), $C_K(y)$ is abelian. Thus H is abelian.

In the remainder of this paper, we will investigate $\mathcal{MC}$-p-groups. The following proposition shows the class of $\mathcal{MC}$-p-groups is quite large.

Proposition 3.2 *Let G be a finite non-abelian p-group. If G is one of the following groups, then G is an $\mathcal{MC}$-group.*

(1) *The p-groups with an abelian maximal subgroup.*

(2) *The p-groups G in which $Z(H) = Z(G)$ for each non-abelian subgroup H of G.*

(3) *The p-groups G in which $|G : Z(G)| \le p^3$.*

(4) *The p-groups G in which $HZ(G)/Z(G)$ is cyclic for each abelian subgroup H of G.*

Proof (1) It follows from Lemma 2.3 (2).

(2) It follows by Theorem 3.1 (4).

(3) Since G is non-abelian, $|G : Z(G)| = p^2$ or p^3. If $|G : Z(G)| = p^2$, then G has an abelian maximal subgroup. It follows from (1) that G is an $\mathcal{MC}$-group. Assume $|G : Z(G)| = p^3$. Let H be a non-abelian subgroup of G and $L = HZ(G)$. Then $Z(G) \le Z(L)$ and $Z(H) \le Z(L)$. If $L = G$, then $Z(H) \le Z(G)$. If $L < G$, then $p^2 \le |L : Z(L)| \le |L : Z(G)| \le p^2$. Thus $Z(L) = Z(G)$, and hence $Z(H) \le Z(G)$. Therefore G is an $\mathcal{MC}$-group by Theorem 3.1 (4).

(4) By [6, Proposition 1], $\overline{G} = G/Z(G)$ is dihedral or of exponent p. Assume A and B are two distinct maximal abelian subgroups of G. By the hypothesis and Correspondence Theorem, $\overline{A} = A/Z(G)$ and $\overline{B} = B/Z(G)$ are two maximal cyclic subgroups of $\overline{G}$. Thus $\overline{A} \cap \overline{B} = \overline{A \cap B} = \overline{1}$. Hence $A \cap B \leq Z(G)$. So $A \cap B = Z(G)$. It follows from Theorem 3.1 (8) that G is an $\mathcal{MC}$-group.

Remark 3.1 The groups of (2) listed in Proposition 3.2 were classified by Wang et al. in [13–14] and the groups of (4) listed in Proposition 3.2 were determined by Mann in [6].

Lemma 3.2 *Let G be an $\mathcal{MC}$-p-group. Then*

(1) *If A is a maximal abelian subgroup of G, then $|A| \geq |G : G'|$.*

(2) *$\Phi(G)Z_{c-1}(G)$ is abelian, where $c = c(G)$. In particular, G is metabelian.*

Proof (1) Let $x \in A \setminus Z(G)$. Then $A \leq C_G(x)$. Since G is an $\mathcal{MC}$-group, $C_G(x)$ is abelian by Theorem 3.1 (5). Thus $A = C_G(x)$ by the maximality of A. It follows that $|G : C_G(x)| = |x^G| \leq |xG'| = |G'|$. Therefore $|A| = |C_G(x)| \geq |G : G'|$.

(2) We first prove that $\Phi(G)$ is abelian. Obviously, $K_c(G) \leq Z(G)$ and $K_{c-1}(G) \nleq Z(G)$. Let $x \in K_{c-1}(G) \setminus Z(G)$ and $x^p \in Z(G)$. Then $[x^p, g] = 1$ for all $g \in G$. Since $[x, g] \in K_c(G)$ and $K_c(G) \leq Z(G)$, $[x, g^p] = [x, g]^p = [x^p, g] = 1$. It follows that $[x, \mho_1(G)] = 1$. On the other hand, $[K_{c-1}(G), K_2(G)] \leq K_{c+1}(G) = 1$. This means $[x, K_2(G)] = 1$. Since $\Phi(G) = G'\mho_1(G)$, $x \in Z(\Phi(G))$. Thus $Z(\Phi(G)) \nleq Z(G)$. Therefore $\Phi(G)$ is abelian by Theorem 3.1 (4). It follows that G' is abelian and so G is metabelian.

Now we prove that $\Phi(G)Z_{c-1}(G)$ is abelian. Obviously, $[K_{c-1}(G), Z_{c-1}(G)] = 1$ and $[K_{c-1}(G), K_2(G)] = 1$. If G has an abelian maximal subgroup A, then $[K_{c-1}(G), \mho_1(G)] \leq [A, A] = 1$. If G has no abelian maximal subgroup, then $\mho_1(G) \leq Z(G)$ by Theorem 3.1 (7) and Lemma 2.6. Thus $[K_{c-1}(G), \mho_1(G)] = 1$. It follows that $[K_{c-1}(G), \Phi(G)] = 1$. So $K_{c-1}(G) \leq C_G(\Phi(G)Z_{c-1}(G))$. Note that $K_{c-1}(G) \nleq Z(G)$. Since G is an $\mathcal{MC}$-group, $\Phi(G)Z_{c-1}(G)$ is abelian.

An $\mathcal{MC}$-p-group G is called an $\mathcal{MC}_1$-group if G has no abelian maximal subgroup.

Lemma 3.3 *Let G and H be two p-groups. If G and H are isoclinic, then G is an $\mathcal{MC}_1$-group if and only if H is an $\mathcal{MC}_1$-group.*

Proof Since G and H are isoclinic, G has abelian maximal subgroups if and only if H has abelian maximal subgroups. Now the lemma follows directly from Theorem 3.3.

Lemma 3.4 *Let G be an $\mathcal{MC}_1$-group. Then $|G : Z(G)| \geq p^3$, $|G'| \geq p^2$ and $Z(G)$ is non-cyclic.*

Proof If $|G : Z(G)| \leq p^2$, then G has an abelian maximal subgroup. This contradicts that G is an $\mathcal{MC}_1$-group. Thus $|G : Z(G)| \geq p^3$.

If $|G'| = p$, then, by Lemma 2.4,

$$G = (A_1 * A_2 * \cdots * A_s)Z(G),$$

where A_i is minimal non-abelian for each $1 \leq i \leq s$. We assert that $s = 1$. Otherwise, $s \geq 2$, then there exists a subgroup K of G such that $K = A_1 * A_2$. By the minimality of A_1 and A_2, assume that $A_1 = \langle a, b\rangle$ and $A_2 = \langle c, d\rangle$ with $[a, b] \neq 1$, $[c, d] \neq 1$. Let $H = \langle a, b, c\rangle$. Clearly, H is non-abelian and $c \in Z(H) \setminus Z(G)$. This contradicts that G is an $\mathcal{MC}$-group by Theorem 3.1 (4). Thus $G = A_1Z(G)$ and $|G : Z(G)| = p^2$, which contradicts with $|G : Z(G)| \geq p^3$. Thus $|G'| \geq p^2$.

Lastly, we prove that $Z(G)$ is non-cyclic. Since G has no abelian maximal subgroup, G is not a 2-group of maximal class. Hence there exists a normal subgroup N of G such that $N \cong \mathrm{C}_p^2$. By N/C Theorem, $|G : C_G(N)| \leq p$. It follows that there exists a maximal subgroup M such that $N \leq M \leq C_G(N)$. Thus $N \leq Z(M)$. In particular, $Z(M)$ is non-cyclic. Since G is an $\mathcal{MC}_1$-group, M is non-abelian. By Theorem 3.1 (4), $Z(M) \leq Z(G)$. Therefore $Z(G)$ is non-cyclic.

Lemma 3.5 *Let G be an $\mathcal{MC}_1$-group with $Z(G) \leq G'$. Then*

(1) $|G| \leq |G'|^3$.

(2) $\mho_1(G) \leq Z(G) \leq G' = \Phi(G)$. *Moreover, if $p = 2$ or $c(G) = 2$, then G is a special p-group.*

Proof (1) Let A and B be two distinct maximal abelian subgroups of G. Then $A \cap B = Z(G)$ by Theorem 3.1 (8). It follows from Lemma 3.2 (1) that

$$|G| \geq |AB| = \frac{|A||B|}{|A \cap B|} \geq \frac{|G : G'|^2}{|Z(G)|} \geq \frac{|G : G'|^2}{|G'|} = \frac{|G|^2}{|G'|^3}.$$

So $|G| \leq |G'|^3$.

(2) By Theorem 3.1 (7) and Lemma 2.6, $\mho_1(G) \leq Z(G)$. Since $Z(G) \leq G'$, $G' = \Phi(G)$. If $p = 2$, then $\mho_1(G) = \Phi(G)$ and so $\mho_1(G) = Z(G) = G' = \Phi(G)$. Therefore G is a special 2-group. If $c(G) = 2$, then $G' \leq Z(G)$ and hence $Z(G) = G' = \Phi(G)$. Therefore G is a special p-group.

Now we are ready for a characterization of $\mathcal{MC}$-p-groups.

Theorem 3.4 *Let G be a finite non-abelian p-group. Then G is an $\mathcal{MC}$-group if and only if G is isoclinic to a p-group with an abelian maximal subgroup or a metabelian $\mathcal{MC}$-p-group H satisfying $\mho_1(H) \leq Z(H) \leq H' = \Phi(H)$.*

In particular, if $p = 2$ or $c(G) = 2$, then G is an $\mathcal{MC}$-group if and only if G is isoclinic to a p-group with an abelian maximal subgroup or a special $\mathcal{MC}$-p-group.

Proof ($\Rightarrow$) Let H be a stem group of the isoclinism family containing group G. Then $Z(H) \leq H'$ by Lemma 2.7. Since G is an $\mathcal{MC}$-group, H is also an $\mathcal{MC}$-group by Theorem 3.3. If H has no abelian maximal subgroup, then H is an $\mathcal{MC}_1$-group. It follows from Lemma 3.2 (2) and Lemma 3.5 (2) that H is a metabelian group satisfying $\mho_1(H) \leq Z(H) \leq H' = \Phi(H)$. In particular, if H is a 2-group or $c(H) = 2$, then H is special.

($\Leftarrow$) It follows from Proposition 3.2 (1).

Remark 3.2 Hall [4] gave a method to classify finite p-groups having an abelian maximal subgroup in isoclinism sense. The class of metabelian p-groups G satisfying $\mho_1(G) \leq Z(G) \leq G' = \Phi(G)$, including special p-groups, metabelian p-groups of maximal class of order less than p^{p+1}, metabelian minimal non-regular p-groups of exponent p^2, is fairly large, it seems like a very difficult thing to characterize or classify metabelian $\mathcal{MC}$-p-groups H satisfying $\mho_1(H) \le Z(H) \le H' = \Phi(H)$. We will concentrate on studying special $\mathcal{MC}$-p-groups in the next section.

4 Special $\mathcal{MC}$-p-Groups

In this section, we will give a criterion for a special p-group to be an $\mathcal{MC}$-group. Moreover, we also provide a method to classify special $\mathcal{MC}$-p-groups up to isoclinism. Recall that $c(G) = 2$ and $\exp(G/G') = p$ for any special p-group G.

Let G be a finite p-group with $c(G) = 2$ and $\exp(G/G') = p$. Then G' and G/G' are elementary abelian. Assume that

$$\overline{G} = G/G' = \langle \overline{a}_1 \rangle \times \langle \overline{a}_2 \rangle \times \cdots \times \langle \overline{a}_n \rangle \quad \text{and} \quad G' = \langle t_1 \rangle \times \langle t_2 \rangle \times \cdots \times \langle t_r \rangle.$$

Then $G = \langle a_1, a_2, \cdots, a_n \rangle$. Since $c(G) = 2$, we have

$$[a_i, t_u] = [t_l, t_u] = 1, \quad 1 \le i \le n,\ 1 \le u, l \le r.$$

Suppose that $[a_i, a_j] = \prod\limits_{u=1}^{r} t_u^{x_{uij}}$, $x_{uij} \in \mathbb{F}_p$, $1 \le i, j \le n$. Then $\boldsymbol{X}_u = (x_{uij})_{nn}$ is a skew-symmetric matrix over $\mathbb{F}_p$, for each $1 \le u \le r$. In this case, $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r$ are non-zero. We call the $n \times nr$ matrix $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r)$ a commutator matrix of G.

For sake of convenience, we denote

$$\boldsymbol{A} \diamond \boldsymbol{X} \diamond \boldsymbol{B} = \boldsymbol{A} \diamond (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r) \diamond \boldsymbol{B} = (\boldsymbol{A}\boldsymbol{X}_1\boldsymbol{B}, \boldsymbol{A}\boldsymbol{X}_2\boldsymbol{B}, \cdots, \boldsymbol{A}\boldsymbol{X}_r\boldsymbol{B})$$

and

$$\boldsymbol{X} \circ \boldsymbol{C} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r) \circ \boldsymbol{C} = \Big(\sum_{k=1}^{r} c_{k1}\boldsymbol{X}_k,\ \sum_{k=1}^{r} c_{k2}\boldsymbol{X}_k, \cdots, \sum_{k=1}^{r} c_{kr}\boldsymbol{X}_k \Big)$$

for any $\boldsymbol{A} \in \mathbb{F}_p^{l \times n}$, $\boldsymbol{B} \in \mathbb{F}_p^{n \times m}$ and $\boldsymbol{C} = (c_{ij})_{rr} \in \mathbb{F}_p^{r \times r}$. In fact, the operations "$\diamond$" is just the multiplication of partitioned matrices, and the operation "$\circ$" is just the multiplication of "formal matrices" (i.e., consider $(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r)$ as a $1 \times r$ matrix). It is easy to verify that $\boldsymbol{E} \diamond \boldsymbol{X} \diamond \boldsymbol{B} = \boldsymbol{X} \diamond \boldsymbol{B}$, $\boldsymbol{A} \diamond \boldsymbol{X} \diamond \boldsymbol{E} = \boldsymbol{A} \diamond \boldsymbol{X}$, $\boldsymbol{E} \diamond \boldsymbol{X} \diamond \boldsymbol{E} = \boldsymbol{X}$, $\boldsymbol{X} \circ \boldsymbol{E} = \boldsymbol{X}$ and $\boldsymbol{X} \circ \boldsymbol{C} \circ \boldsymbol{C}^{-1} = \boldsymbol{X}$ if $\boldsymbol{C}$ is non-singular.

We first give a characterization of special p-groups.

Proposition 4.1 *Let G be a finite p-group of class* 2, $\exp(G/G') = p$ *and* $\boldsymbol{X} \in \mathbb{F}_p^{n \times nr}$ *is a commutator matrix of G. Then G is a special p-group if and only if* $\operatorname{rank}(\boldsymbol{X}) = n$.

Proof By the definition of special p-groups, to prove our result, it is enough to prove

$$Z(G) \le G' \Leftrightarrow \operatorname{rank}(\boldsymbol{X}) = n.$$

Let $g, h \in G$ be such that $\overline{g} = \prod\limits_{i=1}^{n} \overline{a}_i^{k_i}$ and $\overline{h} = \prod\limits_{j=1}^{n} \overline{a}_j^{l_j}$ with $\boldsymbol{k} = (k_1, k_2, \cdots, k_n)^{\mathrm{T}}$, $\boldsymbol{l} = (l_1, l_2, \cdots, l_n)^{\mathrm{T}} \in \mathbb{F}_p^n$. Note that $G' \le Z(G)$ and $\exp(G/G') = p$. We have

$$[g, h] = \Big[\prod_{i=1}^{n} a_i^{k_i}, \prod_{j=1}^{n} a_j^{l_j} \Big] = \prod_{i,j=1}^{n} [a_i, a_j]^{k_i l_j} = \prod_{i,j=1}^{n} \prod_{u=1}^{r} t_u^{x_{uij} k_i l_j}$$

$$= \prod_{u=1}^{r} \prod_{i,j=1}^{n} t_u^{x_{uij}k_i l_j} = \prod_{u=1}^{r} t_u^{\sum\limits_{i,j=1}^{n} x_{uij}k_i l_j}$$

and

$$[g,h] = 1 \Leftrightarrow \sum_{i,j=1}^{n} x_{uij}k_i l_j = 0, \quad 1 \le u \le r \Leftrightarrow \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} = \boldsymbol{0}. \tag{$*$}$$

It follows that

$$g \in Z(G) \Leftrightarrow [g,h] = 1 \quad \text{for all } h \in G \Leftrightarrow \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} = \boldsymbol{0} \quad \text{for all } \boldsymbol{l} \in \mathbb{F}_p^n \Leftrightarrow \boldsymbol{k}^{\mathrm{T}} \boldsymbol{X} = \boldsymbol{0}.$$

Therefore

$$\begin{aligned} Z(G) \le G' \Leftrightarrow & \text{ the system of homogeneous linear equations } \boldsymbol{X}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0} \text{ has only} \\ & \quad \text{the trivial solution in } \mathbb{F}_p^n \\ \Leftrightarrow & \operatorname{rank}(\boldsymbol{X}) = n, \end{aligned}$$

as required.

Now we can give a characterization of special $\mathcal{MC}$-p-groups.

Theorem 4.1 *Let G be a finite special p-group and $\boldsymbol{X} \in \mathbb{F}_p^{n\times nr}$ a commutator matrix of G. Then G is an $\mathcal{MC}$-group if and only if*

$$\operatorname{rank}(\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l}) = n \quad \textit{for all } \boldsymbol{k}, \boldsymbol{l} \in \mathbb{F}_p^n \textit{ with } \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} \ne \boldsymbol{0}.$$

Proof By Theorem 3.1 (2), G is an $\mathcal{MC}$-group if and only if

$$C_G(\langle g,h\rangle) = Z(G) \quad \text{for all } g,h \in G \text{ with } [g,h] \ne 1.$$

Let $g,h \in G$ be such that $\overline{g} = \prod\limits_{i=1}^{n} \overline{a}_i^{k_i}$ and $\overline{h} = \prod\limits_{j=1}^{n} \overline{a}_j^{l_j}$ with $\boldsymbol{k} = (k_1,k_2,\cdots,k_n)^{\mathrm{T}}$, $\boldsymbol{l} = (l_1,l_2,\cdots,l_n)^{\mathrm{T}} \in \mathbb{F}_p^n$. Then by Equation $(*)$ in the proof of Proposition 4.1, we have

$$[g,h] \ne 1 \Leftrightarrow \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} \ne \boldsymbol{0}.$$

Suppose that $[g,h] \ne 1$. Let $b \in G$ such that $\overline{b} = \prod\limits_{i=1}^{n} \overline{a}_i^{b_i}$ with $\boldsymbol{b} = (b_1,b_2,\cdots,b_n)^{\mathrm{T}} \in \mathbb{F}_p^n$. Then by Equation $(*)$, we have

$$\begin{aligned} b \in C_G(\langle g,h\rangle) &\Leftrightarrow [b,g] = [b,h] = 1 \\ &\Leftrightarrow \boldsymbol{b}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{k} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} = \boldsymbol{0} \\ &\Leftrightarrow \boldsymbol{b}^{\mathrm{T}}(\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l}) = \boldsymbol{0}. \end{aligned}$$

Note that $G' = Z(G)$. It follows that

$$\begin{aligned} C_G(\langle g,h\rangle) = Z(G) &\Leftrightarrow (\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0} \text{ has only the trivial solution in } \mathbb{F}_p^n \\ &\Leftrightarrow \operatorname{rank}(\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l}) = n. \end{aligned}$$

Therefore

$$G \text{ is an } \mathcal{MC}\text{-group } \Leftrightarrow \operatorname{rank}(\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l}) = n \quad \text{for all } \boldsymbol{k}, \boldsymbol{l} \in \mathbb{F}_p^n \text{ with } \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} \neq \boldsymbol{0},$$

as required.

The following result gives a necessary and sufficient condition for two special p-groups being isoclinic.

Proposition 4.2 *Let G and H be two finite special p-groups with commutator matrices $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathbb{F}_p^{n\times nr}$, respectively. Then G and H are isoclinic if and only if there exist two non-singular matrices $\boldsymbol{C} \in \mathbb{F}_p^{n\times n}$ and $\boldsymbol{D} \in \mathbb{F}_p^{r\times r}$ such that*

$$(\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{C}) \circ \boldsymbol{D} = \boldsymbol{X}.$$

Proof Let G be a special p-group with the assumption given at the beginning of this section, and let H be a special p-group such that

$$\overline{H} = H/H' = \langle \overline{b}_1 \rangle \times \langle \overline{b}_2 \rangle \times \cdots \times \langle \overline{b}_n \rangle \quad \text{and} \quad H' = \langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_r \rangle.$$

Then $H = \langle b_1, b_2, \cdots, b_n \rangle$. Assume $[b_i, b_j] = \prod\limits_{u=1}^{r} s_u^{y_{uij}}$, $y_{uij} \in \mathbb{F}_p$, $1 \le i, j \le n$, $\boldsymbol{Y}_u = (y_{uij})_{nn}$ and $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_r)$. Since G and H are special p-groups, we have $H' = Z(H)$ and $G' = Z(G)$.

If G and H are isoclinic, by Definition 2.1, there exist the following isomorphisms

$$\begin{aligned} \alpha : G/Z(G) &\to H/Z(H) \\ Z(G)a &\mapsto Z(H)a' \end{aligned} \quad \text{and} \quad \beta : G' \to H'$$

such that $[a, b]^\beta = [a', b']$ for $a, b \in G$. Assume $(Z(G)a_i)^\alpha = Z(H) \prod\limits_{j=1}^{n} b_j^{a_{ji}}$ and $t_i^\beta = \prod\limits_{j=1}^{r} s_j^{b_{ji}}$. Let $\boldsymbol{A} = (a_{ij})_{nn} \in \mathbb{F}_p^{n\times n}$ and $\boldsymbol{B} = (b_{ij})_{rr} \in \mathbb{F}_p^{r\times r}$. Then $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-singular matrices since α and β are isomorphisms. Now

$$[a_i, a_j]^\beta = \Big(\prod_{u=1}^{r} t_u^{x_{uij}}\Big)^\beta = \prod_{u=1}^{r} (t_u^\beta)^{x_{uij}} = \prod_{u=1}^{r} \Big(\prod_{l=1}^{r} s_l^{b_{lu}}\Big)^{x_{uij}} = \prod_{u,l=1}^{r} s_l^{b_{lu}x_{uij}} = \prod_{l=1}^{r} s_l^{\sum\limits_{u=1}^{r} b_{lu}x_{uij}}.$$

Note that $Z(H)a_i' = Z(H) \prod\limits_{k=1}^{n} b_k^{a_{ki}}$. Hence

$$\begin{aligned} [a_i', a_j'] &= \Big[\prod_{k=1}^{n} b_k^{a_{ki}}, \prod_{m=1}^{n} b_m^{a_{mj}}\Big] = \prod_{k,m=1}^{n} [b_k, b_m]^{a_{ki}a_{mj}} = \prod_{k,m=1}^{n} \Big(\prod_{l=1}^{r} s_l^{y_{lkm}}\Big)^{a_{ki}a_{mj}} \\ &= \prod_{k,m=1}^{n} \prod_{l=1}^{r} s_l^{y_{lkm}a_{ki}a_{mj}} = \prod_{l=1}^{r} \prod_{k,m=1}^{n} s_l^{y_{lkm}a_{ki}a_{mj}} = \prod_{l=1}^{r} s_l^{\sum\limits_{k,m=1}^{n} y_{lkm}a_{ki}a_{mj}}. \end{aligned}$$

Therefore

$$[a_i, a_j]^\beta = [a_i', a_j'] \Leftrightarrow \prod_{l=1}^{r} s_l^{\sum\limits_{u=1}^{r} b_{lu}x_{uij}} = \prod_{l=1}^{r} s_l^{\sum\limits_{k,m=1}^{n} y_{lkm}a_{ki}a_{mj}}$$

$$\begin{aligned}
&\Leftrightarrow \sum_{u=1}^{r} b_{lu} x_{uij} = \sum_{k,m=1}^{n} y_{lkm} a_{ki} a_{mj} \quad \text{for each } 1 \le l \le r \\
&\Leftrightarrow \boldsymbol{X} \circ \boldsymbol{B}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{A} \\
&\Leftrightarrow \boldsymbol{X} = (\boldsymbol{A}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{A}) \circ \boldsymbol{B}^{-T}.
\end{aligned}$$

Let $\boldsymbol{C} = \boldsymbol{A}$ and $\boldsymbol{D} = (\boldsymbol{B}^{-1})^{\mathrm{T}}$. Then $\boldsymbol{C}$ and $\boldsymbol{D}$ are non-singular and $\boldsymbol{X} = (\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{C}) \circ \boldsymbol{D}$.

Conversely, suppose $\boldsymbol{C} = (a_{ij})_{nn} \in \mathbb{F}_p^{n\times n}$ and $(\boldsymbol{D}^{-1})^{\mathrm{T}} = (b_{ij})_{rr} \in \mathbb{F}_p^{r\times r}$. Define two maps

$$\alpha : G/Z(G) \to H/Z(H) \quad \text{and} \quad \beta : G' \to H'$$

such that $(Z(G)a_i)^{\alpha} = Z(H) \prod\limits_{j=1}^{n} b_j^{a_{ji}}$ and $t_i^{\beta} = \prod\limits_{j=1}^{r} s_j^{b_{ji}}$, respectively. It is routine matter to verify that α and β are group isomorphisms. Moreover, for any $a, b \in G$, assume that $\overline{a} = \prod\limits_{i=1}^{n} \overline{a}_i^{k_i}$ and $\overline{b} = \prod\limits_{j=1}^{n} \overline{a}_j^{l_j}$. Then

$$[a,b]^{\beta} = \Big[\prod_{i=1}^{n} a_i^{k_i}, \prod_{j=1}^{n} a_j^{l_j}\Big]^{\beta} = \prod_{i,j=1}^{n} ([a_i,a_j]^{k_i l_j})^{\beta} = \prod_{i,j=1}^{n} ([a_i,a_j]^{\beta})^{k_i l_j}.$$

Let $Z(H)a' = (Z(G)a)^{\alpha}$ and $Z(H)a'_i = (Z(G)a_i)^{\alpha}$. Then $Z(H)a' = Z(H) \prod\limits_{i=1}^{n} (a'_i)^{k_i}$. Similarly, we have $Z(H)b' = Z(H) \prod\limits_{j=1}^{n} (a'_j)^{l_j}$. Thus

$$[a',b'] = \Big[\prod_{i=1}^{n} (a'_i)^{k_i}, \prod_{j=1}^{n} (a'_j)^{l_j}\Big] = \prod_{i,j=1}^{n} [a'_i, a'_j]^{k_i l_j}.$$

From the necessity of proof, we have

$$\boldsymbol{X} = (\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{C}) \circ \boldsymbol{D} \Leftrightarrow [a_i,a_j]^{\beta} = [a'_i,a'_j] \quad \text{for } 1 \le i,j \le n.$$

Thus $[a,b]^{\beta} = [a',b']$. It follows from Definition 2.1 that G and H are isoclinic.

Remark 4.1 (1) Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r)$, $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_r) \in \mathbb{F}_p^{n\times nr}$. $\boldsymbol{X}$ and $\boldsymbol{Y}$ are said to be isoclinic if there exist two non-singular matrices $\boldsymbol{C} \in \mathbb{F}_p^{n\times n}$ and $\boldsymbol{D} \in \mathbb{F}_p^{r\times r}$ such that

$$(\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{C}) \circ \boldsymbol{D} = \boldsymbol{X}.$$

It is easy to see that the isoclinic relation is an equivalence relation. Now Proposition 4.2 can be restated as "two finite special p-groups are isoclinic if and only if their commutator matrices are isoclinic".

Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r) \in \mathbb{F}_p^{n\times nr}$ and $\boldsymbol{D}$ be an $r \times r$ elementary matrix over $\mathbb{F}_p$. Considering $\boldsymbol{X}$ as $1 \times r$ formal matrix, it is clear that $\boldsymbol{X} \circ \boldsymbol{D}$ is just the formal matrix obtained from $\boldsymbol{X}$ by performing elementary column transformation. For convenience, we call such elementary column transformation formal elementary column transformation. Clearly, the matrix obtained from $\boldsymbol{X}$ by performing formal elementary column transformation is isoclinic to $\boldsymbol{X}$.

(2) Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r)$, $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_r) \in \mathbb{F}_p^{n\times nr}$. If $\boldsymbol{X}$ is a commutator matrix of some finite special p-group and $\boldsymbol{Y}$ is isoclinic to $\boldsymbol{X}$, then both $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r$ and $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_r$ are linearly independent.

To classify special $\mathcal{MC}$-p-groups, we need the following two lemmas.

Lemma 4.1 *Let G be a finite special p-group. Then G is isoclinic to a special p-group K with $|K| = |G|$ and $K = \Omega_1(K)$.*

Proof Since G is a special p-group, $Z(G) = G' = \Phi(G)$. Assume $G = \langle a_1, a_2, \cdots, a_n\rangle$. Then $a_i^p \in Z(G)$ for all $1 \le i \le n$. Let $A = \langle b_1\rangle \times \langle b_2\rangle \times \cdots \times \langle b_n\rangle$ with $o(b_j) = p^3$ for all $1 \le j \le n$. Considering $L = G \times A$. Then $Z(L) = Z(G) \times A$ and $L' = G'$. By Corollary 2.1, G and L are isoclinic. Take

$$N = \langle (a_ib_i)^p \mid a_i \in G, b_i \in A, 1 \le i \le n\rangle.$$

It is easy to get $N \le Z(L)$ and $N \cap L' = 1$. Let $\overline{L} = L/N$. Then L and $\overline{L}$ are isoclinic by Lemma 2.8. Notice that $\overline{L} = \overline{G} * \overline{A} = K * \overline{A}$, where

$$K = \langle \overline{a_ib_i} \mid \overline{a_ib_i}^p = \overline{1}, [\overline{a_ib_i}, \overline{a_jb_j}] = \overline{[a_i, a_j]}, \overline{a}_i \in \overline{G}, \overline{b}_i \in \overline{A}, 1 \le i, j \le n\rangle.$$

Then $\overline{L}$ and K are isoclinic by Corollary 2.1. Thus G and K are isoclinic, $|K/Z(K)| = |G/Z(G)| = p^n$ and $|K'| = |G'|$. Obviously, $K = \Omega_1(K)$, $\Phi(K) = K' \le Z(K)$ and $|K/K'| = p^n$. It implies $K' = Z(K)$. Therefore K is a special p-group and $|K| = |G|$.

Lemma 4.2 *Let G be a special $\mathcal{MC}$-p-group with $|G'| = p^r$. Then*

$$\begin{cases} |G/G'| = p^2, & r = 1, \\ p^3 \le |G/G'| \le p^{2r}, & r \ge 2. \end{cases}$$

Proof Since G is a special p-group, $Z(G) = G'$. If G has an abelian maximal subgroup, then $|G/G'| = p|Z(G)| = p^{r+1}$ by Lemma 2.5. If G has no abelian maximal subgroup, then $r \ge 2$ and $p^3 \le |G/G'| \le p^{2r}$ by Lemmas 3.4 and 3.5 (1).

Up to now, we can give a method to determine or classify special $\mathcal{MC}$-p-groups in the following remark.

Remark 4.2 Let p be a prime number. Firstly, for each pair (n, r) with $(n, r) = (2, 1)$ or $r \ge 2$ and $3 \le n \le 2r \le n(n-1)$, find all matrices $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r) \in \mathbb{F}_p^{n\times nr}$ with $\boldsymbol{X}_i$ being skew-symmetric, $i = 1, 2, \cdots, r$ such that $\operatorname{rank}(\boldsymbol{X}) = n$ and

$$\operatorname{rank}(\boldsymbol{X} \diamond \boldsymbol{k}, \boldsymbol{X} \diamond \boldsymbol{l}) = n \quad \text{for all } \boldsymbol{k}, \boldsymbol{l} \in \mathbb{F}_p^n \text{ with } \boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{X} \diamond \boldsymbol{l} \ne \boldsymbol{0}.$$

If such a matrix does not exist, then there is no special $\mathcal{MC}$-p-group G with $|G'| = p^r$ and $|G : G'| = p^n$.

Secondly, let $\mathcal{X}_{n,r}$ be the representative system of the isoclinic classes of the matrices determined above. For simplicity, one can translate the matrices in $\mathcal{X}_{n,r}$ into relatively simple forms by using the operations $\diamond$ and $\circ$ as in Proposition 4.2.

Thirdly, for $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_r) \in \mathcal{X}_{n,r}$, let $\boldsymbol{X}_u = (x_{uij}) \in \mathbb{F}_p^{n\times n}$ for $u = 1, 2, \cdots, r$. Define

$$H_{\boldsymbol{X}} = \Big\langle a_1, a_2, \cdots, a_n; t_1, t_2, \cdots, t_r \mid a_i^p = t_u^p = 1, [a_i, a_j] = \prod_{u=1}^{r} t_u^{x_{uij}}, [a_i, t_u] = [t_l, t_u] = 1,$$

$$i,j=1,2,\cdots,n;u,l=1,2,\cdots,r\Big\rangle.$$

Then all isoclinic classes of $H_{\boldsymbol{X}}$, $\boldsymbol{X}\in\mathcal{X}_{n,r}$, give all special $\mathcal{MC}$-p-groups G with $|G'|=p^r$ and $|G:G'|=p^n$.

We give some examples to end our paper. We first consider finite special p-groups having an abelian maximal subgroup. The next result gives a characterization of such groups.

Theorem 4.2 *Let G be a finite special p-group with $|G'|=p^r$. Then G has an abelian maximal subgroup if and only if $|G/G'|=p^{r+1}$ and G is isoclinic to the special p-group H whose commutator matrix is*

$$\boldsymbol{Y}=(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\cdots,\boldsymbol{Y}_r)_{(r+1)\times(r+1)r},\quad \boldsymbol{Y}_u=\begin{pmatrix}0&\boldsymbol{e}_u^{\mathrm{T}}\\-\boldsymbol{e}_u&\boldsymbol{O}\end{pmatrix}_{(r+1)\times(r+1)},\quad 1\le u\le r.$$

Proof ($\Rightarrow$) Since G has an abelian maximal subgroup, by Lemma 2.5, $|G/G'|=p|Z(G)|$. Note that G is a special p-group. Then $Z(G)=G'$ and $|G/G'|=p^{r+1}$.

Let $\overline{G}=G/G'$. Then $\overline{G}$ is an elementary abelian group of order p^{r+1}. Let A be an abelian maximal subgroup of G. Assume that $\overline{A}=\langle\overline{b}_2\rangle\times\langle\overline{b}_3\rangle\times\cdots\times\langle\overline{b}_{r+1}\rangle$. Take $b_1\in G\setminus A$. Then $G=\langle b_1,b_2,\cdots,b_{r+1}\rangle$ and $[b_i,b_j]=1$ for $2\le i,j\le r+1$. Let $\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r)$ be the commutator matrix of G. Then

$$\boldsymbol{Z}_u=\begin{pmatrix}0&\boldsymbol{\beta}_u^{\mathrm{T}}\\-\boldsymbol{\beta}_u&\boldsymbol{O}\end{pmatrix},\quad \boldsymbol{\beta}_u\in\mathbb{F}_p^r,\ 1\le u\le r.$$

Since G is special, $\operatorname{rank}(\boldsymbol{Z})=r+1$ by Proposition 4.1. Note that $\operatorname{rank}(\boldsymbol{Z}_u)\le 2$ for each u. We get $\operatorname{rank}(\boldsymbol{Z}_u)=2$ for $u=1,2,\cdots,r$. On the other hand, $\operatorname{rank}(\boldsymbol{Z})=r+1$ implies that $\operatorname{rank}(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_r)=r$, and so $(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_r)$ has no zero row.

By Remark 4.1 (1), without loss of generality, we assume that the $(2,1)$-th entry of $\boldsymbol{Z}_1$ does not equal 0. Then by some appropriate formal elementary column transformations, we can translate the entries of the second row of $\boldsymbol{Z}$ into 0 except for the $(2,1)$-th entry. Let

$$\boldsymbol{Z}'=(\boldsymbol{Z}_1',\boldsymbol{Z}_2',\cdots,\boldsymbol{Z}_r')$$

be the resulting matrix. Clearly, $\operatorname{rank}(\boldsymbol{Z}')=\operatorname{rank}(\boldsymbol{Z})=r+1$. Then the $(3,1)$-th entry of $\boldsymbol{Z}_s'$ does not equal 0 for some $2\le s\le r$. Without loss of generality, we may assume that $s=2$. Again by some appropriate formal elementary column transformations, we can translate the entries of the third row of $\boldsymbol{Z}'$ into 0 except for the $(3,r+2)$-th entry.

Note that $\operatorname{rank}(\boldsymbol{Z}')=r+1$. Repeat the above process, we get a matrix

$$\boldsymbol{Z}''=(\boldsymbol{Z}_1'',\boldsymbol{Z}_2'',\cdots,\boldsymbol{Z}_r'')$$

such that

$$\boldsymbol{Z}_u''=\begin{pmatrix}0&x_u\boldsymbol{e}_u^{\mathrm{T}}\\-x_u\boldsymbol{e}_u&\boldsymbol{O}\end{pmatrix},\quad 0\ne x_u\in\mathbb{F}_p,\ 1\le u\le r.$$

Let $\boldsymbol{Y}=\boldsymbol{Z}''\circ\boldsymbol{D}=(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\cdots,\boldsymbol{Y}_r)$ with

$$\boldsymbol{D}=\begin{pmatrix}x_1^{-1}&0&\cdots&0\\0&x_2^{-1}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&x_r^{-1}\end{pmatrix}.$$

Then

$$\boldsymbol{Y}_u = \begin{pmatrix} 0 & \boldsymbol{e}_u^{\mathrm{T}} \\ -\boldsymbol{e}_u & \boldsymbol{O} \end{pmatrix}, \quad 1 \le u \le r.$$

Clearly, $\boldsymbol{Y}$ is isoclinic to $\boldsymbol{Z}$. Therefore G is isoclinic to a special p-group H with the commutator matrix $\boldsymbol{Y}$, as required.

($\Leftarrow$) Let H be a special p-group such that

$$\overline{H} = H/H' = \langle \overline{b}_1 \rangle \times \langle \overline{b}_2 \rangle \times \cdots \times \langle \overline{b}_{r+1} \rangle \quad \text{and} \quad H' = \langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_r \rangle$$

with the commutator matrix given in the theorem. Then $H = \langle b_1, b_2, \cdots, b_{r+1} \rangle$ and

$$[b_i, b_j] = 1, \quad 2 \le i, j \le r+1 \quad \text{and} \quad [b_k, s_u] = [s_l,\ s_u] = 1, \quad 1 \le k \le r+1,\ 1 \le u, l \le r.$$

Let $\overline{A} = \langle \overline{b}_2, \overline{b}_3, \cdots, \overline{b}_{r+1} \rangle$. Then $\overline{A}$ is an abelian maximal subgroup of $\overline{H}$. By Correspondence Theorem, the preimage A of $\overline{A}$ under the natural homomorphism $H \to H/H'$ is also a maximal subgroup of H. It is clear that A is abelian. Hence H has an abelian maximal subgroup. Since G is isoclinic to H, it follows that G has an abelian maximal subgroup, as required.

By Theorem 4.2, Lemma 4.1 and Proposition 3.2 (1), we immediately get the following corollary.

Corollary 4.1 *Let G be a finite special p-group. Then G has an abelian maximal subgroup* (*of course an $\mathcal{MC}$-group*) *if and only if there exists a positive integer r such that G is isoclinic to the following group*:

$$\langle a_1, a_2, \cdots, a_{r+1}; t_1, t_2, \cdots, t_r \mid a_i^p = t_u^p = 1, [a_1, a_{k+1}] = t_k, [a_j, a_v] = [a_i, t_u] = [t_l, t_u] = 1,$$
$$i = 1, 2, \cdots, r+1; j, v = 2, 3, \cdots, r+1; u, l, k = 1, 2, \cdots, r \rangle.$$

In particular, for any prime number p and positive integer r, there exist finite special p-groups G having an abelian maximal subgroup such that $|G'| = p^r$, all these groups are isoclinic.

The following theorem gives a classification of finite special $\mathcal{MC}$-p-groups G with $|G/G'| = p^3$.

Theorem 4.3 *Let G be a special p-group with $|G/G'| = p^3$. Then G is an $\mathcal{MC}$-group if and only if $|G'| = p^2$ or p^3, and G is isoclinic to the group* (1) *if $|G'| = p^2$, G is isoclinic to the group* (2) *if $|G'| = p^3$, where*

(1)

$$\langle a_1, a_2, a_3; t_1, t_2 \mid a_i^p = t_u^p = 1, [a_1, a_2] = t_1, [a_1, a_3] = t_2, [a_2, a_3] = [a_i, t_u] = 1,$$
$$[t_l, t_u] = 1, i = 1, 2, 3; u, l = 1, 2 \rangle,$$

with the commutator matrix

$$\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right);$$

(2)

$$\langle a_1,a_2,a_3;t_1,t_2,t_3 \mid a_i^p=t_u^p=1,[a_1,a_2]=t_1,[a_1,a_3]=t_2,[a_2,a_3]=t_3,[a_i,t_u]=1,$$
$$[t_l,t_u]=1, i=1,2,3; u,l=1,2,3\rangle,$$

with the commutator matrix

$$\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2,\boldsymbol{Z}_3)=\left(\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\right).$$

Proof (⇒) Let G be a special $\mathcal{MC}$-p-group with $|G/G'|=p^3$ and $|G'|=p^r$. Then $r\geq 2$ by Lemma 4.2. Let $\boldsymbol{X}=(\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_r)$ with skew-symmetric matrix $\boldsymbol{X}_u\in\mathbb{F}_p^{3\times 3}$ for $u=1,2,\cdots,r$ be a commutator matrix of G. Then $\operatorname{rank}(\boldsymbol{X}_u)=2$ for $u=1,2,\cdots,r$. So by Lemma 2.1 there exists a non-singular matrix $\boldsymbol{A}\in\mathbb{F}_p^{3\times 3}$ such that

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{A}=\boldsymbol{Z}_1 \quad\text{with}\quad \boldsymbol{Z}_1=\begin{pmatrix}\boldsymbol{J}&\boldsymbol{0}\\\boldsymbol{0}&0\end{pmatrix}, \boldsymbol{J}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Let $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}_u\boldsymbol{A}=\boldsymbol{Z}_u$, $u=1,2,\cdots,r$ and $\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r)$. Then every $\boldsymbol{Z}_u$ is skew-symmetric. By Remark 4.1 (1), without loss of generality, we assume that

$$\boldsymbol{Z}_s=\begin{pmatrix}\boldsymbol{O}&\boldsymbol{\eta}_s\\-\boldsymbol{\eta}_s^{\mathrm{T}}&0\end{pmatrix}\quad\text{with}\quad\boldsymbol{\eta}_s=\begin{pmatrix}y_s\\z_s\end{pmatrix},\ s=2,\cdots,r.$$

Thus $\boldsymbol{Z}_2\neq\boldsymbol{O}$. There are two cases.

Case 1 $y_2\neq 0$. Take

$$\boldsymbol{D}=\begin{pmatrix}\boldsymbol{D}_1&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{E}_{r-2}\end{pmatrix}\quad\text{with}\quad\boldsymbol{D}_1=\begin{pmatrix}1&0\\0&y_2^{-1}\end{pmatrix}\text{ and }\boldsymbol{C}=\begin{pmatrix}1&-y_2^{-1}z_2&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Let $\boldsymbol{Z}'=(\boldsymbol{C}^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{C})\circ\boldsymbol{D}$. Then

$$\boldsymbol{Z}'=(\boldsymbol{Z}_1,\boldsymbol{Z}_2',\boldsymbol{Z}_3'\cdots,\boldsymbol{Z}_r'),$$

where

$$\boldsymbol{Z}_2'=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad\boldsymbol{Z}_s'=\begin{pmatrix}\boldsymbol{O}&\boldsymbol{\eta}_s'\\-\boldsymbol{\eta}_s'^{T}&0\end{pmatrix}\quad\text{with}\quad\boldsymbol{\eta}_s'=\begin{pmatrix}y_s'\\z_s'\end{pmatrix},\ 3\leq s\leq r.$$

If $r=2$, then $\boldsymbol{X}$ is isoclinic to the matrix $\boldsymbol{Z}'=(\boldsymbol{Z}_1,\boldsymbol{Z}_2')$, and so G is isoclinic to the group (1) listed in the theorem.

If $r\geq 3$, by Remark 4.1, we assume that $y_s'=0$ for $s\geq 3$. Then $z_3'\neq 0$. Let $\boldsymbol{Z}''=\boldsymbol{Z}'\circ\boldsymbol{D}$ with

$$\boldsymbol{D}=\begin{pmatrix}\boldsymbol{E}_2&\boldsymbol{0}&\boldsymbol{O}\\\boldsymbol{0}&z_3^{-1}&\boldsymbol{0}\\\boldsymbol{O}&\boldsymbol{0}&\boldsymbol{E}_{r-3}\end{pmatrix}.$$

Then

$$\boldsymbol{Z}'' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2', \boldsymbol{Z}_3'', \cdots, \boldsymbol{Z}_r'')$$

with

$$\boldsymbol{Z}_3'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{Z}_s'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_s'' \\ 0 & -z_s'' & 0 \end{pmatrix}, \quad 4 \leq s \leq r.$$

If $r \geq 4$, then $\boldsymbol{Z}''$ should be isoclinic a matrix of the form

$$(\boldsymbol{Z}_1, \boldsymbol{Z}_2', \boldsymbol{Z}_3'', \boldsymbol{O}, \cdots),$$

it contradicts Remark 4.1 (2). Thus $r = 3$ and $\boldsymbol{X}$ is isoclinic to the matrix $\boldsymbol{Z}'' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2', \boldsymbol{Z}_3'')$. Therefore G is isoclinic to the group (2) listed in the theorem.

Case 2 $y_2 = 0$ and $z_2 \neq 0$. Let

$$\boldsymbol{D} = \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_{r-1} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\boldsymbol{Z}' = (\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{C}) \circ \boldsymbol{D}$. Then

$$\boldsymbol{Z}' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2', \cdots, \boldsymbol{Z}_r') \quad \text{with } \boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & 0 \\ -z_2 & 0 & 0 \end{pmatrix}.$$

This is Case 1.

($\Leftarrow$) Let H be the group (1) listed in the theorem. Obviously, $\operatorname{rank}(\boldsymbol{Z}) = 3$. Then H is special by Proposition 4.1. It follows from Theorem 4.2 that H has an abelian maximal subgroup. Thus H is a special $\mathcal{MC}$-p-group by Proposition 3.2 (1). Therefore G is also a special $\mathcal{MC}$-p-group.

Let H be the group (2) listed in the theorem. Obviously, $\operatorname{rank}(\boldsymbol{Z}) = 3$. Then H is special by Proposition 4.1. Let $\boldsymbol{k} = (x_1, x_2, x_3)^{\mathrm{T}}$, $\boldsymbol{l} = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{F}_p^3$. Then

$$\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l} = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$$

and

$$(\boldsymbol{Z} \diamond \boldsymbol{k}, \boldsymbol{Z} \diamond \boldsymbol{l}) = \begin{pmatrix} x_2 & x_3 & 0 & y_2 & y_3 & 0 \\ -x_1 & 0 & x_3 & -y_1 & 0 & y_3 \\ 0 & -x_1 & -x_2 & 0 & -y_1 & -y_2 \end{pmatrix}.$$

If $\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l} \neq \boldsymbol{0}$, without loss of generality, we may assume $x_1y_2 - x_2y_1 \neq 0$. Then $(x_1, x_2) \neq \boldsymbol{0}$ and $(y_1, y_2) \neq \boldsymbol{0}$. Without loss of generality, assume $x_1 \neq 0$.

If $y_2 \neq 0$, it is easy to verify that

$$\operatorname{rank}\begin{pmatrix} x_2 & x_3 & y_2 \\ -x_1 & 0 & -y_1 \\ 0 & -x_1 & 0 \end{pmatrix} = 3.$$

Hence $\operatorname{rank}(\boldsymbol{Z}\diamond\boldsymbol{k},\boldsymbol{Z}\diamond\boldsymbol{l})=3$.

If $y_2=0$, then $x_2\neq 0$ and $y_1\neq 0$. Thus

$$\operatorname{rank}\begin{pmatrix} x_2 & 0 & y_2\\ -x_1 & x_3 & -y_1\\ 0 & -x_2 & 0\end{pmatrix}=3$$

and so $\operatorname{rank}(\boldsymbol{Z}\diamond\boldsymbol{k},\boldsymbol{Z}\diamond\boldsymbol{l})=3$. Whence the commutator matrix of H satisfies the condition of Theorem 4.1. It follows from Proposition 4.1 and Theorem 4.1 that H is a special $\mathcal{MC}$-group. Therefore G is a special $\mathcal{MC}$-p-group.

Proposition 4.3 *Let G be a special p-group with $|G'|=p^r$ and $|G/G'|=p^{2r}$. If G is an $\mathcal{MC}$-group, then the commutator matrix of G is isoclinic to a matrix of the form*

$$\boldsymbol{Z}=(\boldsymbol{J}_{2r},\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r),$$

where $\boldsymbol{J}_{2r},\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r\in\mathbb{F}_p^{2r\times 2r}$ are skew-symmetric, $\boldsymbol{J}_{2r}=\begin{pmatrix} 0 & 1 & \cdots & 0 & 0\\ -1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & -1 & 0\end{pmatrix}$, *such that*

(1) *$\boldsymbol{J}_{2r},\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r$ are linearly independent,*

(2) $\operatorname{rank}(\boldsymbol{Z}_s)=\operatorname{rank}(\boldsymbol{J}_{2r})=2r$, $s=2,\cdots,r$,

(3) $\operatorname{rank}(x_1\boldsymbol{J}_{2r}+x_2\boldsymbol{Z}_2+\cdots+x_r\boldsymbol{Z}_r)=2r$ *for any* $(x_1,x_2,\cdots,x_r)^{\mathrm{T}}\in\mathbb{F}_p^r\setminus\{\boldsymbol{0}\}$.

Proof Let G be a special $\mathcal{MC}$-p-group with $|G'|=p^r$ and $|G/G'|=p^{2r}$. Assume $\boldsymbol{X}=(\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_r)$ is a commutator matrix of G. By Theorem 4.2, our result is true for $r=1$. So we assume $r\geq 2$. Since $\boldsymbol{X}_1\neq\boldsymbol{O}$, $\operatorname{rank}(\boldsymbol{X}_1)=2\nu$ for some ν with $1\leq\nu\leq r$. By Lemma 2.1, there exists a non-singular matrix $\boldsymbol{A}\in\mathbb{F}_p^{2r\times 2r}$ such that

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{A}=\boldsymbol{Z}_1\quad\text{with }\boldsymbol{Z}_1=\begin{pmatrix}\boldsymbol{J}_{2\nu} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}.$$

Let $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}_u\boldsymbol{A}=\boldsymbol{Z}_u$, $u=1,2,\cdots,r$ and $\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r)$. Then $\boldsymbol{Z}=\boldsymbol{A}^{\mathrm{T}}\diamond\boldsymbol{X}\diamond\boldsymbol{A}$ and every $\boldsymbol{Z}_u$ is skew-symmetric.

If the (i,j)-th entry $\boldsymbol{Z}_s[i,j]$ of $\boldsymbol{Z}_s$ does not equal 0 for some $s\in\{2,\cdots,r\}$, $i\in\{1,\cdots,2r\}$, $j\in\{2\nu+1,\cdots,2r\}$, let $\boldsymbol{e}_i$, $\boldsymbol{e}_j\in\mathbb{F}_p^{2r}$ be the elementary unit column vectors. Then $\boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{Z}_s\boldsymbol{e}_j=\boldsymbol{Z}_s[i,j]\neq 0$ and so $\boldsymbol{e}_i^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{e}_j\neq\boldsymbol{0}$. On the other hand, $\boldsymbol{Z}_1\boldsymbol{e}_j=\boldsymbol{0}$, and so $\operatorname{rank}(\boldsymbol{Z}\diamond\boldsymbol{e}_i,\boldsymbol{Z}\diamond\boldsymbol{e}_j)<2r$. It contradicts Theorem 4.1. Therefore the (i,j)-th entry of $\boldsymbol{Z}_s$ equals 0 for $s=2,\cdots,r, i=1,\cdots,2r$, $j=2\nu+1,\cdots,2r$. Similarly argument, we can prove that the (i,j)-th entry of $\boldsymbol{Z}_s$ equals 0 for $s=2,\cdots,r$, $i=2\nu+1,\cdots,2r$, $j=1,\cdots,2r$. Thus

$$\boldsymbol{Z}_s=\begin{pmatrix}\boldsymbol{Y}_s & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}\quad\text{for some skew-symmetric matrix }\boldsymbol{Y}_s\in\mathbb{F}_p^{2\nu\times 2\nu},\ s=2,\cdots,r.$$

Hence $\operatorname{rank}(\boldsymbol{Z})=2\nu$. Note that $\boldsymbol{X}$ and $\boldsymbol{Z}$ are isoclinic. Then $\operatorname{rank}(\boldsymbol{Z})=\operatorname{rank}(\boldsymbol{X})=2r$ by Proposition 4.1. So $\nu=r$ and $\boldsymbol{X}$ is isoclinic to $\boldsymbol{Z}=(\boldsymbol{J}_{2r},\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r)$. That $\boldsymbol{J}_{2r},\boldsymbol{Z}_2,\cdots,\boldsymbol{Z}_r$ are linearly independent follows from Remark 4.1 (2).

Suppose that $\operatorname{rank}(\boldsymbol{Z}_s) = 2k < 2r$ for some $s \in \{2, \cdots, r\}$. Without loss of generality, assume that $s = 2$. There is a non-singular matrix $\boldsymbol{B} \in \mathbb{F}_p^{2r\times 2r}$ such that

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{B} = \begin{pmatrix} \boldsymbol{J}_{2k} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}.$$

Let $\boldsymbol{Y} = \boldsymbol{B}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{B}$. Then $\boldsymbol{Y} = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{J}_{2r}\boldsymbol{B}, \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{B}, \cdots, \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_r\boldsymbol{B})$ is isoclinic to $\boldsymbol{X}$. Let $i = 2k+1$. Since $\operatorname{rank}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{J}_{2r}\boldsymbol{B}) = 2r$, there exists a $j \in \{1, \cdots, 2r\}$ such that $(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{J}_{2r}\boldsymbol{B})[i,j] \neq 0$. Then $\boldsymbol{e}_i^{\mathrm{T}} \diamond (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{J}_{2r}\boldsymbol{B}) \diamond \boldsymbol{e}_j \neq 0$ and so $\boldsymbol{e}_i^{\mathrm{T}} \diamond \boldsymbol{Y} \diamond \boldsymbol{e}_j \neq \boldsymbol{0}$. But $(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_2\boldsymbol{B})\boldsymbol{e}_i = \boldsymbol{0}$ and $\operatorname{rank}(\boldsymbol{Y} \diamond \boldsymbol{e}_i, \boldsymbol{Y} \diamond \boldsymbol{e}_j) < 2r$. It contradicts Theorem 4.1. Therefore $\operatorname{rank}(\boldsymbol{Z}_s) = 2r$ for $s \in \{2, \cdots, r\}$.

That (3) is true follows from (2) and Remark 4.1 (2).

Remark 4.3 (1) We do not know whether the converse part of Proposition 4.3 is true or not.

(2) We also do not know whether the matrix satisfying the condition of Proposition 4.3 exists for any prime number p and any positive integer $r \geq 2$.

The matrix $\boldsymbol{Z} = (\boldsymbol{J}_6, \boldsymbol{Z}_2, \boldsymbol{Z}_3)$ with

$$\boldsymbol{Z}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

satisfies the condition of Proposition 4.3 with respect to $p = 2$ and $r = 3$. It is a routine matter to verify that the special 2-group with the commutator matrix $\boldsymbol{Z}$ is an $\mathcal{MC}$-group.

But the matrix $\boldsymbol{Z} = (\boldsymbol{J}_6, \boldsymbol{Z}_2, \boldsymbol{Z}_3)$ above does not satisfy the condition of Proposition 4.3 with respect to $p = 3$ and $r = 3$ since $\operatorname{rank}(\boldsymbol{J}_6 + \boldsymbol{Z}_2 + \boldsymbol{Z}_3) = 4 < 6$.

The following two propositions give a classification of special $\mathcal{MC}$-p-group G such that $|G'| = p^2$ with $p = 3, 2$, respectively.

Proposition 4.4 *Let G be a special 3-group with $|G'| = 3^2$. Then G is an $\mathcal{MC}$-group if and only if G is isoclinic to the one of the following groups*:

(1)

$$\begin{aligned} &\langle a_1, a_2, a_3; t_1, t_2 \mid a_i^3 = t_u^3 = 1, [a_1, a_2] = t_1, [a_1, a_3] = t_2, [a_2, a_3] = 1, \\ &[a_i, t_u] = [t_l, t_u] = 1, i = 1, 2, 3; u, l = 1, 2\rangle, \end{aligned}$$

with the commutator matrix

$$\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right);$$

(2)

$$\langle a_1, a_2, a_3, a_4; t_1, t_2 \mid a_i^3 = t_u^3 = 1, [a_1, a_2] = [a_3, a_4] = t_1, [a_1, a_4] = [a_2, a_3] = t_2,$$

$$[a_1,a_3]=[a_2,a_4]=[a_i,t_u]=[t_l,t_u]=1, i=1,2,3,4; u,l=1,2\rangle,$$

with the commutator matrix

$$\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)=\left(\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\right).$$

Proof (⇒) Since G is a special $\mathcal{MC}$-3-group with $|G'|=3^2$, by Lemma 4.2, $3^3\leq|G/G'|\leq 3^4$. If $|G/G'|=3^3$, then by Theorem 4.3, we can get the group (1) listed in the proposition. If $|G/G'|=3^4$, then by Proposition 4.3, G is isoclinic to the special 3-group whose commutator matrix $\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)$ with

$$\boldsymbol{Z}_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\quad\text{and}\quad\boldsymbol{Z}_2\in\mathbb{F}_3^{4\times4}\text{ is a skew-symmetric matrix}$$

satisfies the condition of Proposition 4.3. By Remark 4.1 (1), without loss of generality, we assume that

$$\boldsymbol{Z}_2=\begin{pmatrix}0&0&a&b\\0&0&c&d\\-a&-c&0&e\\-b&-d&-e&0\end{pmatrix}\quad\text{with } a,b,c,d,e\in\mathbb{F}_3.$$

Case 1 $a=0$. Let $\boldsymbol{e}_1,\boldsymbol{e}_2\in\mathbb{F}_3^4$ be the elementary unit volumn vectors. Then

$$\boldsymbol{e}_2^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{e}_1=(-1,0)\neq\boldsymbol{0}\quad\text{and}\quad(\boldsymbol{Z}\diamond\boldsymbol{e}_2,\boldsymbol{Z}\diamond\boldsymbol{e}_1)=\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&-c&0&0\\0&-d&0&-b\end{pmatrix}.$$

By Theorem 4.1, $\operatorname{rank}(\boldsymbol{Z}\diamond\boldsymbol{e}_2,\boldsymbol{Z}\diamond\boldsymbol{e}_1)=4$ implies that $b,c\neq0$.

Let

$$\boldsymbol{C}=\begin{pmatrix}1&-b^{-1}d&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and $\boldsymbol{Z}'=\boldsymbol{C}^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{C}$. Then

$$\boldsymbol{Z}'=(\boldsymbol{Z}_1,\boldsymbol{Z}_2')$$

with

$$\boldsymbol{Z}_2'=\begin{pmatrix}0&0&0&b\\0&0&c&0\\0&-c&0&e\\-b&0&-e&0\end{pmatrix}.$$

By Remark 4.1 (1), without loss of generality, we assume that $b = 1$, that is,

$$\boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \\ 0 & -c & 0 & e \\ -1 & 0 & -e & 0 \end{pmatrix}.$$

Then

$$\boldsymbol{Z}_1 + \boldsymbol{Z}_2' = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & c & 0 \\ 0 & -c & 0 & 1+e \\ -1 & 0 & -1-e & 0 \end{pmatrix} \quad \text{and} \quad 2\boldsymbol{Z}_1 + \boldsymbol{Z}_2' = \begin{pmatrix} 0 & 2 & 0 & 1 \\ -2 & 0 & c & 0 \\ 0 & -c & 0 & 2+e \\ -1 & 0 & -2-e & 0 \end{pmatrix}.$$

Now rank $(\boldsymbol{Z}_1 + \boldsymbol{Z}_2') = 4$ implies $1 + e + c \neq 0$, and $\operatorname{rank}(2\boldsymbol{Z}_1 + \boldsymbol{Z}_2') = 4$ implies $1 + 2e + c \neq 0$.

If $e = 0$, then $c \neq -1$ and so $c = 1$, hence

$$\boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore G is isoclinic to the group (2) listed in the proposition.

If $e \neq 0$, whence $c = -1$. If $e = 1$, then

$$\boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

Let

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\boldsymbol{Z}'' = (\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Z}' \diamond \boldsymbol{C}) \circ \boldsymbol{D}$. Then

$$\boldsymbol{Z}'' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2'') \quad \text{with} \quad \boldsymbol{Z}_2'' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore G is isoclinic to the group (2) listed in the proposition.

If $e = -1$, then

$$\boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

Let

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\boldsymbol{Z}'' = (\boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Z}' \diamond \boldsymbol{C}) \circ \boldsymbol{D}$. Then

$$\boldsymbol{Z}'' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2'') \quad \text{with} \quad \boldsymbol{Z}_2'' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

Similar argument as above, we get that G is isoclinic to the group (2) listed in the proposition.

Case 2 $a \neq 0$. Let

$$\boldsymbol{C} = \begin{pmatrix} a^{-1}c & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\boldsymbol{Z}' = \boldsymbol{C}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{C}$. Then

$$\boldsymbol{Z}' = (\boldsymbol{Z}_1, \boldsymbol{Z}_2'),$$

where

$$\boldsymbol{Z}_2' = \begin{pmatrix} 0 & 0 & 0 & b' \\ 0 & 0 & c' & d' \\ 0 & -c' & 0 & e' \\ -b' & -d' & -e' & 0 \end{pmatrix} \quad \text{with } b', c', d', e' \in \mathbb{F}_3.$$

This is Case 1.

($\Leftarrow$) Let H be the group (1) listed in the proposition. Obviously, rank $(\boldsymbol{Z}) = 3$. Then H is special by Proposition 4.1. It follows from Theorem 4.2 that H has an abelian maximal subgroup. Thus H is a special $\mathcal{MC}$-3-group by Proposition 3.2 (1). Therefore G is also a special $\mathcal{MC}$-3-group.

Let H be the group (2) listed in the proposition. Obviously, rank$(\boldsymbol{Z}) = 4$. Then H is special by Proposition 4.1. Let $\boldsymbol{k} = (x_1, x_2, x_3, x_4)^{\mathrm{T}}$, $\boldsymbol{l} = (y_1, y_2, y_3, y_4)^{\mathrm{T}} \in \mathbb{F}_3^4$. Then

$$\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l} = (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3, x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)$$

and

$$(\boldsymbol{Z} \diamond \boldsymbol{k}, \boldsymbol{Z} \diamond \boldsymbol{l}) = \begin{pmatrix} x_2 & x_4 & y_2 & y_4 \\ -x_1 & x_3 & -y_1 & y_3 \\ x_4 & -x_2 & y_4 & -y_2 \\ -x_3 & -x_1 & -y_3 & -y_1 \end{pmatrix}.$$

Suppose that rank $(\boldsymbol{Z} \diamond \boldsymbol{k}, \boldsymbol{Z} \diamond \boldsymbol{l}) < 4$. Then the column vectors of $(\boldsymbol{Z} \diamond \boldsymbol{k}, \boldsymbol{Z} \diamond \boldsymbol{l})$ are linearly dependent, hence there exist $k_1, k_2, k_3, k_4 \in \mathbb{F}_3$ with $(k_1, k_2, k_3, k_4) \neq \boldsymbol{0}$ such that

$$\begin{cases} k_1x_2 + k_2x_4 + k_3y_2 + k_4y_4 = 0, \\ -k_1x_1 + k_2x_3 - k_3y_1 + k_4y_3 = 0, \\ k_1x_4 - k_2x_2 + k_3y_4 - k_4y_2 = 0, \\ -k_1x_3 - k_2x_1 - k_3y_3 - k_4y_1 = 0. \end{cases}$$

Note that $\boldsymbol{l}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{k} = -\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l}$. Only the case of $k_1 \neq 0$ or $k_2 \neq 0$ needs to be considered.

If $k_1 \neq 0$, without loss of generality, we may assume that $k_1 = 1$. Then

$$\begin{cases} x_2 = -k_2x_4 - k_3y_2 - k_4y_4, \\ x_1 = k_2x_3 - k_3y_1 + k_4y_3, \\ x_4 = k_2x_2 - k_3y_4 + k_4y_2, \\ x_3 = -k_2x_1 - k_3y_3 - k_4y_1. \end{cases}$$

It follows that

$$x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 = -k_2(x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)$$

and

$$x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2 = k_2(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3).$$

That is

$$(1 + k_2^2)(x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2) = 0.$$

Then $x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2 = 0$ and so $x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 = 0$. Hence $\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l} = \boldsymbol{0}$.

If $k_2 \neq 0$, without loss of generality, we may assume that $k_2 = 1$. Then

$$\begin{cases} x_4 = -k_1x_2 - k_3y_2 - k_4y_4, \\ x_3 = k_1x_1 + k_3y_1 - k_4y_3, \\ x_2 = k_1x_4 + k_3y_4 - k_4y_2, \\ x_1 = -k_1x_3 - k_3y_3 - k_4y_1. \end{cases}$$

It follows that

$$x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 = k_1(x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)$$

and

$$x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2 = -k_1(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3).$$

Similar argument as above, we get $\boldsymbol{k}^{\mathrm{T}} \diamond \boldsymbol{Z} \diamond \boldsymbol{l} = \boldsymbol{0}$.

Summing up, we have proved that the commutator matrix of H satisfies the condition of Theorem 4.1. It follows that H is a special $\mathcal{MC}$-3-group. Therefore G is also a special $\mathcal{MC}$-3-group.

Proposition 4.5 *Let G be a special 2-group with $|G'| = 2^2$. Then G is an $\mathcal{MC}$-group if and only if G is isoclinic to one of the following groups* :

(1)

$$\begin{aligned} &\langle a_1, a_2, a_3; t_1, t_2 \mid a_i^2 = t_u^2 = 1, [a_1, a_2] = t_1, [a_1, a_3] = t_2, [a_2, a_3] = 1, \\ &[a_i, t_u] = [t_l, t_u] = 1, i = 1, 2, 3; u, l = 1, 2\rangle \end{aligned}$$

with the commutator matrix

$$\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)=\left(\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\right);$$

(2)

$$\langle a_1,a_2,a_3,a_4;t_1,t_2 \mid\ a_i^2=t_u^2=1,[a_1,a_2]=t_1,[a_1,a_4]=[a_2,a_3]=t_2,[a_3,a_4]=t_1t_2,$$
$$[a_1,a_3]=[a_2,a_4]=[a_i,t_u]=[t_l,t_u]=1,i=1,2,3,4;u,l=1,2\rangle$$

with the commutator matrix

$$\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)=\left(\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}\right).$$

Proof (⇒) Since G is a special $\mathcal{MC}$-2-group with $|G'|=2^2$, by Lemma 4.2, $2^3\le|G/G'|\le 2^4$. If $|G/G'|=2^3$, then by Theorem 4.3, we can get the group (1) listed in the proposition. If $|G/G'|=2^4$, then by Proposition 4.3, G is isoclinic to the special 2-group whose commutator matrix $\boldsymbol{Z}=(\boldsymbol{Z}_1,\boldsymbol{Z}_2)$ with

$$\boldsymbol{Z}_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad\text{and}\quad \boldsymbol{Z}_2\in\mathbb{F}_2^{4\times 4}\ \text{is a skew-symmetric matrix}$$

satisfies the condition of Proposition 4.3. By Remark 4.1 (1), without loss of generality, we assume that

$$\boldsymbol{Z}_2=\begin{pmatrix}0&0&a&b\\0&0&c&d\\a&c&0&e\\b&d&e&0\end{pmatrix}\quad\text{with}\quad a,b,c,d,e\in\mathbb{F}_2.$$

Case 1 $a=0$. Let $\boldsymbol{e}_1,\boldsymbol{e}_2\in\mathbb{F}_2^4$ be the elementary unit volumn vectors. Then

$$\boldsymbol{e}_2^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{e}_1=(1,0)\neq\boldsymbol{0}\quad\text{and}\quad(\boldsymbol{Z}\diamond\boldsymbol{e}_2,\boldsymbol{Z}\diamond\boldsymbol{e}_1)=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&c&0&0\\0&d&0&b\end{pmatrix}.$$

By Theorem 4.1, $\operatorname{rank}(\boldsymbol{Z}\diamond\boldsymbol{e}_2,\boldsymbol{Z}\diamond\boldsymbol{e}_1)=4$ implies that $b,c\neq0$. So $b=c=1$.

If $d=0$, then

$$\boldsymbol{Z}_1+\boldsymbol{Z}_2=\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1+e\\1&0&1+e&0\end{pmatrix}.$$

Now $\operatorname{rank}(\boldsymbol{Z}_1+\boldsymbol{Z}_2)=4$ implies $e=1$. Hence

$$\boldsymbol{Z}_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}.$$

So G is isoclinic to the group (2) listed in the proposition.

If $d=1$, let

$$\boldsymbol{C}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

and $\boldsymbol{Z}'=\boldsymbol{C}^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{C}$. Then

$$\boldsymbol{Z}'=(\boldsymbol{Z}_1,\boldsymbol{Z}_2')$$

with

$$\boldsymbol{Z}_2'=\left(\begin{array}{cccc}0&0&0&1\\0&0&1&0\\0&1&0&e\\1&0&e&0\end{array}\right).$$

Similar argument as above, we get $e=1$ and G is isoclinic to the group (2) listed in the proposition.

Case 2 $a=1$. Let

$$\boldsymbol{C}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix},\quad \text{if } b=1;\quad \boldsymbol{C}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad \text{if } b=0$$

and $\boldsymbol{Z}'=\boldsymbol{C}^{\mathrm{T}}\diamond\boldsymbol{Z}\diamond\boldsymbol{C}$. Then

$$\boldsymbol{Z}'=(\boldsymbol{Z}_1,\boldsymbol{Z}_2'),$$

where

$$\boldsymbol{Z}_2'=\begin{pmatrix}0&0&0&b'\\0&0&c'&d'\\0&c'&0&e'\\b'&d'&e'&0\end{pmatrix}\quad \text{with } b',c',d',e'\in\mathbb{F}_2.$$

This is Case 1.

($\Leftarrow$) Similar arguments as in the proof of the sufficiency of Proposition 4.4, we can prove that G is a special $\mathcal{MC}$-2-group.

Remark 4.4 By Theorem 4.3 and Propositions 4.4–4.5, we know that the bound of $|G/G'|$ for special $\mathcal{MC}$-p-groups given in Lemma 4.2 is sharp, we do not know the distribution of $|G/G'|$ between p^3 and p^{2r}.

Acknowledgements The authors would like to express their gratitude to the referees for their valuable comments and suggestions which improve the paper, and to Professor Qinhai Zhang for his helpful suggestions.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] An, L. J., Qu, H. P. and Zhang, J. Q., Finite 2-groups all of whose non-abelian subgroups are self-centralizing, *J. Algebra Appl.*, **20**, 2021, 2150219, 18 pp.

[2] Berkovich, Y., Groups of Prime Power Order, **1**, Walter de Gruyter, Berlin, New York, 2008.

[3] Berkovich, Y. and Janko, Z., Groups of Prime Power Order, **5**, Walter de Gruyter, Berlin, Boston, 2016.

[4] Hall, P., The classification of prime-power groups, *J. Reine Angew. Math.*, **182**, 1940, 130–141.

[5] Huppert, B., Endliche Gruppen, I, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[6] Mann, A., Groups with small abelian subgroups, *Arch. Math.* (*Basel*), **50**, 1988, 210–213.

[7] Robinson, D. J. S., A Course in the Theory of Groups, Second edition, Springer-Verlag, New York, 1996.

[8] Schmidt, R., Zentralisatorverbände endlicher Gruppen, *Rend. Sem. Mat. Univ. Padova.*, **44**, 1970, 97–131.

[9] Schmidt, R., Subgroup Lattices of Groups, Walter de Gruyter, Berlin, New York, 1994.

[10] Tuan, H. F., A theorem about p-groups with abelian subgroups of index p, *Acad. Sinica Science Record*, **3**, 1950, 17–23.

[11] Wan, Z. X., Geometry of Classical Groups over Finite Fields, Second edition, Science Press, Beijing, 2002.

[12] Wang, J. and Guo, X. Y., Finite p-groups and centralizers of non-cyclic abelian subgroups, *Bull. Iranian Math. Soc.*, **43**, 2017, 171–192.

[13] Wang, L. F., Finite p-groups whose non-abelian subgroups have the same center, *J. Korean Math. Soc.*, **54**, 2017, 1109–1120.

[14] Wang, L. F. and Zhang, Q. H., Finite 2-groups whose non-abelian subgroups have the same center, *J. Group Theory*, **17**, 2014, 689–703.

[15] West, D. B., Introduction to Graph Theory (2nd ed.), Prentice Hall, Upper Saddle River, NJ, 2001.

[16] Zassenhaus, H. J., A group-theoretic proof of a theorem of Maclagan-Wedderburn, *Proc. Glasgow Math. Assoc.*, **1**, 1952, 53–63.

[17] Zhang, L. H., Wang, J. and Qu, H. P., Finite p-groups whose non-central cyclic subgroups have cyclic quotient groups in their centralizers, *Bull. Korean Math. Soc.*, **52**, 2015, 367–376.