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Classification of Certain Weakly Integral Fusion Categories*

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Abstract The author proves that braided fusion categories of Frobenius-Perron dimension $p^m q^n d$ or $p^2q^2r^2$ are weakly group-theoretical, where p, q, r are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. As an application, the author obtains that weakly integral braided fusion categories of Frobenius-Perron dimension less than 1800 are weakly group-theoretical, and weakly integral braided fusion categories of odd dimension less than 33075 are solvable. For the general case, the author proves that fusion categories (not necessarily braided) of Frobenius-Perron dimension 84 and 90 are either solvable or group-theoretical. Together with the results in the literature, this shows that every weakly integral fusion category of Frobenius-Perron dimension less than 120 is either solvable or group-theoretical. Thus the author completes the classification of all these fusion categories in terms of Morita equivalence.

Keywords Solvable fusion categories, Group-theoretical fusion categories, Weakly group-theoretical fusion categories, Frobenius property
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1 Introduction

A fusion category $\mathcal{C}$ is a k-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional vector spaces of morphisms and such that the unit object $\mathbf{1}$ is simple. The theory of fusion categories arises from many areas of mathematics and physics, such as semisimple Hopf algebras, quantum groups (see [1]), vertex operator algebras (see [4]) and topological quantum field theory (see [30]).

The notion of a weakly group-theoretical fusion category is introduced in [15]. By definition, a fusion category $\mathcal{C}$ is called weakly group-theoretical if it is Morita equivalent to a nilpotent fusion category. In particular, every weakly group-theoretical fusion category is weakly integral, i.e., it has integral Frobenius-Perron dimension. This fact motivates the conjecture that every weakly integral fusion category is weakly group-theoretical (see [15]). Some examples of weakly group-theoretical fusion categories were obtained in [5–8, 10, 15]. Natale also obtained some examples under the assumption of nondegeneracy (see [26]). In fact, all known weakly integral fusion categories are weakly group-theoretical.

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People's interest in weakly group-theoretical fusion categories also comes from the fact that they have the strong Frobenius property (see [15, Theorem 1.5]). That is, for every weakly group-theoretical fusion category $\mathcal{C}$ and every indecomposable $\mathcal{C}$-module category $\mathcal{M}$ and any simple object X in $\mathcal{M}$, the number $\frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(X)}$ is an algebraic integer, where FPdim stands for the Frobenius-Perron dimension. If we take $\mathcal{M} = \mathcal{C}$, then the strong Frobenius property of a fusion category implies the usual Frobenius property which was conjectured by Kaplansky in the setting of semisimple Hopf algebras (see [20]).

The class of solvable or group-theoretical fusion categories is a special case of weakly group-theoretical fusion categories. They are both interesting examples which come from finite group theory. Moreover, the latter one can be completely classified by finite groups and their cohomology (see [14]). So it is an interesting task to determine which class of weakly group-theoretical fusion categories is solvable or group-theoretical.

The present paper is devoted to extending the results obtained in [15] and discarding the assumption of nondegeneracy in [26]. Our first result is listed below.

Theorem 1.1 *Let $\mathcal{C}$ be a braided fusion category of Frobenius-Perron dimension $p^m q^n d$ or $p^2q^2r^2$, where p, q, r are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. Then $\mathcal{C}$ is weakly group-theoretical.*

Subsequently, all weakly integral braided fusion categories of dimension less than 1800 are weakly group-theoretical. In particular, weakly integral braided fusion categories of odd dimension less than 33075 are solvable.

Our second result is the following theorem.

Theorem 1.2 *Fusion categories* (*not necessarily braided*) *of Frobenius-Perron dimension* 84 *and* 90 *are solvable or group-theoretical.*

Our conclusion, together with results in the literature, shows that every weakly integral fusion category of Frobenius-Perron dimension less than 120 is either solvable or group-theoretical.

The paper is organized as follows. In Section 2, we recall some basic definitions and results which will be used throughout. In Section 3, we study braided fusion categories of Frobenius-Perron dimension $p^m q^n d$ and $p^2q^2r^2$. In Subsection 4.1, we study the existence of nontrivial symmetric categories in the Drinfeld center of a fusion category. Fusion categories of dimension 84 and 90 will be studied in Subsections 4.2 and 4.3, respectively.

Throughout this paper, we shall work over an algebraically closed field k of characteristic 0. We refer to [13] for the main notions about fusion categories.

2 Preliminaries

2.1 Frobenius-Perron dimensions

Let $\mathcal{C}$ be a fusion category and $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of simple objects in $\mathcal{C}$. Then $\mathrm{Irr}(\mathcal{C})$ is a basis of the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$. The Frobenius-Perron dimension $\operatorname{FPdim}(X)$ of $X \in \operatorname{Irr}(\mathcal{C})$ is defined as the largest eigenvalue of the matrix of left multiplication by X in the Grothendieck ring with respect to the basis. The Frobenius-Perron dimension of $\mathcal{C}$ is the number

$$\operatorname{FPdim}(\mathcal{C}) = \sum_{X \in \operatorname{Irr}(\mathcal{C})} (\operatorname{FPdim} X)^2.$$

A fusion category $\mathcal{C}$ is called weakly integral if $\operatorname{FPdim}(\mathcal{C})$ is an integer. A fusion category $\mathcal{C}$ is called integral if $\operatorname{FPdim}(X)$ is an integer for every $X \in \operatorname{Irr}(\mathcal{C})$.

A simple object $X \in \mathcal{C}$ is called invertible if $\operatorname{FPdim}(X) = 1$. A pointed fusion category is a fusion category $\mathcal{C}$ whose simple objects are all invertible. If $\mathcal{C}$ is a pointed fusion category, then $\mathcal{C}$ is equivalent to the category of G-graded vector spaces with associativity constraint given by a 3-cocycle $\omega \in H^3(G, k^\times)$, denoted by $\operatorname{Vec}_{G,\omega}$. This fact is originally due to Eilenberg and Mac Lane. See also [17] for details.

We use $\mathcal{C}_{pt}$ to denote the largest pointed fusion subcategory in $\mathcal{C}$. Let $G(\mathcal{C})$ be the group of isomorphism classes of invertible simple objects of a fusion category $\mathcal{C}$. Then $|G(\mathcal{C})| = \operatorname{FPdim}(\mathcal{C}_{pt})$.

Let Y be an object of $\mathcal{C}$ and write $Y = \sum_{X \in \operatorname{Irr}(\mathcal{C})} m(X, Y)X$, where $m(X, Y) \in \mathbb{Z}$. The integer $m(X, Y)$ is called the multiplicity of X in Y. By [28, Theorems 9–10], we have the following results which will be used in our computation.

Let $\mathcal{C}$ be an integral fusion category and X, Y, Z be objects of $\mathcal{C}$. Then we have $m(X, Y) = m(X^*, Y^*)$ and

$$m(X, Y \otimes Z) = m(Y^*, Z \otimes X^*) = m(Y, X \otimes Z^*).$$

Let g be an element in $G(\mathcal{C})$. Then we have $m(g, X \otimes Y) = 1$ if and only if $Y = X^* \otimes g$, otherwise it is 0. In particular, $m(g, X \otimes Y) = 0$ if $\operatorname{FPdim} X \neq \operatorname{FPdim} Y$. Let $X \in \operatorname{Irr}(\mathcal{C})$. Then for all $g \in G(\mathcal{C})$, $m(g, X \otimes X^*) > 0$ if and only if $m(g, X \otimes X^*) = 1$ if and only if $g \otimes X = X$. The set of isomorphism classes of such invertible objects will be denoted by $G[X]$. Thus $G[X]$ is a subgroup of $G(\mathcal{C})$ of order dividing $(\operatorname{FPdim} X)^2$. In particular, for all $X \in \operatorname{Irr}(\mathcal{C})$, we have a decomposition

$$X \otimes X^* = \bigoplus_{g \in G[X]} g \oplus \sum_{Y \in \operatorname{Irr}(\mathcal{C}) - G[X]} m(Y, X \otimes X^*)Y. \tag{2.1}$$

It is known that the group $G(\mathcal{C})$ acts on the set $\operatorname{Irr}(\mathcal{C})$ by left tensor multiplication. This action preserves Frobenius-Perron dimensions and, for $X \in \operatorname{Irr}(\mathcal{C})$, $G[X]$ is the stabilizer of X in $G(\mathcal{C})$.

In fact, these results and Theorem 2.1 below were established in the case where $\mathcal{C}$ is the category of finite-dimensional representations of a semisimple Hopf algebra. Because their proofs only make use of the properties of the Grothendieck ring, these proofs also work mutatis mutandis in the fusion category setting, and thus we omit their proof.

Let $1 = d_0, d_1, \cdots, d_s$, $s \geq 0$, be positive real numbers such that $1 = d_0 < d_1 < \cdots < d_s$, and $n_1, n_2, \cdots, n_s$ be positive integers. We shall say that $\mathcal{C}$ is of type $(d_0, n_0; d_1, n_1; \cdots; d_s, n_s)$ if, for all $i = 0, \cdots, s$, n_i is the number of the non-isomorphic simple objects of Frobenius-Perron dimension d_i.

The following theorem is a restatement of [28, Theorem 11] in the context of fusion categories.

Theorem 2.1 *Let* $\mathcal{C}$ *be an integral fusion category and* X *be a simple object of dimension* 2. *Then at least one of the following holds*:

(1) $G[X] \neq \mathbf{1}$.

(2) $\mathcal{C}$ *has a fusion subcategory* $\mathcal{D}$ *of type* $(1, 2; 2, 1; 3, 2)$, *such that* $X \notin \mathrm{Irr}(\mathcal{D})$, *and* $\mathcal{D}$ *has an invertible object* g *of order* 2 *such that* $g \otimes X \neq X$.

(3) $\mathcal{C}$ *has a fusion subcategory of type* $(1, 3; 3, 1)$ *or* $(1, 1; 3, 2; 4, 1; 5, 1)$.

In particular, if $G[X] = \mathbf{1}$, *then* $\mathcal{C}$ *contains a fusion subcategory of dimension* 12, 24 *or* 60.

2.2 Extensions of a fusion category

Let G be a finite group. A G-grading of $\mathcal{C}$ is a decomposition of $\mathcal{C}$ as a direct sum of full Abelian subcategories $\mathcal{C} = \bigoplus\limits_{g\in G} \mathcal{C}_g$, such that $(\mathcal{C}_g)^* = \mathcal{C}_{g^{-1}}$ and the tensor product $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$. The grading $\mathcal{C} = \bigoplus\limits_{g\in G} \mathcal{C}_g$ is called faithful if $\mathcal{C}_g \neq 0$ for all $g \in G$. We say that $\mathcal{C}$ is a G-extension of $\mathcal{D}$ if the grading is faithful and the trivial component is $\mathcal{D}$.

By [18, Theorem 3.10], every weakly integral fusion category is a G-extension of an integral fusion category $\mathcal{D}$, where G is an elementary Abelian 2-group.

Let $\mathcal{C}$ be a fusion category. The fusion subcategory of $\mathcal{C}$ generated by simple objects in $X \otimes X^*$ for all $X \in \mathrm{Irr}(\mathcal{C})$ is called the adjoint subcategory of $\mathcal{C}$ and is denoted by $\mathcal{C}_{ad}$. By [18, Corollary 3.7] every fusion category has a canonical faithful grading $\mathcal{C} = \bigoplus\limits_{g\in \mathcal{U}(\mathcal{C})} \mathcal{C}_g$ with trivial component $\mathcal{C}_{ad}$. This grading is called the universal grading of $\mathcal{C}$ and $\mathcal{U}(\mathcal{C})$ is called the universal grading group of $\mathcal{C}$.

Let $\mathcal{C} = \bigoplus\limits_{g\in G} \mathcal{C}_g$ be a G-extension of $\mathcal{D}$. Then $\mathrm{FPdim}(\mathcal{C}_g) = \mathrm{FPdim}(\mathcal{C}_h)$ for all $g, h \in G$ and $\mathrm{FPdim}(\mathcal{C}) = |G|\mathrm{FPdim}(\mathcal{D})$ (see [14, Proposition 8.20]).

2.3 Weakly group-theoretical fusion categories

Let $\mathcal{C}$ be a fusion category and $\mathcal{M}$ be an indecomposable right $\mathcal{C}$-module category. Let $\mathcal{C}^*_{\mathcal{M}}$ denote the category of $\mathcal{C}$-module endofunctors of $\mathcal{M}$. Then $\mathcal{C}^*_{\mathcal{M}}$ is a fusion category, called the dual of $\mathcal{C}$ with respect to $\mathcal{M}$ (see [14, 29]). Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if $\mathcal{D}$ is equivalent to $\mathcal{C}^*_{\mathcal{M}}$ for some indecomposable right $\mathcal{C}$-module category $\mathcal{M}$.

A fusion category $\mathcal{C}$ is said to be (cyclically) nilpotent if there is a sequence of fusion categories

$$\mathcal{C}_0 = \mathrm{Vec}, \quad \mathcal{C}_1, \cdots, \mathcal{C}_n = \mathcal{C} \tag{2.2}$$

and a sequence of finite (cyclic) groups $G_1, \cdots, G_n$ such that $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ by a G_i-extension.

A fusion category is called weakly group-theoretical if it is Morita equivalent to a nilpotent fusion category. A fusion category is called group-theoretical if it is Morita equivalent to a pointed fusion category. A fusion category is called solvable if it is Morita equivalent to a cyclically nilpotent fusion category.

Let us recall an old result in group theory before giving the proof of the proposition below. Let G be a finite group. If $|G|$ is odd or $2n$ then G is solvable, where n is odd (see e.g., [19, Theorem 1.35]).

Proposition 2.1 *Let $\mathcal{C}$ be a weakly group-theoretical fusion category. Assume that* $\mathrm{FPdim}(\mathcal{C})$ *is $2n$, where n is odd. Then $\mathcal{C}$ is solvable.*

Proof By the definition above, $\mathcal{C}$ is Morita equivalent to a nilpotent fusion category $\mathcal{D}$. If $\mathcal{D}$ is pointed then $\mathcal{D} = \mathrm{Vec}_{G,\omega}$ for a finite group G with $|G| = \mathrm{FPdim}(\mathcal{C})$ and a 3-cocycle ω. Since G is solvable, $\mathcal{D}$ is solvable and so is $\mathcal{C}$, by [15, Proposition 4.5]. In fact, $\mathcal{C}$ is also group-theoretical by the definition of a group-theoretical fusion category.

We then assume that $\mathcal{D}$ is not pointed and hence there is a sequence of fusion categories

$$\mathcal{D}_0 = \mathrm{Vec}, \quad \mathcal{D}_1, \cdots, \mathcal{D}_n = \mathcal{D} \tag{2.3}$$

and a sequence of finite groups $G_1, \cdots, G_n$ such that $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by a G_i-extension.

It is clear that the order of G_i divides $\mathrm{FPdim}(\mathcal{C})$ and hence G_i is solvable, $1 \leq i \leq n$. Moreover, the fusion subcategory $\mathcal{D}_1$ must be pointed and hence $\mathcal{D}_1 = \mathrm{Vec}_{K,\omega}$, where K is a finite group with order dividing $\mathrm{FPdim}(\mathcal{C})$. By [15, Proposition 4.5(ii)], $\mathcal{D}_1$ is solvable. Then $\mathcal{D}_i$ in the sequence is solvable by [15, Proposition 4.5(ii)]. Again by [15, Proposition 4.5(ii)], $\mathcal{C}$ is solvable.

Corollary 2.1 *Let $p < q < r$ be distinct prime numbers, and $\mathcal{C}$ be a fusion category of dimension pqr. Then $\mathcal{C}$ is solvable.*

Proof If $\mathcal{C}$ is integral then $\mathcal{C}$ is group-theoretical by [15, Theorem 9.2]. Hence $\mathcal{C}$ is solvable by Proposition 2.1.

If $\mathcal{C}$ is not integral, then $\mathcal{C} = \bigoplus\limits_{g \in \mathbb{Z}_2} \mathcal{C}_g$ by [18, Theorem 3.10], where $\mathcal{C}_e$ is a fusion category of dimension qr. Hence $\mathcal{C}$ is solvable by [15, Proposition 4.5].

2.4 Braided fusion categories

A fusion category $\mathcal{C}$ is called braided if it admits a braiding c, where the braiding c is a family of natural isomorphisms: $c_{X,Y}: X \otimes Y \to Y \otimes X$ satisfying the hexagon axioms for all $X, Y \in \mathcal{C}$ (see [21]).

Let $\mathcal{C}$ be a braided fusion category and $\mathcal{D} \subseteq \mathcal{C}$ be a fusion subcategory. The Müger centralizer $\mathcal{D}'$ of $\mathcal{D}$ in $\mathcal{C}$ is the category of all objects $Y \in \mathcal{C}$ such that $c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X \in \mathcal{D}$. The centralizer $\mathcal{D}'$ is again a fusion subcategory of $\mathcal{C}$. The Müger center of $\mathcal{C}$ is the Müger centralizer $\mathcal{Z}_2(\mathcal{C}) := \mathcal{C}'$ of $\mathcal{C}$ itself. A braided fusion category $\mathcal{C}$ is called nondegenerate if its

Müger center $\mathcal{Z}_2(\mathcal{C})$ is trivial, and it is called slightly degenerate if its Müger center $\mathcal{Z}_2(\mathcal{C})$ is equivalent to the category Svec of super vector spaces.

Let $\mathcal{C}$ be a fusion category. The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is defined as the category whose objects are pairs $(X, c_{-,X})$, where X is an object of $\mathcal{C}$ and $c_{-,X}$ is a natural family of isomorphisms $c_{V,X} : V \otimes X \to X \otimes V$, $V \in \mathcal{C}$, satisfying certain compatibility conditions (see [21, Definition XIII.4.1]). It is shown in [14, Theorem 2.15, Proposition 8.12] that $\mathcal{Z}(\mathcal{C})$ is a braided fusion category and $\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$. In addition, $\mathcal{Z}(\mathcal{C})$ is nondegenerate by [12, Corollary 3.9].

A braided fusion category $\mathcal{C}$ is called symmetric if $c_{Y,X}c_{X,Y} = \mathrm{id_{X\otimes Y}}$ for all objects $X, Y \in \mathcal{C}$. A symmetric fusion category $\mathcal{C}$ is said to be Tannakian if it is equivalent to $\mathrm{Rep}(G)$ for some finite group G as symmetric categories.

Let G be a finite group and $u \in G$ be a central element of order 2. Then the category $\mathrm{Rep}(G)$ has a braiding $c^u_{X,Y}$ as follows: For all $x \in X$, $y \in Y$,

$$c^u_{X,Y}(x \otimes y) = (-1)^{mn} y \otimes x \quad \text{if } ux = (-1)^m x,\ uy = (-1)^n y.$$

Let $\mathrm{Rep}(G, u)$ be the fusion category $\mathrm{Rep}(G)$ equipped with the new braiding $c^u_{X,Y}$. Deligne proved that any symmetric fusion category is equivalent to some $\mathrm{Rep}(G, u)$ (see [3]).

Lemma 2.1 (see [12, Corollary 2.50]) *Let $\mathcal{C}$ be a symmetric fusion category* $\mathrm{Rep}(G, u)$. *Then one of the following holds*:

(1) *$\mathcal{C}$ is a Tannakian category*;

(2) $\mathrm{Rep}(G/\langle u\rangle) \subseteq \mathcal{C}$ *is a maximal Tannakian subcategory with dimension* $\frac{1}{2}\mathrm{FPdim}(\mathcal{C})$.

In particular, if $\mathrm{FPdim}(\mathcal{C})$ *is bigger than* 2, *then $\mathcal{C}$ always has a nontrivial Tannakian subcategory.*

The following theorem is known as the Müger Decomposition Theorem since it is due to Müger [22, Theorem 4.2] when $\mathcal{C}$ is modular.

Theorem 2.2 (see [12, Theorem 3.13]) *Let $\mathcal{C}$ be a braided fusion category and $\mathcal{D}$ be a nondegenerate subcategory of $\mathcal{C}$. Then $\mathcal{C}$ is braided equivalent to $\mathcal{D} \boxtimes \mathcal{D}'$, where $\mathcal{D}'$ is the centralizer of $\mathcal{D}$ in $\mathcal{C}$.*

2.5 Equivariantizations and braided G-crossed fusion categories

Let G be a finite group and $\mathcal{C}$ be a fusion category. Let $\underline{G}$ denote the monoidal category whose objects are elements of G, morphisms are identities and tensor product is given by the multiplication in G. Let $\underline{\mathrm{Aut}_{\otimes}}\mathcal{C}$ denote the monoidal category whose objects are tensor autoequivalences of $\mathcal{C}$, morphisms are isomorphisms of tensor functors and tensor product is given by the composition of functors.

An action of G on $\mathcal{C}$ is a monoidal functor

$$T : \underline{G} \to \underline{\mathrm{Aut}_{\otimes}}\mathcal{C}, \quad g \mapsto T_g$$

with the isomorphism $f^X_{g,h} : T_g(X) \otimes T_h(X) \cong T_{gh}(X)$, for every X in $\mathcal{C}$.

Let $\mathcal{C}$ be a fusion category with an action of G. Then the fusion category $\mathcal{C}^G$, called the G-equivariantization of $\mathcal{C}$, is defined as follows (see [2, 12, 23]):

(1) An object in $\mathcal{C}^G$ is a pair $(X, (u^X_g)_{g\in G})$, where X is an object of $\mathcal{C}$ and $u^X_g : T_g(X) \to X$ is an isomorphism such that

$$u^X_g T_g(u^X_h) = u^X_{gh} f^X_{g,h} \quad \text{for all } g, h \in G.$$

(2) A morphism $\phi : (Y, u^Y_g) \to (X, u^X_g)$ in $\mathcal{C}^G$ is a morphism $\phi : Y \to X$ in $\mathcal{C}$ such that $\phi u^Y_g = u^X_g \phi$ for all $g \in G$.

(3) The tensor product in $\mathcal{C}^G$ is defined as $(Y, u^Y_g) \otimes (X, u^X_g) = (Y \otimes X, (u^Y_g \otimes u^X_g) j_g|_{Y,X})$, where $j_g|_{Y,X} : T_g(Y \otimes X) \to T_g(Y) \otimes T_g(X)$ is the isomorphism giving the monoidal structure on T_g.

There is a procedure opposite to equivariantization. Let $\mathcal{C}$ be a fusion category and $\mathrm{Rep}(G) \subseteq \mathcal{Z}(\mathcal{C})$ be a Tannakian subcategory which embeds into $\mathcal{C}$ via the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$. Let $A = \mathrm{Fun}(G)$ be the algebra of functions on G. It is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ under the embedding above. Let $\mathcal{C}_G$ be the category of left A-modules in $\mathcal{C}$. It is a fusion category which is called the de-equivariantization of $\mathcal{C}$ by $\mathrm{Rep}(G)$.

The two procedures above are inverse to each other; that is, there are canonical equivalences $(\mathcal{C}_G)^G \cong \mathcal{C}$ and $(\mathcal{C}^G)_G \cong \mathcal{C}$. Moreover, we have

$$\mathrm{FPdim}(\mathcal{C}^G) = |G|\mathrm{FPdim}(\mathcal{C}) \quad \text{and} \quad \mathrm{FPdim}(\mathcal{C}_G) = \frac{\mathrm{FPdim}(\mathcal{C})}{|G|}.$$

A braided G-crossed fusion category is a fusion category $\mathcal{C}$ endowed with a G-grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ and an action of G by tensor autoequivalences $\rho : \underline{G} \to \mathrm{Aut}_\otimes \mathcal{C}$, such that $\rho^g(\mathcal{C}_h) \subseteq \mathcal{C}_{ghg^{-1}}$ for all $g, h \in G$, and a G-braiding $c : X \otimes Y \to \rho^g(Y) \otimes X$, $g \in G$, $X \in \mathcal{C}_g$, $Y \in \mathcal{C}$, subject to appropriate compatibility conditions.

Let $\mathcal{C}$ be a braided fusion category and $\mathrm{Rep}(G) \subseteq \mathcal{C}$ be a Tannakian subcategory. The de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathrm{Rep}(G)$ is a braided G-crossed fusion category. The category $\mathcal{C}_G$ is not braided in general. But the neutral component $(\mathcal{C}_G)_e$ of the associated G-grading of $\mathcal{C}_G$ is braided. By [12, Proposition 4.56], $\mathcal{C}$ is nondegenerate if and only if $(\mathcal{C}_G)_e$ is nondegenerate and the associated grading of $\mathcal{C}_G$ is faithful. In this case, $|G|^2$ divides $\mathrm{FPdim}(\mathcal{C})$.

3 Braided Fusion Categories of Dimension $p^m q^n d$ and $p^2q^2r^2$

In this section, we study the braided fusion categories of dimension $p^m q^n d$ and $p^2q^2r^2$, where p, q, r are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. Then we apply the results obtained to weakly integral braided fusion categories of dimension less than 1800. Our results show that all fusion categories involved are weakly group-theoretical.

Proposition 3.1 *Let p, q be distinct prime numbers. Assume that $\mathcal{C}$ is a slightly degenerate fusion category of Frobenius-Perron dimension $p^m q^n d$, where p, q are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. Then $\mathcal{C}$ is solvable.*

Proof By [32, Corollary 3.4], $\frac{\mathrm{FPdim}(\mathcal{C})}{2\mathrm{FPdim}(X)^2}$ is an algebraic integer for all $X \in \mathrm{Irr}(\mathcal{C})$. Hence the Frobenius-Perron dimensions of integral simple objects have the form $p^a q^b$, where $a, b \geq 0$. Then $\mathcal{C}$ is weakly group-theoretical by [32, Proposition 3.14]. It follows from the arguments of [27, Proposition 5.3] that every Tannakian subcategory of $\mathcal{C}$ is solvable if it exists. Furthermore, in this case $\mathcal{C}$ is solvable by [27, Theorem 5.1]. In the rest of the paper, we aim to prove the existence of a nontrivial Tannakian subcategory.

We may assume that $\mathcal{C}$ contains no nontrivial nondegenerate fusion subcategories. In fact, if $\mathcal{C}$ contains such a fusion category $\mathcal{D}$ then $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{D}'$ by Theorem 2.2, where $\mathcal{D}'$ is also slightly degenerate. By induction, $\mathcal{D}'$ is solvable. By [27, Corollary 5.4], $\mathcal{D}$ is solvable. Hence $\mathcal{C}$ is solvable.

We assume on the contrary that $\mathcal{C}$ contains no nontrivial Tannakian subcategory. Then $G[X] = \{\mathbf{1}\}$ for all simple objects in $\mathcal{C}$ by [26, Lemma 7.1]. Moreover, every fusion subcategory of $\mathcal{C}$ is slightly degenerate, also by [26, Lemma 7.1].

Suppose first that $\mathcal{C}$ is integral. If the Frobenius-Perron dimensions of simple objects of $\mathcal{C}$ have a common prime factor p or q then the order of the group $G[X]$ is divisible by such a prime number. This is a contradiction. Therefore, there exists a simple object whose Frobenius-Perron dimension is a power of p and also there exists a simple object whose Frobenius-Perron dimension is a power of q. It follows from [15, Proposition 7.4] that $\mathcal{C}$ contains a nontrivial Tannakian subcategory, a contradiction.

Suppose then that $\mathcal{C}$ is not integral. Then $\mathcal{C}$ is faithfully graded by an elementary Abelian 2-group whose trivial component is an integral fusion category $\mathcal{C}_e$ (see [18, Theorem 3.10]). By the discussion above, $\mathcal{C}_e$ is also slightly degenerate. By the same arguments in the previous paragraph, we can prove that $\mathcal{C}_e$ contains a nontrivial Tannakian subcategory, also a contradiction.

Theorem 3.1 *Let p, q be distinct prime numbers. Assume that $\mathcal{C}$ is a braided fusion category of Frobenius-Perron dimension $p^m q^n d$, where d is a square-free natural number such that $(pq, d) = 1$. Then $\mathcal{C}$ is weakly group-theoretical.*

Proof We may assume that $\mathcal{C}$ is neither nondegenerate nor slightly degenerate. Indeed, if $\mathcal{C}$ is nondegenerate or slightly degenerate, then $\mathcal{C}$ is solvable by [27, Corollary 5.4] and Proposition 3.1.

Let $\mathcal{E} = \mathrm{Rep}(G) \subseteq \mathcal{C}'$ be the maximal Tannakian subcategory of $\mathcal{C}'$. Then the de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathrm{Rep}(G)$ is nondegenerate if $\mathcal{E} = \mathcal{C}'$, or slightly degenerate if $\mathcal{E} \subsetneq \mathcal{C}'$ (see [12, Corollary 4.31]). Since the dimension of $\mathcal{C}_G$ still has the form $p^{m'} q^{n'} d'$, $\mathcal{C}_G$ is solvable by [27, Corollary 5.4] and Proposition 3.1. Hence $\mathcal{C}$ is weakly group-theoretical by [15, Proposition 4.1].

Remark 3.1 In the proof above, we obtain that $\mathcal{C}_G$ is solvable, but we are not sure whether the group G is solvable. Hence we cannot determine the solvability of $\mathcal{C}$. In fact, there exists a braided fusion category whose dimension has the form p^mq^nd, but it is not solvable. For example, let $\mathbb{A}_5$ be the alternating group of order $60 = 2^2 \times 3 \times 5$, and $\mathrm{Rep}(\mathbb{A}_5)$ be the category of finite-dimensional representations of $\mathbb{A}_5$. Since $\mathbb{A}_5$ is a simple group, $\mathrm{Rep}(\mathbb{A}_5)$ is not solvable by [15, Proposition 4.5].

Corollary 3.1 *Let p, q, r be distinct prime numbers. Assume that $\mathcal{C}$ is a braided fusion category of Frobenius-Perron dimension $p^2q^2r^2$. Then $\mathcal{C}$ is weakly group-theoretical.*

Proof Clearly, if $\mathcal{C}$ has a faithful grading then the trivial component matches the assumption of Theorem 3.1 and hence it is weakly group-theoretical. Thus $\mathcal{C}$ is weakly group-theoretical by [15, Proposition 4.1].

We may then assume that $\mathcal{C} = \mathcal{C}_{ad}$. In particular, $\mathcal{C}$ is integral. We shall prove that $\mathcal{C}$ contains a nontrivial Tannakian subcategory. In fact, if $\mathcal{C}$ contains a nontrivial Tannakian subcategory $\mathcal{E} = \mathrm{Rep}(G)$ then the de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathrm{Rep}(G)$ is a braided G-crossed fusion category whose trivial component $(\mathcal{C}_G)_e$ is braided and it is weakly group-theoretical by Theorem 3.1. Therefore $\mathcal{C}$ is weakly group-theoretical by [15, Proposition 4.1].

If $\mathcal{C}$ is nondegenerate, then the proof of [15, Theorem 9.2] shows that $\mathcal{C}$ contains a nontrivial Tannakian subcategory.

If $\mathcal{C}$ is slightly degenerate, then $\mathrm{FPdim}(\mathcal{C})$ is even. We may assume that $p = 2$ in this case. By [32, Corollary 3.4], the dimensions of all simple objects of $\mathcal{C}$ are odd. Here we have two possibilities. The first possibility is that the dimensions of non-invertible simple objects X are divisible by qr. Then qr divides the order of the group $G[X]$. By [10, Lemma 2.4], $\mathcal{C}_{ad}$ contains a nontrivial Tannakian subcategory. The second possibility is that there exist simple objects whose Frobenius-Perron dimensions are powers of q or r. It follows from [15, Proposition 7.4] that $\mathcal{C}$ contains a nontrivial Tannakian subcategory.

If $\mathcal{C}$ is neither nondegenerate nor slightly degenerate, then the Müger center $\mathcal{C}'$ contains a nontrivial Tannakian subcategory. This completes the proof.

Corollary 3.2 *Let $\mathcal{C}$ be a weakly integral braided fusion category of Frobenius-Perron dimension less than* 1800. *Then $\mathcal{C}$ is weakly group-theoretical.*

Proof Let n be a natural number less than 1800. Then n factors in the form p^mq^nd or $p^2q^2r^2$, where p, q, r are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. The result then follows from Theorem 3.1 and Corollary 3.1.

Corollary 3.3 *Let $\mathcal{C}$ be a weakly integral braided fusion category. Assume that* $\mathrm{FPdim}(\mathcal{C}) <$ 33075 *is odd. Then $\mathcal{C}$ is solvable.*

Proof By Proposition 2.1, it suffices to prove that $\mathcal{C}$ is weakly group-theoretical.

Let n be an odd natural number less than 33075. Then n factors in the form p^mq^nd except the case when $n = 11025$, where p, q are distinct prime numbers, d is a square-free natural number such that $(pq, d) = 1$. By Theorem 3.1, it is enough to consider the case when $n = 11025$. Notice that if $\mathrm{FPdim}(\mathcal{C}) = 11025$, then $\mathcal{C}$ cannot be slightly degenerate. If $\mathcal{C}$ is nondegenerate, then we are done by [26, Theorem 8.2]. If $\mathcal{C}$ is degenerate, then the Müger center $\mathcal{C}'$ is a nontrivial Tannakian subcategory $\mathcal{E} = \mathrm{Rep}(G)$. Then the de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathrm{Rep}(G)$ is a braided G-crossed fusion category whose trivial component $(\mathcal{C}_G)_e$ is braided and it is weakly group-theoretical. Hence $\mathcal{C}$ is weakly group-theoretical.

4 Non-braided Fusion Categories of Dimension 84 and 90

We start this section with the existence of nontrivial symmetric categories in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

4.1 Existence of nontrivial symmetric categories

There is an obvious forgetful tensor functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. By [16, Proposition 3.39], the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is surjective.

Lemma 4.1 (see [9, Lemma 2.1]) *Let $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$ be the group homomorphism induced by the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Then the following hold*:

(1) *$\mathcal{C}$ is faithfully graded by the group $\widehat{N}$, where N is the kernel of F_0.*

(2) *Suppose that $\mathcal{U}(\mathcal{C})$ is trivial. Then the group homomorphism F_0 is injective.*

Since $\mathcal{Z}(\mathcal{C})$ is semisimple, the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. By [15, Lemma 3.2], $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ has a natural structure of commutative algebra. By [14, Proposition 5.4], $\mathrm{FPdim}(I(\mathbf{1})) = \mathrm{FPdim}(\mathcal{C})$.

Lemma 4.2 *Assume that $I(\mathbf{1})$ contains nontrivial invertible simple objects of $\mathcal{Z}(\mathcal{C})$. Then $\mathcal{C}$ is faithful graded by some finite group.*

Proof Let $g \in I(\mathbf{1})$ be a nontrivial invertible object. Then $\mathrm{Hom}_{\mathcal{C}}(g, I(\mathbf{1})) = \mathrm{Hom}_{\mathcal{C}}(F(g), \mathbf{1}) \neq 0$ implies that $F(g) = \mathbf{1}$. Hence the kernel of $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$ is not trivial. By Lemma 4.1, $\mathcal{C}$ is faithfully graded by the group $\widehat{N}$, where N is the kernel of F_0.

The following lemma is contained in the proof of [15, Lemma 9.17]. We explicitly state it for the reader's convenience.

Lemma 4.3 *Let $\mathcal{D} \subset \mathcal{C}$ be a fusion subcategory. Then $I(\mathbf{1})$ contains a subalgebra B corresponding to $\mathcal{D}$ such that* $\mathrm{FPdim}(B) = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{D})}$.

Lemma 4.4 *Assume that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric category $\mathcal{E}$. If the order of $G(\mathcal{C})$ is odd, then $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.*

Proof We may assume that $\mathcal{C}$ has a trivial universal grading, otherwise [15, Proposition 2.9(ii)] shows that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.

Consider the group homomorphism $F_0 : G(\mathcal{Z}(\mathcal{C})) \to G(\mathcal{C})$. Since $\mathcal{U}(\mathcal{C})$ is trivial, F_0 is injective by Lemma 4.1. It follows that the order of $G(\mathcal{Z}(\mathcal{C}))$ is odd since the order of $G(\mathcal{C})$

is odd. Hence, $\mathcal{Z}(\mathcal{C})$ cannot contain a fusion subcategory of dimension 2. It follows that the dimension of $\mathcal{E}$ is greater than 2. By Lemma 2.1, $\mathcal{E}$ contains a nontrivial Tannakian subcategory.

Let $\mathcal{C}$ be a pre-modular category with a spherical structure ψ. Let Tr denote the trace corresponding to ψ. The S-matrix $S = \{s_{X,Y}\}_{X,Y\in\mathrm{Irr}(\mathcal{C})}$ of $\mathcal{C}$ is defined by $s_{X,Y} = \mathrm{Tr}(\mathrm{c}_{\mathrm{Y,X}}\mathrm{c}_{\mathrm{X,Y}})$.

Theorem 4.1 *Let p, q, r be distinct prime numbers and n be a positive integer. Assume that $\mathcal{C}$ is an integral nondegenerate fusion category of dimension $p^{2n}q^2r^2$. Then one of the following holds true:*

(1) $\mathcal{C}$ contains a nontrivial symmetric subcategory.

(2) The Frobenius-Perron dimensions of simple objects of $\mathcal{C}$ cannot be divisible by p^n.

Proof Let X be a simple object of $\mathcal{C}$. Then $\mathrm{FPdim}(X)$ divides p^nqr by [15, Theorem 2.11]. If $\mathrm{FPdim}(X)$ is a power of p, q or r then $\mathcal{C}$ contains a nontrivial symmetric category by [15, Corollary 7.2]. This proves (1).

In the rest of the proof, we assume that $\mathrm{FPdim}(X)$ is not a power of a prime number and prove $\mathrm{FPdim}(X)$ cannot be p^nd, where $d = 1, q$ or r.

We first consider the case when $\mathcal{C}$ does not contain nontrivial invertible simple objects. In this case, $\mathcal{C}$ must contain a simple object of dimension qr. In fact, if not, all possible dimensions of nontrivial simple objects have a common factor p which implies that $\mathcal{C}$ has a pointed fusion subcategory with dimension divisible by p. This contradicts our assumption.

Assume on the contrary that there exists a simple object X_0 of dimension p^nd. By the orthogonality of columns of the S-matrix, we have

$$\sum_{X\in\mathrm{Irr}(\mathcal{C})} \frac{s_{X_0,X}}{\mathrm{FPdim}(X_0)}\mathrm{FPdim}(X) = 0. \tag{4.1}$$

Hence there exists $X_1 \in \mathrm{Irr}(\mathcal{C})$ of dimension qr such that $s_{X_0,X_1} \neq 0$. In fact, if there does not exist such a simple object, then the simple object X such that $s_{X_0,X} \neq 0$ is either 1 or its dimension has a prime factor p. It follows that the left side of (4.1) is equal to 1 modulo p, a contradiction.

The ratios $\frac{s_{X_0,X_1}}{\mathrm{FPdim}(X_0)}$ and $\frac{s_{X_0,X_1}}{\mathrm{FPdim}(X_1)}$ are both algebraic integers, and so is $\frac{s_{X_0,X_1}}{p^nqr}$. This implies that $\frac{s_{X_0,X_1}}{\mathrm{FPdim}(X_0)}$ is divisible by $t := \frac{qr}{d}$.

On the other hand, we have

$$\sum_{X\in\mathrm{Irr}(\mathcal{C})} \left|\frac{s_{X_0,X}}{\mathrm{FPdim}(X_0)}\right|^2 = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(X_0)^2} = t^2. \tag{4.2}$$

As we know, $s_{X_0,X}$ is a sum of roots of unity, hence every summand on the left side is a totally positive algebraic integer. The summand corresponding to $X = \mathbf{1}$ is 1 and the summand a corresponding to $X = X_1$ is an algebraic integer divisible by t^2. Hence there is a Galois automorphism δ such that $\delta(a) \geq t^2$. Applying δ to (4.2), we get that the left side $\geq 1 + t^2$, a contradiction.

Now we consider the case when $\mathcal{C}$ has nontrivial invertible simple objects. Let $\mathcal{B} = \mathcal{C}_{pt}$ be the maximal pointed fusion subcategory of $\mathcal{C}$. If $\mathcal{B}$ is degenerate then the Müger center of $\mathcal{B}$ is a nontrivial symmetric subcategory.

If $\mathcal{B}$ is nondegenerate then $\mathcal{C} = \mathcal{B} \boxtimes \mathcal{B}'$ by Theorem 2.2, where $\mathcal{B}'$ is the Müger centralizer of $\mathcal{B}$ in $\mathcal{C}$. In particular, $\mathcal{B}'$ is nondegenerate and does not contain nontrivial invertible simple objects. By the proof of the first case, the Frobenius-Perron dimensions of simple objects of $\mathcal{B}'$ and hence $\mathcal{C}$ cannot be $p^n d$, where $d = 1, q$ or r. This completes the proof.

Proposition 4.1 *Let $\mathcal{C}$ be a fusion category of dimension $2qp^n$, where p, q are distinct odd primes and $n \geq 1$. If $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory, then $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory.*

Proof Assume on the contrary that $\mathcal{Z}(\mathcal{C})$ has a unique nontrivial symmetric subcategory $\mathcal{E}$ which is equivalent to the category of super vectors. Let $\mathcal{A} = \mathcal{E}' \subset \mathcal{Z}(\mathcal{C})$ be the Müger centralizer of $\mathcal{E}$ in $\mathcal{Z}(\mathcal{C})$. Then the Müger center $\mathcal{Z}_2(\mathcal{A})$ of $\mathcal{A}$ is $\mathcal{E}$ and hence $\mathcal{A}$ is a slightly degenerate fusion category of dimension $2q^2p^{2n}$.

By [15, Proposition 7.4], we may assume that $\mathcal{A}$ cannot contain simple objects of odd prime power dimension. On the other hand, [32, Corollary 3.4] shows that $\frac{\mathrm{FPdim}(\mathcal{A})}{2\mathrm{FPdim}(X)^2}$ is an integer for all noninvertible simple objects X of $\mathcal{A}$. Hence $\mathrm{FPdim}(X) = p^i q^j$ for $i, j \geq 1$. The decomposition of $X \otimes X^* \in \mathcal{Z}(\mathcal{C})_{ad}$ implies that the order of $G[X]$ is divisible by pq. By [10, Lemma 2.4], the fusion subcategory generated by $G[X]$ is a symmetric subcategory which must be Tannakian because it has odd dimension. This is a contradiction.

The proof of the proposition below is similar to that of [15, Lemma 9.18].

Proposition 4.2 *Let $\mathcal{C}$ be a slightly degenerate fusion category. Suppose that $\mathcal{C}$ contains a simple object of dimension 2 or 4. Then $\mathcal{C}$ contains a nontrivial Tannakian subcategory.*

Proposition 4.3 *Let p, q, r be distinct prime numbers, and $\mathcal{C}$ be a fusion category of dimension p^2qr. Assume that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory $\mathcal{E} = \mathrm{Rep}(G)$. Then we have the following results*:

(1) *If $\mathrm{FPdim}(\mathcal{E}) < p^2qr$, then $\mathcal{C}$ is solvable.*

(2) *If $\mathrm{FPdim}(\mathcal{E}) = p^2qr$, then $\mathcal{C}$ is group-theoretical.*

Proof Assume that $\mathrm{FPdim}(\mathcal{E})$ is a power of p. Then G is a solvable group. It follows that G has a quotient group H such that $|H|$ is p. Hence $\mathcal{E}$ has a subcategory $\mathrm{Rep}(H)$. Under the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, $\mathrm{Rep}(H)$ either maps to Vec or embeds to $\mathcal{C}$. Hence $\mathcal{C}$ is an H-extension of some fusion category $\mathcal{D}_1$ by [15, Proposition 2.9(i)], or $\mathcal{C}$ is an H-equivariantization of some fusion category $\mathcal{D}_2$ by [15, Proposition 2.10(i)]. In both cases, $\mathrm{FPdim}(\mathcal{D}_1) = \mathrm{FPdim}(\mathcal{D}_2) = pqr$. By Corollary 2.1, $\mathcal{D}_1$ and $\mathcal{D}_2$ are both solvable, hence $\mathcal{C}$ is solvable by [15, Proposition 4.5].

Assume that $\mathrm{FPdim}(\mathcal{E})$ has a prime factor q or r. We consider the de-equivariantization $\mathcal{Z}(\mathcal{C})_G$ of $\mathcal{Z}(\mathcal{C})$ by $\mathcal{E}$. Set $\mathcal{D} = \mathcal{Z}(\mathcal{C})_G$. Then $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is faithfully graded by G, see [12, Proposition 4.56]. Hence $\mathrm{FPdim}(\mathcal{D}_e)=\frac{p^4q^2r^2}{\mathrm{FPdim}(\mathcal{E})^2}$.

If $\mathrm{FPdim}(\mathcal{E})<p^2qr$, then either $\mathrm{FPdim}(\mathcal{E})=pqr$ or $\mathrm{FPdim}(\mathcal{E})$ has at most 2 distinct prime factors. This implies that G is a solvable group. In this case, $\mathrm{FPdim}(\mathcal{D}_e)=pqr$ or $\mathrm{FPdim}(\mathcal{D}_e)$ has at most 2 distinct prime factors. By Corollary 2.1 and [15, Theorem 1.6], $\mathcal{D}_e$ is solvable in both cases. Therefore, $\mathcal{Z}(\mathcal{C})$ and hence $\mathcal{C}$ are solvable by [15, Proposition 4.5].

If $\mathrm{FPdim}(\mathcal{E})=p^2qr$, then $\mathrm{FPdim}(\mathcal{D}_e)=1$, and hence $\mathcal{D}$ is pointed. It follows that $\mathcal{Z}(\mathcal{C})$, being an equivariantization of a pointed fusion category, is group-theoretical by [24, Theorem 7.2]. So $\mathcal{C}$ is group-theoretical by [11, Theorem 1.5].

Corollary 4.1 *Let p,q,r be distinct prime numbers. Assume that $\mathcal{C}$ is a weakly group-theoretical fusion category of dimension p^2qr. Then $\mathcal{C}$ is group-theoretical or solvable.*

Proof By [15, Proposition 4.2], $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory. The result then follows from Proposition 4.3.

4.2 Fusion categories of dimension 84

For any fusion category $\mathcal{C}$, we use $\deg(\mathcal{C})$ to denote the set of Frobenius-Perron dimensions of simple objects of $\mathcal{C}$.

Lemma 4.5 *Let $\mathcal{C}$ be a fusion category. Assume that* $\deg(\mathcal{C})=\{1,6,14,21,42\}$ *and $G[X]$ is trivial for all $X\in\mathrm{Irr}(\mathcal{C})$. Then $\mathcal{C}$ has at least* 6 *nontrivial invertible simple objects. Moreover, if $\mathcal{C}$ is the Drinfeld center of a fusion category $\mathcal{D}$ and the universal grading group $\mathcal{U}(\mathcal{D})$ is trivial, then $\mathcal{D}$ also has at least* 6 *nontrivial invertible simple objects.*

Proof Let X_6 be a simple object of dimension 6. Then $X_6\otimes X_6^*=\mathbf{1}+X_{14}+X_{21}$, where X_{14},X_{21} are simple objects of dimension 14 and 21, respectively. From $m(X_{14},X_6\otimes X_6^*)=m(X_6,X_{14}\otimes X_6)=1$, we can write $X_{14}\otimes X_6=X_6+W$, where W is a direct sum of simple objects of dimension $6,14,21$ or 42 and $\mathrm{FPdim}(W)=78$. Then we have an equation $6a_1+14a_2+21a_3+42a_4=78$ which shows that $a_1=6$ or 8. This implies that there are 6 or 8 6-dimensional simple objects in the decomposition of W.

Let X_6' be a simple object of dimension 6 such that $m(X_6',X_{14}\otimes X_6)\geq 1$. Then $m(X_6',X_{14}\otimes X_6)=m(X_{14},X_6'\otimes X_6^*)\leq 2$. If $m(X_{14},X_6'\otimes X_6^*)=1$, then $X_6'\otimes X_6^*=X_{14}+W$ where $\mathrm{FPdim}(W)=22$. Considering the possible decomposition of W, we get that W at least contains one invertible simple object. If $m(X_{14},X_6'\otimes X_6^*)=2$, then $X_6'\otimes X_6^*=2X_{14}+W$ where $\mathrm{FPdim}(W)=8$. Considering the possible decomposition of W, we get that W at least contains two invertible simple objects. It is easy to check all invertible objects obtained above are pairwise different. It follows that $\mathcal{C}$ has at least 6 nontrivial invertible simple objects.

Assume that $\mathcal{C}=\mathcal{Z}(\mathcal{D})$. Since $\mathcal{U}(\mathcal{D})$ is trivial, the group homomorphism $F_0:G(\mathcal{C})\to G(\mathcal{D})$ is injective by Lemma 4.1. Hence $\mathcal{D}$ also has at least 6 nontrivial invertible simple objects.

Lemma 4.6 *Let $\mathcal{C}$ be an integral fusion category of dimension* 84. *Assume that $\mathcal{C}$ has a subcategory $\mathcal{B}$ of dimension ≥ 7. Then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ contains a nontrivial* *symmetric subcategory.*

Proof By Lemma 4.3, there exists a subalgebra $D \subset I(\mathbf{1})$ corresponding to the fusion subcategory $\mathcal{B}$ such that $\mathrm{FPdim}(D) = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{B})} \leq 12$. In view of [15, Theorem 2.11], the Frobenius-Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ divide 84. The possible decompositions of D as an object of $\mathcal{Z}(\mathcal{C})$ shows that D either contains nontrivial invertible simple objects, or contains simple objects of prime power dimension. If the former case holds true, then $\mathcal{C}$ has a nontrivial grading by Lemma 4.2, and hence $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory by [15, Corollary 7.2]. If the latter case holds true, then $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory by [15, Corollary 7.2].

Lemma 4.7 *Let $\mathcal{C}$ be an integral fusion category of dimension* 84. *Then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ contains a nontrivial symmetric subcategory.*

Proof By Theorem 4.1, we may assume that the Frobenius-Perron dimensions of simple objects of $\mathcal{Z}(\mathcal{C})$ are $1, 6, 14, 21$ or 42. If there exists $X \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ such that $G[X]$ is not trivial, then $G[X] \subset \mathcal{Z}(\mathcal{C})_{ad} \cap \mathcal{Z}(\mathcal{C})_{pt}$, and hence it is a symmetric subcategory by [10, Lemma 2.4]. We are done in this case. This fact also implies that we may assume that $\mathcal{Z}(\mathcal{C})$ has simple objects of dimension 6, otherwise the order of $G[X]$ is divisible by 7 for any noninvertible simple object $X \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$. This is because the Frobenius-Perron dimensions of noninvertible simple objects have a common prime factor 7.

Now we can assume that $\deg(\mathcal{Z}(\mathcal{C})) = \{1, 6, 14, 21, 42\}$ and $G[X]$ is trivial for any noninvertible simple object X. In addition, we also assume that $\mathcal{U}(\mathcal{C})$ is trivial, otherwise [15, Proposition 2.9(ii)] shows that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial Tannakian subcategory. Therefore, we match the assumption of Lemma 4.5. Hence $G(\mathcal{C})$ has at least 7 invertible simple objects. It follows from Lemma 4.6 that $\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory.

Theorem 4.2 *Let $\mathcal{C}$ be a fusion category of dimension* 84. *Then $\mathcal{C}$ is solvable or group-theoretical.*

Proof If $\mathcal{C}$ is not integral then $\mathcal{C}$ is a G-extension of an integral fusion subcategory $\mathcal{D}$ with lower dimension by [18, Theorem 3.10], where G is an elementary Abelian 2-group. Then $\mathrm{FPdim}(\mathcal{D}) = 21$ or 42. By Corollary 2.1 and [15, Theorem 1.6], $\mathcal{D}$ is solvable. Hence $\mathcal{C}$ is solvable by [15, Proposition 4.5].

We therefore assume that $\mathcal{C}$ is integral. By Lemma 4.7, $\mathcal{Z}(\mathcal{C})$ has a nontrivial symmetric subcategory. In view of Lemma 2.1, we may assume that $\mathcal{Z}(\mathcal{C})$ has a unique nontrivial symmetric subcategory $\mathcal{E}$ which is equivalent to the category of super vectors. Let $\mathcal{B} = \mathcal{E}' \subset \mathcal{Z}(\mathcal{C})$ be the Müger centralizer of $\mathcal{E}$ in $\mathcal{Z}(\mathcal{C})$. Then the Müger center $\mathcal{Z}_2(\mathcal{B})$ of $\mathcal{B}$ is $\mathcal{E}$ and hence $\mathcal{B}$ is a slightly degenerate fusion category.

Let $\mathcal{A}$ be the category spanned by the invertible objects of $\mathcal{B}$. By [15, Proposition 2.6(ii)], $\mathcal{A} = \mathcal{E} \boxtimes \mathcal{A}_0$, where $\mathcal{A}_0$ is a nondegenerate pointed category. Then $\mathcal{B} = \mathcal{A}_0 \boxtimes \mathcal{A}_0'$ by Theorem 2.2. It is known that $\mathcal{A}_0'$ is a slightly degenerate fusion category and contains only two invertible objects: $\mathbf{1}$ and the generator δ of $\mathcal{E}$.

We claim that $\mathcal{A}_0'$ contains a simple object X of prime power dimension. In fact, if $\mathcal{A}_0'$ does not contain simple objects of prime dimension then $\deg(\mathcal{A}_0') = \{1, 6, 14, 21, 42\}$. Then $\mathcal{A}_0'$ contains at least 7 invertible simple objects by Lemma 4.5 which contradicts the fact $G(\mathcal{A}_0') = \{\mathbf{1}, \delta\}$. If $\mathrm{FPdim}(X)$ is odd, then $\mathcal{A}_0'$ has a nontrivial Tannakian subcategory by [15, Proposition 7.4]. If $\mathrm{FPdim}(X)$ is even, then $\mathcal{A}_0'$ has a nontrivial Tannakian subcategory by Proposition 4.2. In both cases, $\mathcal{Z}(\mathcal{C})$ has a nontrivial Tannakian subcategory. The result then follows from Lemma 4.3.

4.3 Fusion categories of dimension 90

Lemma 4.8 *Let $\mathcal{C}$ be an integral fusion category of dimension* 90*. If $\mathcal{C}$ has a fusion subcategory $\mathcal{D}$ of dimension* ≥ 6*. Then one of the following holds true*:

(1) *$\mathcal{C}$ is faithfully graded by a nontrivial finite group.*

(2) *$\mathcal{Z}(\mathcal{C})$ contains a nontrivial symmetric subcategory. In particular, $\mathcal{C}$ is weakly group-theoretical.*

Proof Let $B \subseteq A = I(\mathbf{1})$ be the subalgebra of A corresponding to the fusion subcategory $\mathcal{D}$. Then $\mathrm{FPdim}(B) = \frac{90}{\mathrm{FPdim}(\mathcal{D})} \leq 15$. In view of [6, Theorem 2.11], the Frobenius-Perron dimension of simple object of $\mathcal{Z}(\mathcal{C})$ divides 90. Hence the possible decomposition of B as an object of $\mathcal{Z}(\mathcal{C})$ shows that B (and hence A) contains a nontrivial invertible simple object or a simple object of prime power dimension. If the former possibility holds, then $\mathcal{C}$ is faithfully graded by a nontrivial finite group by Lemma 4.2. In this case, the trivial component has dimension ≤ 45 and hence is weakly group-theoretical. Thus $\mathcal{C}$ is weakly group-theoretical.

If the latter possibility holds, then $Z(\mathcal{C})$ contains a nontrivial symmetric subcategory by [15, Corollary 7.2]. In this case, $\mathcal{C}$ is weakly group-theoretical by Proposition 4.1 and Lemma 4.3.

Theorem 4.3 *Let $\mathcal{C}$ be a fusion category of dimension* 90*. Then $\mathcal{C}$ is weakly group-theoretical.*

Proof If $\mathcal{C}$ is not integral, then $\mathcal{C}$ is a $\mathbb{Z}_2$-extension of an integral fusion subcategory $\mathcal{D}$ with dimension 45, by [18, Theorem 3.10]. By [15, Theorem 1.6], $\mathcal{D}$ is solvable. Hence $\mathcal{C}$ is weakly group-theoretical by [15, Proposition 4.1].

If $\mathcal{C}$ is integral, then it suffices to prove that $\mathcal{C}$ has a fusion subcategory of dimension ≥ 6 by Lemma 4.8.

By Theorem 4.2, $\mathcal{C}$ has the following possible types:

$$(1,2;2,4;3,8),(1,2;2,4;6,2),(1,2;2,4;3,4;6,1),(1,2;2,22),$$
$$(1,6;2,3;3,8),(1,6;2,3;6,2),(1,6;2,3;3,4;6,1),(1,6;2,21),$$
$$(1,9;3,9),(1,9;3,1;6,2),(1,9;3,5;6,1),(1,9;9,1),(1,10;2,20),(1,15;5,3),$$
$$(1,18;2,18),(1,18;3,8),(1,18;3,4;6,1),(1,18;6,2),(1,30;2,15),(1,45;3,5).$$

Assume that $\mathcal{C}$ has the first 3 types. Then $G[X] = G(\mathcal{C})$ for any simple object X of dimension 2. In fact, if there exists a 2-dimensional simple object X such that $G[X] = \{1\}$, then Theorem 2.1 shows that $\mathcal{C}$ has a fusion subcategory of dimension $12, 24$ or 60. It is impossible since they cannot divide 90. It follows from [9, Lemma 3.2] that all simple objects of dimension 1 and 2 generate a fusion subcategory $\mathcal{D}$ of dimension 18.

Assume that $\mathcal{C}$ has the fourth type. Then $X \otimes X^* = 1 + g + X_2$ for every 2-dimensional simple object X, where $G(\mathcal{C}) = \{1, g\}$ and X_2 is a 2-dimensional simple object. Hence X_2 is a self-dual simple object. Let $\mathcal{C}\langle X_2\rangle$ be the fusion subcategory generated by X_2. If $\mathcal{C}\langle X_2\rangle$ is a proper subcategory of $\mathcal{C}$, then $\mathrm{FPdim}(\mathcal{C}\langle X_2\rangle) \geq 6$. If $\mathcal{C}\langle X_2\rangle = \mathcal{C}$, then $\mathcal{C}$ is Grothendieck equivalent to the category $\mathrm{Rep}(\mathcal{D}_{45})$ by [31, Theorem 1.2], where $\mathcal{D}_{45}$ is the dihedral group of order 90. Hence $\mathcal{C}$ is group-theoretical by [25, Proposition 4.7(1)].

For the remaining types, the largest pointed fusion subcategory $\mathcal{C}_{pt}$ has dimension ≥ 6. This completes the proof.

Combining Proposition 2.1 and Theorem 4.3, we get the following corollary.

Corollary 4.2 *Let $\mathcal{C}$ be a fusion category of dimension* 90. *Then $\mathcal{C}$ is solvable.*

4.4 Main results

Theorem 4.4 *Let $\mathcal{C}$ be a weakly integral fusion category of dimension less than* 120. *Then $\mathcal{C}$ is either group-theoretical or solvable.*

Proof Suppose $\mathrm{FPdim}(\mathcal{C}) = n$ is a natural number and $n < 120$.

If $n = p^a q^b$, where p, q are prime numbers, $a, b \geq 0$, then $\mathcal{C}$ is solvable by [15, Theorem 1.6].

If $n = 84$ or 90, then $\mathcal{C}$ is solvable by Theorem 4.2 and Corollary 4.2.

If $n = pqr$, where p, q and r are distinct prime numbers, then $\mathcal{C}$ is solvable by Corollary 2.1.

It remains to consider the case when $n = 60$. We shall follow the line of the proof of Proposition 2.1. By [15, Theorems 9.16], $\mathcal{C}$ is weakly group-theoretical and hence it is Morita equivalent to a nilpotent fusion category $\mathcal{D}$. If $\mathcal{D}$ is pointed, then $\mathcal{C}$ is group-theoretical. We then assume that $\mathcal{D}$ is not pointed and hence there exists a sequence of fusion subcategories

$$\mathcal{D}_0 = \mathrm{Vec}, \quad \mathcal{D}_1, \cdots, \mathcal{D}_n = \mathcal{D} \tag{4.3}$$

and a sequence of finite groups $G_1, \cdots, G_n$ such that $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by a G_i-extension.

For every group G_i, $1 \leq i \leq n$, the order $|G_i|$ divides 60 and hence it is solvable. Moreover, the fusion subcategory $\mathcal{D}_1$ must be pointed and hence $\mathcal{D}_1 = \mathrm{Vec}_{K,\omega}$ for some 3-cocycle $\omega \in H^3(G, k^\times)$ and a solvable group K. Then $\mathcal{C}$ is solvable by the same reason as in the proof of Proposition 2.1.

Corollary 4.3 *Let $\mathcal{C}$ be a strictly weakly integral fusion category of dimension less than* 120. *Then $\mathcal{C}$ is solvable.*

Proof The result follows from Theorem 4.4 and the fact that a group-theoretical fusion category is integral, see [14, Corollary 8.43].

By Theorem 4.4 and [15, Theorems 1.5], we can recover the main result in [9].

Corollary 4.4 *Let $\mathcal{C}$ be a weakly integral fusion category of dimension less than* 120*. Then $\mathcal{C}$ has the strong Frobenius property.*

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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