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# Cohomology of $\mathfrak{q}(2)$ in Prime Characteristic*

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**Abstract** Over an algebraically closed field of characteristic $p > 2$, the 0-dimensional and 1-dimensional cohomology of the queer Lie superalgebra $\mathfrak{q}(2)$ with coefficients in all baby Verma modules and all simple modules are determined.

**Keywords** Queer Lie superalgebras, Baby Verma modules, Simple modules, Cohomology
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## 1 Introduction

The Lie superalgebra cohomology is widely used in mathematics and physics, which is of great importance for studying extensions of modules as well as extensions of Lie superalgebras themselves. For instance, relative cohomology is fundamental in the Borel-Weil-Bott theory (see [4]) and cohomology of nilpotent radicals of parabolic subalgebras is crucial in the Kazhdan-Lusztig theory (see [2]). It is the second Hochschild cohomology group, with coefficients in the algebra itself regarded as a bi-module, that classifies the formal deformations of an associative algebra. Lie superalgebra cohomology is the foundation of the deformation theory of universal enveloping algebras of Lie superalgebras. In addition, the Hochschild cohomology group for the universal enveloping algebra of a Lie superalgebra can be shown to be isomorphic to the second Lie superalgebra cohomology group with its coefficient module being its universal enveloping algebra under the adjoint action. In 1977, Kac posed a challenging question: Determine the low-dimensional cohomology of simple Lie superalgebras with coefficients in arbitrary simple modules over a field of characteristic 0. For the simple Lie superalgebras $\mathfrak{sl}(m,n)$ and $\mathfrak{osp}(2,2n)$, Schunert, Su and Zhang have answered Kac's question in [5, p. 5052] and [6, Theorems 1.2–1.3].

Our concern is the modular-version of Kac's question above. This paper is a sequel to [7], in which $\mathrm{H}^1(\mathfrak{sl}(2,1),M)$ is determined for any finite-dimensional simple $\mathfrak{sl}(2,1)$-module $M$ over a field of prime characteristic. This paper aims to determine the 0-dimensional and 1-dimensional cohomology of the smallest queer Lie superalgebra $\mathfrak{q}(2)$ with coefficients in all baby Verma modules and all simple modules over an algebraically closed field of characteristic

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$p > 2$. Note that the 0-dimensional cohomology of $\mathfrak{q}(2)$ with coefficients in all simple modules is trivial by definition. Hence we list our main results in the following two theorems.

**Theorem 1.1** *Over an algebraically closed field of characteristic $p > 2$, let $Z_\chi(\lambda)$ be the baby Verma module of $\mathfrak{q}(2)$ with highest weight $\lambda$ and $p$-character $\chi$. Then*

$$\text{sdim H}^0(\mathfrak{q}(2), Z_\chi(\lambda)) = \begin{cases} 0 \mid 1, & (\lambda, \chi) = (0,0), \\ 0 \mid 0, & \textit{otherwise}, \end{cases} \tag{1.1}$$

$$\text{sdim H}^1(\mathfrak{q}(2), Z_\chi(\lambda)) = \begin{cases} 1 \mid 1, & (\lambda, \chi) = (0,0), \\ 0 \mid 0, & \textit{otherwise}. \end{cases} \tag{1.2}$$

**Theorem 1.2** *Over an algebraically closed field of characteristic $p > 2$, let $L_\chi(\lambda)$ be the simple module of $\mathfrak{q}(2)$ with highest weight $\lambda$ and $p$-character $\chi$. Then*

$$\text{sdim H}^1(\mathfrak{q}(2), L_\chi(\lambda)) = \begin{cases} 2 \mid 2, & (\lambda, \chi) = (0,0), \\ 0 \mid 1, & \lambda = (1, p-1),\ \chi = 0, \\ 2 \mid 0, & \lambda = (p-1, 1),\ \chi = 0, \\ 0 \mid 0, & \textit{otherwise}. \end{cases}$$

## 2 Preliminaries

In this paper, we write $\mathbb{F}$ for the underlying field and $\mathbb{F}_p$ for the prime subfield of $\mathbb{F}$. All vector spaces, algebras and (sub)modules are assumed to be $\mathbb{Z}_2$-graded and finite-dimensional over $\mathbb{F}$. Hereafter $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the field of two elements. We make some conventions for a vector (super)space $V = V_{\bar{0}} \oplus V_{\bar{1}}$:

(1) For $v \in V_{\bar{0}} \cup V_{\bar{1}}$, write $|v| \in \mathbb{Z}_2$ for the parity ($\mathbb{Z}_2$-degree) of $v$ and the symbol $|a|$ always implies that $a$ is $\mathbb{Z}_2$-homogeneous in a vector space.

(2) Write sdim $V = \dim V_{\bar{0}} \mid \dim V_{\bar{1}}$.

(3) Write $V = \langle v_1, \cdots, v_m \mid w_1, \cdots, w_n \rangle$, which means that $\{v_1, \cdots, v_m \mid w_1, \cdots, w_n\}$ is a $\mathbb{Z}_2$-homogeneous basis of $V$. In case $m = 0$ or $n = 0$, write $V = \langle 0 \mid w_1, \cdots, w_n \rangle$ or $\langle v_1, \cdots, v_m \mid 0 \rangle$, respectively.

### 2.1 The low-dimensional cohomology of a Lie superalgebra

Let $L$ be a Lie superalgebra and $M$ be an $L$-module. Recall that a $\mathbb{Z}_2$-homogeneous linear mapping $\varphi : L \to M$ is a derivation of parity $|\varphi|$ if

$$\varphi([x, y]) = (-1)^{|\varphi||x|} x\varphi(y) - (-1)^{|y|(|\varphi|+|x|)} y\varphi(x) \quad \text{for } x, y \in L.$$

Denote by Der$(L, M)$ the vector space spanned by all the $\mathbb{Z}_2$-homogeneous derivations from $L$ to $M$; each element therein is called a derivation. For a $\mathbb{Z}_2$-homogeneous element $m \in M$, the map $\mathfrak{D}_m$ from $L$ to $M$ is defined by

$$\mathfrak{D}_m(x) = (-1)^{|x||m|} xm, \quad \text{where } x \in L.$$

Then $\mathfrak{D}_m$ is a $\mathbb{Z}_2$-homogeneous derivation of parity $|m|$. Write $\mathrm{Ider}(L,M)$ for the vector space spanned by all $\mathfrak{D}_m$ with $\mathbb{Z}_2$-homogeneous elements $m \in M$; each element therein is called an inner derivation. In general, $L$-module $\mathrm{Hom}_{\mathbb{F}}(L,M)$ (consisting of all linear maps from $L$ to $M$) contains $\mathrm{Ider}(L,M)$ and $\mathrm{Der}(L,M)$ as submodules.

Let $\mathfrak{h}$ be a Cartan subalgebra of $L$. Suppose that $L$ and $M$ possess weight space decompositions with respect to $\mathfrak{h}_{\bar{0}}$:

$$L = \bigoplus_{\gamma \in \mathfrak{h}_{\bar{0}}^*} L_\gamma, \quad M = \bigoplus_{\gamma \in \mathfrak{h}_{\bar{0}}^*} M_\gamma.$$

Hereafter, denote by $L^*$ the space consisting of all linear maps from $L$ to $\mathbb{F}$ for Lie superalgebra $L$. Write

$$\mathrm{Hom}_{\mathbb{F}}(L,M)_{(0)} = \{\phi \in \mathrm{Hom}_{\mathbb{F}}(L,M) \mid \phi(L_\alpha) \subset M_\alpha, \forall \alpha \in \mathfrak{h}_{\bar{0}}^*\},$$
$$\mathrm{Der}(L,M)_{(0)} = \{\phi \in \mathrm{Der}(L,M) \mid \phi(L_\alpha) \subset M_\alpha, \forall \alpha \in \mathfrak{h}_{\bar{0}}^*\}.$$

A linear map (resp. derivation) in $\mathrm{Hom}_{\mathbb{F}}(L,M)_{(0)}$ (resp. $\mathrm{Der}(L,M)_{(0)}$) is called a weight-map (resp. weight-derivation) with respect to $\mathfrak{h}$. It is a standard fact that

$$\mathrm{Der}(L,M) = \mathrm{Der}(L,M)_{(0)} + \mathrm{Ider}(L,M) \tag{2.1}$$

for a standard proof of which the reader can see [1, Lemma 3.2] or [7, Lemma 2.1].

By definition, the 1-dimensional cohomology of $L$ with coefficients in $M$ is

$$\mathrm{H}^1(L,M) = \mathrm{Der}(L,M)/\mathrm{Ider}(L,M) \tag{2.2}$$

and the 0-dimensional cohomology is

$$\mathrm{H}^0(L,M) = \{m \in M \mid xm = 0, \forall x \in L\}.$$

Two 1-cocycles (elements in $\mathrm{Der}(L,M)$) are said to be cohomologous if their images in $H^1(L,M)$ are equal. From (2.1)–(2.2), we get the following lemma, which gives a useful reduction method in computing the 1-dimensional cohomology of Lie superalgebras.

**Lemma 2.1** *Retain the above notations. Let $\varphi \in \mathrm{H}^1(L,M)$. Then $\varphi$ is cohomologous to a weight-derivation. In particular, $\varphi(h)$ lies in $\mathrm{H}^0(L,M)$ for any $h \in \mathfrak{h}_{\bar{0}}$.*

**Proof** It is sufficient to show the last assertion. Since $\varphi$ is cohomologous to a weight-derivation, we may regard $\varphi$ as a weight-derivation. Let $h \in \mathfrak{h}_{\bar{0}}$ and $x \in L_\alpha$ for any $\alpha \in \mathfrak{h}_{\bar{0}}^*$. Then

$$\alpha(h)\varphi(x) = \varphi([h,x]) = h\varphi(x) - (-1)^{|x||\varphi|}x\varphi(h) = \alpha(h)\varphi(x) - (-1)^{|x||\varphi|}x\varphi(h).$$

It follows that $x\varphi(h) = 0$. This implies $\varphi(h) \in \mathrm{H}^0(L,M)$ since both $x$ and $\alpha$ are arbitrary.

### 2.2 The queer Lie superalgebra $\mathfrak{q}(2)$ and its representation theory

We follow the reference [8] for the structure and representation theory of $\mathfrak{q}(2)$. For the convenience of the readers, we summarize some information as follows.

For $k = 1, 2$, set $\dot{k} = 2 + k$ for convenience. Write

$$h_1 := E_{11} + E_{\dot{1}\dot{1}}, \quad h_2 := E_{22} + E_{\dot{2}\dot{2}}, \quad e := E_{12} + E_{\dot{1}\dot{2}}, \quad f := E_{21} + E_{\dot{2}\dot{1}},$$
$$H_1 := E_{1\dot{1}} + E_{\dot{1}1}, \quad H_2 := E_{2\dot{2}} + E_{\dot{2}2}, \quad E := E_{1\dot{2}} + E_{\dot{1}2}, \quad F := E_{2\dot{1}} + E_{\dot{2}1}.$$

Hereafter $E_{ij}$ is the $4 \times 4$ matrix unit. The queer Lie superalgebra

$$\mathfrak{q}(2) = \langle h_1, h_2, e, f \mid H_1, H_2, E, F \rangle$$

and we write it as $\mathfrak{g}$ for short. We call $\mathfrak{h} := \langle h_1, h_2 \mid H_1, H_2 \rangle$ the standard Cartan subalgebra of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}_{\bar{0}}^*$. If $\lambda(h_i) = \lambda_i$ for $i = 1, 2$, write $\lambda = (\lambda_1, \lambda_2)$ for short. With respect to $\mathfrak{h}_{\bar{0}}$, all weight spaces of $\mathfrak{g}$ are listed below:

$$\mathfrak{g}_0 = \langle h_1, h_2 \mid H_1, H_2 \rangle, \quad \mathfrak{g}_{(1,-1)} = \langle e \mid E \rangle, \quad \mathfrak{g}_{(-1,1)} = \langle f \mid F \rangle.$$

Letting $\mathfrak{n}^- = \langle f \mid F \rangle$ and $\mathfrak{n}^+ = \langle e \mid E \rangle$, we have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Recall that a restricted Lie superalgebra is a Lie superalgebra where the even part is a restricted Lie algebra and the odd part is a restricted module of the even part by the adjoint action. Then $\mathfrak{g}$ is a restricted Lie superalgebra with a $p$-mapping $[p]$ which is the usual $p$th power. Let $V$ be a simple $\mathfrak{g}$-module. Then there exists $\chi \in \mathfrak{g}_{\bar{0}}^*$, such that

$$x^p v - x^{[p]} v = \chi(x)^p v, \quad \forall x \in \mathfrak{g}_{\bar{0}},\ v \in V.$$

In this case we also call $V$ a simple $\mathfrak{g}$-module with $p$-character $\chi$. Fix $\chi \in \mathfrak{g}_{\bar{0}}^*$. Denote by $I_\chi$ the ideal of $U(\mathfrak{g})$ generated by the elements $x^p - x^{[p]} - \chi(x)^p$ for all $x \in \mathfrak{g}_{\bar{0}}$. Write $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/I_\chi$, which is called the reduced enveloping superalgebra with $p$-character $\chi$. Note that a simple $\mathfrak{g}$-module with $p$-character $\chi$ is the same as a simple $U_\chi(\mathfrak{g})$-module. Any $p$-character $\chi'$ is $G$-conjugate to a $p$-character $\chi$ with $\chi(\mathfrak{n}_{\bar{0}}^+) = 0$ and $U_{\chi'}(\mathfrak{g}) = U_\chi(\mathfrak{g})$, where $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$. Therefore the study of simple $\mathfrak{g}$-modules is reduced to the problem of studying simple ones with $p$-character $\chi$ when $\chi$ runs over the representatives of coadjoint $G$-orbits in $\mathfrak{g}_{\bar{0}}^*$ (see [3, Remark 2.3]). Recall that there are three coadjoint $G$-orbits with the following representatives (see [8, Sec. 6]):

(1) nilpotent: $\chi(e) = \chi(h_1) = \chi(h_2) = 0$ and $\chi(f) = 1$.

(2) semisimple: $\chi(e) = \chi(f) = 0, \chi(h_1) = a, \chi(h_2) = b$ for some $a, b \in \mathbb{F}$.

(3) mixed: $\chi(e) = 0, \chi(f) = 1, \chi(h_1) = \chi(h_2) = a$ for some $a \in \mathbb{F}\backslash\{0\}$.

Hereafter the symbol $\chi$ implies that $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $\chi$ is either nilpotent or semisimple or mixed.

Below we recall simple $U_\chi(\mathfrak{h})$-modules constructed in [8, Sec. 2.3]. Let $\lambda \in \mathfrak{h}_{\bar{0}}^*$ and $\mathfrak{h}_1$ a maximal isotropic subspace with respect to the following bilinear form on $\mathfrak{h}_{\bar{1}}$

$$(a, b)_\lambda := \lambda([a, b]), \quad \forall a,\ b \in \mathfrak{h}_{\bar{1}}.$$

Then $\lambda$ can be extended to a one-dimensional $(\mathfrak{h}_{\bar{0}} + \mathfrak{h}_1)$-module $\mathbb{F}_\lambda$ by letting $\mathfrak{h}_1 \mathbb{F}_\lambda = 0$. Write

$$\Lambda_\chi = \{\lambda \in \mathfrak{h}_{\bar{0}}^* \mid \lambda(h)^p - \lambda(h) = \chi(h)^p, h \in \mathfrak{h}_{\bar{0}}\}.$$

Then $\mathbb{F}_\lambda$ is a $U_\chi(\mathfrak{h}_{\overline{0}}+\mathfrak{h}_1)$-module if and only if $\lambda \in \Lambda_\chi$. Write

$$V_\chi(\lambda)=U_\chi(\mathfrak{h})\bigotimes_{U_\chi(\mathfrak{h}_{\overline{0}}+\mathfrak{h}_1)}\mathbb{F}_\lambda,\quad \lambda\in\Lambda_\chi.$$

Then $V_\chi(\lambda)$ is a simple $U_\chi(\mathfrak{h})$-module. Recall that the baby Verma module of $U_\chi(\mathfrak{g})$ with highest weight $\lambda$ and $p$-character $\chi$ is

$$Z_\chi(\lambda):=U_\chi(\mathfrak{g})\bigotimes_{U_\chi(\mathfrak{h}\oplus\mathfrak{n}^+)}V_\chi(\lambda),\quad \lambda\in\Lambda_\chi.$$

Denote by $L_\chi(\lambda)$ the unique simple quotient of $Z_\chi(\lambda)$, which is also of highest weight $\lambda$ and $p$-character $\chi$. Write $v_\lambda$ for the highest-weight vector of weight $\lambda$ in $V_\chi(\lambda)$ and set $|v_\lambda|=\overline{0}$. For convenience, write $(i,j,k)$ and $[i,j,k]$ for the elements $f^iF^jH_1^kv_\lambda$ and $f^iF^jH_2^kv_\lambda$ in $Z_\chi(\lambda)$, respectively. In this paper, the symbols $f^a,(a,j,k),[a,j,k]$ and $\underline{a}$ always imply that $a$ is the smallest nonnegative integer in the residue class containing $a$ modulo $p$. We also use $(i,j,k)$ or $[i,j,k]$ to represent the residue class containing $(i,j,k)$ or $[i,j,k]$ in $L_\chi(\lambda)$. If $x=\sum\limits_{i=0}^{p-1}\sum\limits_{j=0}^{1}\sum\limits_{k=0}^{1}a_{ijk}(i,j,k)$, we write $x^{(i,j,k)}$ for $a_{ijk}$.

**Remark 2.1** (see [8, Sec. 5]) Let $\chi\in\mathfrak{g}_{\overline{0}}^*$ and $\lambda=(\lambda_1,\lambda_2)\in\Lambda_\chi$.
(1) If $\lambda=0$, then $Z_\chi(\lambda)$ has a basis

$$\{(a,j,0)\mid j=0,1,0\le a\le p-1\}.$$

(2) If $\lambda_1\ne 0$, then $Z_\chi(\lambda)$ has a basis

$$\{(a,j,k)\mid j,k=0,1,0\le a\le p-1\}.$$

(3) If $\lambda_1=0$ and $\lambda_2\ne0$, then $Z_\chi(\lambda)$ has a basis

$$\{[a,j,k]\mid j,k=0,1,0\le a\le p-1\}.$$

## 3 $\mathrm{H}^0(\mathfrak{g},Z_\chi(\lambda))$

In this paper, the symbol $\delta_P$ means 1 if a proposition $P$ is true, and 0 otherwise. We list some formulas about the $\mathfrak{g}$-action on $Z_\chi(\lambda)$ (see [8, Sec. 5] for details) in the following.

**Remark 3.1** The $\mathfrak{g}$-action on $Z_\chi(\lambda)$ is given in the following.
(1) If $\lambda=(0,\lambda_2)\ne0$, then

$$\begin{aligned}
F[a,j,k]&=\delta_{j=0}[a,1,k],\\
f[a,j,k]&=\delta_{a\ne p-1}[a+1,j,k]+\delta_{a=p-1}\chi(f)^p[0,j,k],\\
H_i[a,0,k]&=(-1)^ia[a-1,1,k]+\delta_{i=2}(\delta_{k=0}+\lambda_2\delta_{k=1})[a,0,\delta_{k=0}],\\
E[a,1,k]&=a(\delta_{k=0}+\lambda_2\delta_{k=1})[a-1,1,\delta_{k=0}]+\lambda_2[a,0,k],\\
E[a,0,k]&=-a(\delta_{k=0}+\lambda_2\delta_{k=1})[a-1,0,\delta_{k=0}]-a(a-1)[a-2,1,k], \qquad (3.1)\\
e[a,j,k]&=-a(a-(-1)^j+\lambda_2)[a-1,j,k]-\delta_{(j,k)=(1,0)}[a,0,1]-\delta_{(j,k)=(1,1)}\lambda_2[a,0,0],
\end{aligned}$$

$$H_i[a,1,k] = \delta_{a\neq p-1}[a+1,0,k] + \delta_{a=p-1}\chi(f)^p[0,0,k] - \delta_{i=2}(\delta_{k=0}+\lambda_2\delta_{k=1})[a,1,\delta_{k=0}].$$

(2) If $\lambda = (\lambda_1, 0) \neq 0$, then

$$\begin{aligned}
F(a,j,k) &= \delta_{j=0}(a,1,k),\\
f(a,j,k) &= \delta_{a\neq p-1}(a+1,j,k) + \delta_{a=p-1}\chi(f)^p(0,j,k),\\
H_i(a,0,k) &= (-1)^i a(a-1,1,k) + \delta_{i=1}(\delta_{k=0}+\lambda_1\delta_{k=1})(a,0,\delta_{k=0}),\\
E(a,1,k) &= -a(\delta_{k=0}+\lambda_1\delta_{k=1})(a-1,1,\delta_{k=0}) + \lambda_1(a,0,k),\\
E(a,0,k) &= a(\delta_{k=0}+\lambda_1\delta_{k=1})(a-1,0,\delta_{k=0}) - a(a-1)(a-2,1,k),\\
e(a,j,k) &= a(\lambda_1 - a + (-1)^j)(a-1,j,k)\\
&\quad + \delta_{(j,k)=(1,0)}(a,0,1) + \delta_{(j,k)=(1,1)}\lambda_1(a,0,0),\\
H_i(a,1,k) &= \delta_{a\neq p-1}(a+1,0,k) + \delta_{a=p-1}\chi(f)^p(0,0,k)\\
&\quad - \delta_{i=1}(\delta_{k=0}+\lambda_1\delta_{k=1})(a,1,\delta_{k=0}).
\end{aligned} \tag{3.2}$$

(3) If $\lambda_1 = \lambda_2 \neq 0$ or $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$, then

$$\begin{aligned}
F(a,j,k) &= \delta_{j=0}(a,1,k),\\
f(a,j,k) &= \delta_{a\neq p-1}(a+1,j,k) + \delta_{a=p-1}\chi(f)^p(0,j,k),\\
E(a,1,k) &= -a(\delta_{k=0}(1+\mu^{-1}) + (\lambda_1+\mu\lambda_2)\delta_{k=1})(a-1,1,\delta_{k=0})\\
&\quad + (\lambda_1+\lambda_2)(a,0,k),\\
E(a,0,k) &= a(\delta_{k=0}(1+\mu^{-1}) + (\lambda_1+\mu\lambda_2)\delta_{k=1})(a-1,0,\delta_{k=0})\\
&\quad - a(a-1)(a-2,1,k),\\
e(a,j,k) &= \delta_{(j,k)=(1,1)}(\lambda_1+\mu\lambda_2)(a,0,0) + \delta_{(j,k)=(1,0)}(1+\mu^{-1})(a,0,1)\\
&\quad + a(\lambda_1-\lambda_2-a+(-1)^j)(a-1,j,k),\\
H_i(a,0,k) &= (-1)^i a(a-1,1,k) + \delta_{i=1}(\delta_{k=0}+\lambda_1\delta_{k=1})(a,0,\delta_{k=0})\\
&\quad - \delta_{i=2}(\delta_{k=0}\mu^{-1}+\mu\lambda_2\delta_{k=1})(a,0,\delta_{k=0}),\\
H_i(a,1,k) &= \delta_{a\neq p-1}(a+1,0,k) + \delta_{a=p-1}\chi(f)^p(0,0,k)\\
&\quad - \delta_{i=1}(\delta_{k=0}+\lambda_1\delta_{k=1})(a,1,\delta_{k=0})\\
&\quad + \delta_{i=2}(\delta_{k=0}\mu^{-1}+\mu\lambda_2\delta_{k=1})(a,1,\delta_{k=0}),
\end{aligned} \tag{3.3}$$

where $\mu^2 = -1$ if $\lambda_1 = \lambda_2 \neq 0$, and $\mu\lambda_2 + \lambda_1\mu^{-1} = 0$ if $0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0$.

(4) If $\lambda = (\lambda_1, -\lambda_1)$, then

$$h_i(a,j,k) = (-1)^i(a+j-\lambda_1)(a,j,k), \tag{3.4}$$

$$F(a,j,k) = \delta_{j=0}(a,1,k), \tag{3.5}$$

$$f(a,j,k) = \begin{cases} \chi(f)^p(0,j,k), & \text{if } a = p-1,\\ (a+1,j,k), & \text{if } a \neq p-1. \end{cases} \tag{3.6}$$

Besides, the following formulas are true.

- If $\lambda_1 \neq 0$, then

$$e(a,j,k) = a(2\lambda_1 - a + (-1)^j)(a-1,j,k) + \delta_{(j,k)=(1,0)}2(a,0,1), \tag{3.7}$$

$$H_i(a,1,k) = ((-1)^i\delta_{k=0} - \lambda_1\delta_{k=1})(a,1,\delta_{k=0}) + \begin{cases} \chi(f)^p(0,0,k), & \text{if } a = p-1, \\ (a+1,0,k), & \text{if } a \neq p-1, \end{cases} \tag{3.8}$$

$$H_i(a,0,k) = ((-1)^{i+1}\delta_{k=0} + \lambda_1\delta_{k=1})(a,0,\delta_{k=0}) + (-1)^i a(a-1,1,k), \tag{3.9}$$

$$E(a,j,k) = \delta_{(j,k)\neq(1,1)}(\delta_{k=0}(-1)^j 2a(a-1,j,1) - \delta_{j=0}a(a-1)(a-2,1,k)). \tag{3.10}$$

- If $\lambda_1 = 0$, then

$$e(a,j,0) = -a(a-(-1)^j)(a-1,j,0), \tag{3.11}$$

$$H_i(a,1,0) = \begin{cases} \chi(f)^p(0,0,0), & \text{if } a = p-1, \\ (a+1,0,0), & \text{if } a \neq p-1, \end{cases} \tag{3.12}$$

$$H_i(a,0,0) = (-1)^i a(a-1,1,0), \tag{3.13}$$

$$E(a,j,0) = -\delta_{j=0}a(a-1)(a-2,1,0).$$

In the following, we give a proof of the first part of Theorem 1.1.

**Proof of Formula (1.1) in Theorem 1.1** Let $x \in \mathrm{H}^0(\mathfrak{g}, Z_\chi(\lambda))$. Since $\mathrm{H}^0(\mathfrak{g}, Z_\chi(\lambda))$ is a weight-module, $x$ may be viewed as one of the following forms:

$$x_1(a+1,0,0) + x_2(a,1,1) \quad \text{or} \quad x_3(a+1,0,1) + x_4(a,1,0), \quad \text{if} \ \ \lambda_1 \neq 0;$$
$$y_1[a+1,0,0] + y_2[a,1,1] \quad \text{or} \quad y_3[a+1,0,1] + y_4[a,1,0], \quad \text{if} \ \ \lambda = (0,\lambda_2) \neq 0;$$
$$z_1(a+1,0,0) \quad \text{or} \quad z_2(a,1,0), \quad \text{if} \ \ \lambda = 0,$$

where $x_i, y_i, z_i \in \mathbb{F}$. By $Fx = 0$, we have

$$x_1 = x_3 = y_1 = y_3 = z_1 = 0.$$

**Case $\boldsymbol{\lambda = (0, \lambda_2) \neq 0}$** Note that

$$0 = Ey_2[a,1,1] \overset{(3.1)}{=} y_2\lambda_2(a[a-1,1,0] + [a,0,1]),$$
$$0 = Ey_4[a,1,0] \overset{(3.1)}{=} y_4(a[a-1,1,1] + \lambda_2[a,0,0]).$$

It follows that $y_2 = y_4 = 0$.

**Case $\boldsymbol{\lambda = 0}$** Note that

$$0 = H_1 z_2(a,1,0) \overset{(3.12)}{=} z_2(\delta_{a=p-1}\chi(f)^p(0,0,0) + \delta_{a\neq p-1}(a+1,0,0)).$$

It follows that $z_2 = 0$ in case $a \neq p-1$ or $\chi(f) \neq 0$. As a result, $x = z_2(p-1,1,0)$ if $\chi = 0$. It is clear that $(p-1,1,0)$ is in $\mathrm{H}^0(\mathfrak{g}, Z_\chi(\lambda))$ in case $(\lambda,\chi) = (0,0)$. Then $\mathrm{H}^0(\mathfrak{g}, Z_\chi(\lambda)) = \langle 0 \mid (p-1,1,0)\rangle$ if $(\lambda,\chi) = (0,0)$, and 0 otherwise.

**Case $\boldsymbol{\lambda_1 = \lambda_2 \neq 0}$ or $\boldsymbol{0 \neq \lambda_1^2 \neq \lambda_2^2 \neq 0}$** Note that

$$\begin{aligned} 0 = H_1 x_2(a,1,1) &\overset{(3.3)}{=} x_2(\delta_{a\neq p-1}(a+1,0,1) + \delta_{a=p-1}\chi(f)^P(0,0,1) - \lambda_1(a,1,0)), \\ 0 = H_1 x_4(a,1,0) &\overset{(3.3)}{=} x_4(\delta_{a\neq p-1}(a+1,0,0) + \delta_{a=p-1}\chi(f)^P(0,0,0) - (a,1,1)). \end{aligned} \tag{3.14}$$

It follows that $x_2 = x_4 = 0$.

**Case $\boldsymbol{\lambda = (\lambda_1, 0) \neq 0}$** By using (3.2), we may get equation (3.14), which implies that $x_2 = x_4 = 0$.

**Case $\boldsymbol{\lambda = (\lambda_1, -\lambda_1) \neq 0}$** By using (3.8), we may get equation (3.14), which implies that $x_2 = x_4 = 0$.

It follows that

$$\mathrm{H}^0(\mathfrak{g}, \mathrm{Z}_\chi(\lambda)) = \begin{cases} \langle 0 \mid (p-1, 1, 0)\rangle, & (\chi, \lambda) = (0, 0), \\ 0, & \text{otherwise}. \end{cases} \tag{3.15}$$

As a result formula (1.1) is true.

## 4 $\mathbf{H}^1(\mathfrak{g}, Z_\chi(\lambda))$ and $\mathbf{H}^1(\mathfrak{g}, L_\chi(\lambda))$

The following proposition determines the unique simple quotient $L_\chi(\lambda)$ of $Z_\chi(\lambda)$ in case $\lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2 := \mathbb{F}_p \times \mathbb{F}_p$.

**Proposition 4.1** *Let* $\lambda = (\lambda_1, -\lambda_1)$.

(1) *Let* $\lambda_1 \in \mathbb{F}_p \setminus \{0\}$. *Denote by* $M_1$ *the subspace of* $Z_\chi(\lambda)$ *with a basis*

$$(a, 1, 1), \quad (c, 0, 0), \quad (c, 0, 1), \quad (c, 1, 0), \quad (b+1, 0, 1) - \lambda_1(b, 1, 0), \quad (2\lambda_1, 1, 0),$$

*where*

$$0 \le a \le p-1, \quad 0 \le b \le \underline{2\lambda_1 - 1}, \quad \underline{2\lambda_1 + 1} \le c \le p-1.$$

*Denote by* $M_2$ *the subspace of* $Z_\chi(\lambda)$ *with a basis*

$$\{(a, 1, 1), (a+1, 0, 1) - \lambda_1(a, 1, 0) \mid 0 \le a \le p-1\}.$$

*Then*

$$L_\chi(\lambda) = \begin{cases} Z_\chi(\lambda)/M_1, & \chi = 0, \\ Z_\chi(\lambda)/M_2, & \chi \text{ is nilpotent}. \end{cases}$$

(2) *Let* $\lambda = 0$. *Denote by* $M_3$ *the subspace of* $Z_\chi(\lambda)$ *with a basis*

$$\{(a, 0, 0), (a, 1, 0), (0, 1, 0) \mid 1 \le a \le p-1\}.$$

*Then*

$$L_\chi(\lambda) = \begin{cases} Z_\chi(\lambda)/M_3, & \chi = 0, \\ Z_\chi(\lambda), & \chi \text{ is nilpotent}. \end{cases}$$

**Proof** Let $i = 1, 2$ or 3. From Remark 3.1(4), it is routine to show that $M_i$ is a submodule of $Z_\chi(\lambda)$ under the corresponding condition.

It is sufficient to show that $Z_\chi(\lambda)/M_i$ can be generated by any nonzero element. $M_i$ is a weight module, so is $Z_\chi(\lambda)/M_i$. Then it is sufficient to show that $Z_\chi(\lambda)/M_i$ can be generated by any $\mathbb{Z}_2$-homogeneous weight vector.

Note that $Z_0(\lambda)/M_1$ has a basis consisting of the following $\mathbb{Z}_2$-homogeneous weight vectors

$$\{(a, 0, 0), (a, 0, 1) \mid 0 \le a \le \underline{2\lambda_1}\}.$$

Let $0 \le a \le \underline{2\lambda_1}$. By (3.7), one sees that $Z_0(\lambda)/M_1$ can be generated by $(a,0,0)$. Then by $H_1(a,0,1) \overset{(3.9)}{=} \lambda_1(a,0,0)$ in $Z_0(\lambda)/M_1$ and $\lambda_1 \neq 0$, one sees that $Z_0(\lambda)/M_1$ can also be generated by $(a,0,1)$.

Note that $Z_\chi(\lambda)/M_2$ has a basis consisting of the following $\mathbb{Z}_2$-homogeneous weight vectors

$$\{(a,0,0),(a,0,1) \mid 0 \le a \le p-1\}.$$

Let $0 \le a \le p-1$. Hence $Z_\chi(\lambda)/M_2$ can be generated by $(a,0,0)$ in case $\chi$ is nilpotent because of (3.6). Then by $H_1(a,0,1) \overset{(3.9)}{=} \lambda_1(a,0,0)$ in $Z_\chi(\lambda)/M_2$ and $\lambda_1 \neq 0$, one sees that $Z_\chi(\lambda)/M_2$ can also be generated by $(a,0,1)$.

Note that $Z_\chi(\lambda)/M_3$ has a basis $\{(0,0,0)\}$, which can generate $Z_\chi(\lambda)/M_3$, obviously.

Let $\chi$ be nilpotent. Note that $Z_\chi(0)$ has a basis consisting of the following $\mathbb{Z}_2$-homogeneous weight vectors

$$\{(a,0,0),(a,1,0) \mid 0 \le a \le p-1\}.$$

Let $0 \le a \le p-1$. Hence $Z_\chi(0)$ can be generated by $(a,0,0)$ in case $\chi$ is nilpotent because of (3.6). Then by (3.12), $Z_\chi(0)$ can be generated by $(a,1,0)$.

### 4.1 Target-weight spaces

In view of Lemma 2.1, in order to determine $\mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)}$ and $\mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0)}$, we shall give the weight-spaces $Z_\chi(\lambda)_\alpha$ and $L_\chi(\lambda)_\alpha$ for $\alpha = (1,-1),(-1,1),0$. The weight-s $(1,-1),(-1,1)$ and $0$ are called the target-weights of $Z_\chi(\lambda)$ or $L_\chi(\lambda)$. The following two lemmas determine all the target-weight spaces of $Z_\chi(\lambda)$ and $L_\chi(\lambda)$.

**Lemma 4.1** *For $\lambda = (\lambda_1, \lambda_2)$, the target-weight spaces of $Z_\chi(\lambda)$ and $L_\chi(\lambda)$ are listed below.*
(1) *Let $\lambda_1 = -\lambda_2 \notin \mathbb{F}_p$ or $\lambda_1 \neq -\lambda_2$. Then*

$$Z_\chi(\lambda)_\alpha = L_\chi(\lambda)_\alpha = 0, \quad \alpha = 0,\ (1,-1),\ (-1,1).$$

(2) *Let $\lambda_1 = -\lambda_2 \in \mathbb{F}_p \backslash \{0\}$. Then*

$$Z_\chi(\lambda)_\alpha = \begin{cases} \langle (\lambda_1,0,0),(\lambda_1-1,1,1) \mid (\lambda_1-1,1,0),(\lambda_1,0,1)\rangle, & \alpha = 0, \\ \langle (\lambda_1+1,0,0),(\lambda_1,1,1) \mid (\lambda_1,1,0),(\lambda_1+1,0,1)\rangle, & \alpha = (-1,1), \\ \langle (\lambda_1-1,0,0),(\lambda_1-2,1,1) \mid (\lambda_1-2,1,0),(\lambda_1-1,0,1)\rangle, & \alpha = (1,-1). \end{cases}$$

(3) *Let $\lambda = 0$. Then*

$$Z_\chi(\lambda)_\alpha = \begin{cases} \langle (0,0,0) \mid (p-1,1,0)\rangle, & \alpha = 0, \\ \langle (p-1,0,0) \mid (p-2,1,0)\rangle, & \alpha = (1,-1), \\ \langle (1,0,0) \mid (0,1,0)\rangle, & \alpha = (-1,1), \end{cases}$$

$$L_0(\lambda)_\alpha = \begin{cases} \langle (0,0,0) \mid 0\rangle, & \alpha = 0, \\ 0, & \alpha = (1,-1),(-1,1). \end{cases}$$

**Proof** Note that in $Z_\chi(\lambda)$, $(a,j,k)$ or $[a,j,k]$ has the weight $(\lambda_1-a-j,\lambda_2+a+j)$, since

$$h_i(a,j,k)=(\delta_{i=1}(\lambda_1-a-j)+\delta_{i=2}(\lambda_2+a+j))(a,j,k).$$

Then the condition $\lambda_1+\lambda_2\neq 0$ or $\lambda_1=-\lambda_2\notin\mathbb{F}_p$ implies that

$$Z_\chi(\lambda)_\alpha=0,\quad \alpha=0,\ (1,-1),\ (-1,1).$$

Then

$$L_\chi(\lambda)_\alpha=0,\quad \alpha=0,\ (1,-1),\ (-1,1)$$

in case $\lambda_1+\lambda_2\neq 0$ or $\lambda_1=-\lambda_2\notin\mathbb{F}_p$. Hence (1) is true. By a direct computation, the conclusions on $Z_\chi(\lambda)$ in (2)–(3) are true from Remark 2.1 and (3.4).

Let $\lambda=0$. Then the conclusions on $L_\chi(\lambda)$ in (3) are true from Proposition 4.1.

**Lemma 4.2** *Let* $\lambda=(\lambda_1,-\lambda_1)\in\mathbb{F}_p^2$.

(1) *If* $\lambda_1=1$, *then*

$$L_0(\lambda)_\alpha=\begin{cases}\langle(1,0,0)\mid(0,1,0)\rangle, & \alpha=0,\\ \langle(2,0,0)\mid(1,1,0)\rangle, & \alpha=(-1,1),\\ \langle(0,0,0)\mid(0,0,1)\rangle, & \alpha=(1,-1).\end{cases}$$

(2) *If* $\lambda_1=p-1$, *then*

$$L_0(\lambda)_\alpha=\begin{cases}0, & \alpha=0,\\ \langle(0,0,0)\mid(0,0,1)\rangle, & \alpha=(-1,1),\\ \langle(p-2,0,0)\mid(p-3,1,0)\rangle, & \alpha=(1,-1).\end{cases}$$

(3) *If* $p\geq 5$ *and* $2\leq\lambda_1\leq\frac{p-1}{2}$, *then*

$$L_0(\lambda)_\alpha=\begin{cases}\langle(\lambda_1,0,0)\mid(\lambda_1-1,1,0)\rangle, & \alpha=0,\\ \langle(\lambda_1+1,0,0)\mid(\lambda_1,1,0)\rangle, & \alpha=(-1,1),\\ \langle(\lambda_1-1,0,0)\mid(\lambda_1-2,1,0)\rangle, & \alpha=(1,-1).\end{cases}$$

(4) *If* $p\geq 5$ *and* $\frac{p+1}{2}\leq\lambda_1\leq p-2$, *then* $L_0(\lambda)_\alpha=0$ *for* $\alpha=0,(-1,1),(1,-1)$.

(5) *Let* $\chi$ *be nilpotent and* $\lambda_1\in\mathbb{F}_p\backslash\{0\}$. *Then*

$$L_\chi(\lambda)_\alpha=\begin{cases}\langle(\lambda_1,0,0)\mid(\lambda_1-1,1,0)\rangle, & \alpha=0,\\ \langle(\lambda_1+1,0,0)\mid(\lambda_1,1,0)\rangle, & \alpha=(-1,1),\\ \langle(\lambda_1-1,0,0)\mid(\lambda_1-2,1,0)\rangle, & \alpha=(1,-1).\end{cases}$$

(6) *Let* $(\chi,\lambda)=(0,0)$. *Then*

$$L_\chi(\lambda)_\alpha=\begin{cases}\langle(0,0,0)\mid 0\rangle, & \alpha=0,\\ 0, & \alpha=(-1,1),(1,-1).\end{cases}$$

**Proof** **Case** $\boldsymbol{\chi=0}$ From Proposition 4.1, $L_0(\lambda)$ has a basis

$$\{(a,0,0),(2\lambda_1,0,0),(a,1,0),(0,0,1)\mid 0\leq a\leq \underline{2\lambda_1-1}\}$$

and the following equations hold in $L_0(\lambda)$:

$$\begin{aligned}
&(a+1,0,1) = \lambda_1(a,1,0) \neq 0, \quad 0 \leq a \leq \underline{2\lambda_1 - 1},\\
&(b+1,0,1) = \lambda_1(b,1,0) = 0, \quad \underline{2\lambda_1} \leq b \leq p-2,\\
&(p-1,1,0) = (c,0,0) = (d,1,1) = 0, \quad \underline{2\lambda_1 + 1} \leq c \leq p-1,\ 0 \leq d \leq p-1.
\end{aligned}$$

Then for $x = 0, \pm 1, -2$, it is necessary to compare $\underline{\lambda_1 + x}$ and $\underline{2\lambda_1}$ by Lemma 4.1(2). To that aim, we get the following conclusions:

- If $p \geq 5$ and $2 \leq \lambda_1 \leq \frac{p-1}{2}$, then
$$\underline{\lambda_1 - 2} < \underline{\lambda_1 - 1} < \underline{\lambda_1} < \underline{\lambda_1 + 1} < \underline{2\lambda_1}.$$
- If $p \geq 5$ and $\lambda_1 = 1$, then
$$\underline{\lambda_1 - 1} < \underline{\lambda_1} < \underline{\lambda_1 + 1} = \underline{2\lambda_1} < \underline{\lambda_1 - 2}.$$
- If $p \geq 5$ and $\lambda_1 = p-1$, then
$$\underline{\lambda_1 + 1} < \underline{\lambda_1 - 2} < \underline{\lambda_1 - 1} = \underline{2\lambda_1} < \underline{\lambda_1}.$$
- If $p \geq 5$ and $\lambda_1 = p-2$, then
$$\underline{\lambda_1 - 2} = \underline{2\lambda_1} < \underline{\lambda_1 - 1} < \underline{\lambda_1} < \underline{\lambda_1 + 1}.$$
- If $p \geq 7$ and $\frac{p+1}{2} \leq \lambda_1 \leq p-3$, then
$$\underline{2\lambda_1} < \underline{\lambda_1 - 2} < \underline{\lambda_1 - 1} < \underline{\lambda_1} < \underline{\lambda_1 + 1}.$$
- If $p = 3$ and $\lambda_1 = 1$, then
$$\underline{\lambda_1 - 1} < \underline{\lambda_1} < \underline{\lambda_1 + 1} = \underline{2\lambda_1} = \underline{\lambda_1 - 2}.$$
- If $p = 3$ and $\lambda_1 = 2$, then
$$\underline{\lambda_1 + 1} = \underline{\lambda_1 - 2} < \underline{\lambda_1 - 1} = \underline{2\lambda_1} < \underline{\lambda_1}.$$

Therefore, the conclusions on target-weight spaces of $L_0(\lambda)$ in (1)–(4) are true.

**Case $\chi$ being nilpotent** From Proposition 4.1, $L_\chi(\lambda)$ has a basis

$$\{(a,0,0), (a,1,0) \mid 0 \leq a \leq p-1\}$$

and the following equations hold in $L_\chi(\lambda)$

$$(a+1,0,1) = \lambda_1(a,1,0) \neq 0, \quad (a,1,1) = 0, \quad 0 \leq a \leq p-1.$$

Then (5) holds.

### 4.2 Weight-derivation spaces

In this subsection, we determine all of weight-derivations from $\mathfrak{g}$ to $Z_\chi(\lambda)$ or $L_\chi(\lambda)$.

From Remark 3.1 and Proposition 4.1, we get the following formulas about $L_\chi(\lambda)$ as the $\mathfrak{g}$-module in case $\lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2\backslash\{0\}$, which will be used in the future and only need a direct computation:

$$H_1(0,0,1) = H_2(0,0,1) = \lambda_1(0,0,0), \tag{4.1}$$

$$F(a,j,0) = \delta_{j=0}(\delta_{\chi=0}\delta_{0\le a\le \underline{2\lambda_1-1}} + \delta_{\chi\neq 0})(a,1,0), \tag{4.2}$$

$$\begin{aligned} H_i(a,1,0) &= (\delta_{\chi=0}\delta_{0\le a\le \underline{2\lambda_1-1}} + \delta_{\chi\neq 0})\delta_{a\neq p-1}(a+1,0,0) \\ &\quad + \delta_{\chi\neq 0}\delta_{a=p-1}\chi(f)^p(0,0,0), \end{aligned} \tag{4.3}$$

$$\begin{aligned} H_i(a,0,0) &= (\delta_{\chi=0}\delta_{1\le a\le \underline{2\lambda_1}} + \delta_{\chi\neq 0})(-1)^i(a-\lambda_1)(a-1,1,0) \\ &\quad + \delta_{(a,\chi)=(0,0)}(-1)^{i+1}(0,0,1), \end{aligned} \tag{4.4}$$

$$\begin{aligned} E(a,j,0) &= \delta_{j=0}(\delta_{\chi=0}\delta_{2\le a\le \underline{2\lambda_1}} + \delta_{\chi\neq 0})a(2\lambda_1 - a + 1)(a-2,1,0) \\ &\quad + \delta_{(j,\chi,a)=(0,0,1)}2(0,0,1), \end{aligned} \tag{4.5}$$

$$\begin{aligned} f(a,j,0) &= \delta_{a\neq p-1}(\delta_{\chi=0}\delta_{0\le a< \underline{2\lambda_1-1}} + \delta_{j=0}\delta_{\chi=0}\delta_{a=\underline{2\lambda_1-1}})(a+1,j,0) \\ &\quad + \delta_{\chi\neq 0}\delta_{a\neq p-1}(a+1,j,0) + \delta_{a=p-1}\delta_{\chi\neq 0}\chi(f)^p(0,0,0), \end{aligned} \tag{4.6}$$

$$\begin{aligned} e(a,j,0) &= a(2\lambda_1 - a + (-1)^j)(\delta_{(\chi,k,a)=(0,0,\underline{2\lambda_1})} + \delta_{\chi\neq 0})(a-1,j,0) \\ &\quad + a(2\lambda_1 - a + (-1)^j)\delta_{\chi=0}\delta_{0\le a\le \underline{2\lambda_1-1}}(a-1,j,0) \\ &\quad + \delta_{(\chi,j,k)=(0,1,0)}(2\delta_{a\neq 0}\lambda_1(a-1,1,0) + 2\delta_{a=0}(0,0,1)) \\ &\quad + 2\delta_{\chi\neq 0}\lambda_1(a-1,1,0). \end{aligned} \tag{4.7}$$

**Lemma 4.3** (1) *If $\chi$ is semisimple, then*

$$\mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)} = 0.$$

(2) *Let $\sigma \in \mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)}$ and $\tau \in \mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0)}$. Then*

$$\sigma(h_i) = \delta_{(\chi,\lambda)\neq(0,0)}\tau(h_i) = 0, \quad i = 1,2.$$

(3) *Let $\sigma \in \mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)}$ and $\tau \in \mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0),\bar{1}}$. If*

$$(i_1, j_1, k_1) \neq (\lambda_1, 1, 0), \quad (i_2, j_2, k_2) \neq (\lambda_1, 1, 1), \quad (i_3, j_3, k_3) \neq (\lambda_1 - 2, 1, 1),$$

*then*

$$\begin{aligned} &\delta_{(\chi,\lambda_1)\neq(0,1)}\tau(F) = \delta_{(\chi,\lambda_1)\neq(0,1)}\tau(E) = 0, \\ &\delta_{(\chi,\lambda_1)=(0,1)}(2\tau(F)^{(\lambda_1+1,0,0)} + \tau(E)^{(\lambda_1-1,0,0)}) = 0, \\ &\delta_{|\sigma|=\bar{0}}\sigma(F)^{(i_1,j_1,k_1)} = \delta_{|\sigma|=\bar{1}}\sigma(F)^{(i_2,j_2,k_2)} = \delta_{|\sigma|=\bar{1}}\sigma(E)^{(i_3,j_3,k_3)} = 0. \end{aligned}$$

**Proof** (1) Let $\chi$ be semisimple. There exists $h_i$ such that $\chi(h_i) \neq 0$. Since $\lambda \in \Lambda_\chi$, that is, $\lambda_i^p - \lambda_i = \chi(h_i)^p$, we get $\lambda_i \notin \mathbb{F}_p$. From Lemma 4.1(1),

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, Z_\chi(\lambda))_{(0)} = 0.$$

It follows that

$$\mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)} = 0.$$

(2) From Lemma 2.1, (3.15) and Lemma 4.1, it is sufficient to show

$$\delta_{|\sigma|=\bar{1}}\sigma(h_i)^{(p-1,1,0)} = 0, \quad i = 1, 2$$

in case $(\chi, \lambda) = (0, 0)$. Let $(\lambda, \chi) = (0, 0)$ and $|\sigma| = \bar{1}$. By the definition of derivations,

$$2\sigma(h_i) = \sigma([H_i, H_i]) = -2H_i\sigma(H_i).$$

Then

$$-\sigma(h_i)^{(p-1,1,0)}(p-1, 1, 0) = \sigma(H_i)^{(0,0,0)}H_i(0, 0, 0) \overset{(3.13)}{=} 0.$$

As a result, $\sigma(h_i)^{(p-1,1,0)} = 0$ in case $(\lambda, \chi) = (0, 0)$ and $|\sigma| = \bar{1}$.

(3) From Lemmas 4.1–4.2, it is sufficient to show

$$\begin{aligned}
&\delta_{|\sigma|=\bar{0}}\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1+1,0,1)} = \delta_{|\sigma|=\bar{1}}\sigma(F)^{(\lambda_1+1,0,0)} = \delta_{|\sigma|=\bar{1}}\sigma(E)^{(\lambda_1-1,0,0)} = 0,\\
&\delta_{(\chi,\lambda_1)\neq(0,1)}\tau(F)^{(\lambda_1+1,0,0)} = \delta_{(\chi,\lambda_1)\neq(0,1)}\tau(E)^{(\lambda_1-1,0,0)} = 0,\\
&\delta_{(\chi,\lambda_1)=(0,1)}(2\tau(F)^{(\lambda_1+1,0,0)} + \tau(E)^{(\lambda_1-1,0,0)}) = 0,
\end{aligned}$$

in case $\lambda_1 = -\lambda_2 \in \mathbb{F}_p$. Let $\lambda_1 = -\lambda_2 \in \mathbb{F}_p$ and $|\tau| = \bar{1}$. By the definition of weight-derivations and $[F, F] = 0$,

$$\begin{aligned}
0 = F\sigma(F) &= \delta_{|\sigma|=\bar{0}}(\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1+1,0,1)}F(\lambda_1+1, 0, 1) + \sigma(F)^{(\lambda_1,1,0)}F(\lambda_1, 1, 0))\\
&\quad + \delta_{|\sigma|=\bar{1}}(\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1,1,1)}F(\lambda_1, 1, 1) + \sigma(F)^{(\lambda_1+1,0,0)}F(\lambda_1+1, 0, 0))\\
&\overset{(3.5)}{=} \delta_{|\sigma|=\bar{0}}\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1+1,0,1)}(\lambda_1+1, 1, 1) + \delta_{|\sigma|=\bar{1}}\sigma(F)^{(\lambda_1+1,0,0)}(\lambda_1+1, 1, 0),
\end{aligned}$$

$$0 = F\tau(F) \overset{(4.2)}{=} \tau(F)^{(\lambda_1+1,0,0)}(\lambda_1+1, 1, 0).$$

Hence

$$\begin{gathered}
(\delta_{\chi=0}\delta_{\lambda_1\neq\pm 1} + \delta_{P_1})\tau(F)^{(\lambda_1+1,0,0)} = 0,\\
\delta_{|\sigma|=\bar{0}}\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1+1,0,1)} = \delta_{|\sigma|=\bar{1}}\sigma(F)^{(\lambda_1+1,0,0)} = 0,
\end{gathered}$$

where the first equation is from

$$(\lambda_1+1, 1, 0) \neq 0 \text{ in } L_\chi(\lambda) \Leftrightarrow \chi = 0 \text{ and } \lambda_1 \notin \{1, p-1\}, \text{or } \chi \text{ is nilpotent}.$$

For convenience, hereafter denote by $P_1$ the position that $\chi$ is nilpotent; $P_4$ the one that $2 \leq \lambda_1 \leq \frac{p-1}{2}$.

Let $|\sigma| = |\tau| = \bar{1}$ in the following. Then by $[E, F] = h_1 + h_2$ and (2),

$$\begin{aligned}
0 = -\sigma([E, F]) &= \delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1,1,1)}E(\lambda_1, 1, 1)\\
&\quad + \delta_{\lambda_1\neq 0}\sigma(E)^{(\lambda_1-2,1,1)}F(\lambda_1-2, 1, 1)\\
&\quad + \sigma(E)^{(\lambda_1-1,0,0)}F(\lambda_1-1, 0, 0)
\end{aligned}$$

$$\overset{(3.5),(3.10)}{=} \sigma(E)^{(\lambda_1-1,0,0)}F(\lambda_1-1,1,0),$$

$$\begin{aligned}0 = -\tau([E,F]) &= (\delta_{(\chi,\lambda_1)=(0,1)} + \delta_{(\chi,\lambda_1)=(0,p-1)})2\tau(F)^{(\lambda_1+1,0,0)}E(\lambda_1+1,0,0)\\ &\quad + \tau(E)^{(\lambda_1-1,0,0)}F(\lambda_1-1,0,0)\\ &\overset{(4.2),(4.5)}{=} \delta_{(\chi,\lambda_1)=(0,1)}(2\tau(F)^{(\lambda_1+1,0,0)} + \tau(E)^{(\lambda_1-1,0,0)})(\lambda_1-1,1,0)\\ &\quad + \delta_{\chi=0}\delta_{P_4}\tau(E)^{(\lambda_1-1,0,0)}(\lambda_1-1,1,0)\\ &\quad + \delta_{P_1}\tau(E)^{(\lambda_1-1,0,0)}(\lambda_1-1,1,0),\end{aligned}$$

where the last equation is from

$$(\lambda_1-1,1,0) \neq 0 \text{ in } L_\chi(\lambda) \Leftrightarrow \chi = 0 \text{ and } 1 \leq \lambda_1 \leq \frac{p-1}{2}, \text{ or } \chi \text{ is nilpotent}.$$

It follows that

$$\begin{aligned}&\delta_{|\sigma|=\bar{1}}\sigma(E)^{(\lambda_1-1,0,0)} = 0,\\ &\delta_{(\chi,\lambda_1)=(0,1)}(2\tau(F)^{(\lambda_1+1,0,0)} + \tau(E)^{(\lambda_1-1,0,0)}) = 0,\\ &\delta_{\chi=0}\delta_{P_4}\tau(E)^{(\lambda_1-1,0,0)} = 0,\\ &\delta_{P_1}\tau(E)^{(\lambda_1-1,0,0)} = 0.\end{aligned}$$

It remains to show $\delta_{(\chi,\lambda_1)=(0,p-1)}\tau(E)^{(\lambda_1-1,0,0)} = 0$. By $[E,E] = 0$ and the definition of derivations,

$$\begin{aligned}0 = E\tau(E) &= \tau(E)^{(\lambda_1-1,0,0)}E(\lambda_1-1,0,0)(\delta_{(\chi,\lambda_1)=(0,p-1)} + \delta_{(\chi,\lambda_1)=(0,1)})\\ &\overset{(4.5)}{=} \delta_{(\chi,\lambda_1)=(0,p-1)}\tau(E)^{(\lambda_1-1,0,0)}(\delta_{p=3}2(0,0,1) - \delta_{p\geq5}2(\lambda_1-3,1,0)).\end{aligned}$$

It follows that

$$\delta_{(\chi,\lambda_1)=(0,p-1)}\tau(E)^{(\lambda_1-1,0,0)} = 0.$$

For convenience, define two linear maps from $\mathfrak{g}$ to $Z_\chi(\lambda)$ as follows:

(1) Define $\varphi \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, Z_\chi(\lambda))$ by

$$\varphi(H_1) = \varphi(H_2) = (p-1,1,0), \quad \varphi(x) = 0,$$

where $x = e, E, F, f, h_1, h_2$.

(2) Define $\psi \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, Z_\chi(\lambda))$ by

$$\psi(H_1) = \psi(H_2) = -(0,0,0), \quad \psi(f) = (0,1,0), \quad \psi(x) = 0,$$

where $x = e, E, F, h_1, h_2$.

In the following, we give a proof of the second part of Theorem 1.1.

**Proof of Formula (1.2) in Theorem 1.1** Claim that

$$\mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0)} = \begin{cases}\langle \mathfrak{D}_{(\lambda_1-1,1,1)}, \mathfrak{D}_{(\lambda_1,0,0)} \mid \mathfrak{D}_{(\lambda_1-1,1,0)}, \mathfrak{D}_{(\lambda_1,0,1)}\rangle, & \lambda = (\lambda_1,-\lambda_1) \in \mathbb{F}_p^2\backslash\{0\},\\ \langle \mathfrak{D}_{(0,0,0)}, \varphi \mid \psi\rangle, & (\lambda,\chi) = (0,0),\\ \langle \mathfrak{D}_{(0,0,0)} \mid \mathfrak{D}_{(p-1,1,0)}\rangle, & \lambda = 0, \chi \text{ is nilpotent},\\ 0, & \text{otherwise}.\end{cases}$$

It is routine to show that $\varphi, \phi$ are weight-derivations if $(\lambda, \chi) = (0, 0)$. It is a standard fact that the space $\mathrm{Ider}(\mathfrak{g}, Z_\chi(\lambda))_{(0)} = \langle \mathfrak{D}_{(0,0,0)} \mid 0 \rangle$ in case $(\lambda, \chi) = (0, 0)$, which implies that the derivations $\varphi, \psi$ are not inner.

Firstly we show that the elements are linearly independent in the right sets of the above claim.

**Case $\boldsymbol{\lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2 \backslash \{0\}}$** On one hand, let

$$a\mathfrak{D}_{(\lambda_1-1,1,1)} + b\mathfrak{D}_{(\lambda_1,0,0)} = 0 = c\mathfrak{D}_{(\lambda_1,0,1)} + d\mathfrak{D}_{(\lambda_1-1,1,0)},$$

where $a, b, c, d \in \mathbb{F}$. For convenience, we write

$$\sigma = a\mathfrak{D}_{(\lambda_1-1,1,1)} + b\mathfrak{D}_{(\lambda_1,0,0)}, \quad \tau = c\mathfrak{D}_{(\lambda_1,0,1)} + d\mathfrak{D}_{(\lambda_1-1,1,0)}.$$

Then

$$0 = \sigma(F) = aF(\lambda_1 - 1, 1, 1) + bF(\lambda_1, 0, 0) \overset{(3.5)}{=} b(\lambda_1, 1, 0),$$
$$0 = \tau(F) = cF(\lambda_1, 0, 1) + dF(\lambda_1 - 1, 1, 0) \overset{(3.5)}{=} c(\lambda_1, 1, 1),$$

which implies $b = 0 = c$. Furthermore

$$0 = \sigma(f) = af(\lambda_1 - 1, 1, 1) \overset{(3.6)}{=} a(\lambda_1, 1, 1),$$
$$0 = \tau(f) = df(\lambda_1 - 1, 1, 0) \overset{(3.5)}{=} d(\lambda_1, 1, 0),$$

which implies $a = 0 = d$. Then

$$\{\mathfrak{D}_{(\lambda_1-1,1,1)}, \mathfrak{D}_{(\lambda_1,0,0)}\} \quad \text{or} \quad \{\mathfrak{D}_{(\lambda_1,0,1)}, \mathfrak{D}_{(\lambda_1-1,1,0)}\}$$

is linearly independent in case $\lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2 \backslash \{0\}$.

**Case $\boldsymbol{\lambda = 0}$** On one hand, it is routine to show that $\varphi$ and $\psi$ are weight-derivations. It is obvious that $\varphi, \psi$ and $\mathfrak{D}_{(0,0,0)}$ are linearly independent in case $(\lambda, \chi) = (0, 0)$, so are $\psi = \mathfrak{D}_{(p-1,1,0)}$ and $\varphi = \mathfrak{D}_{(0,0,0)}$ in case $\lambda = 0$ and $\chi$ is nilpotent.

Secondly, we prove that weight-derivations must be in the right sets of the above claim in case $\lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2$.

Let $\sigma \in \mathrm{Der}(\mathfrak{g}, Z_\chi(\lambda))_{(0),\overline{0}}$. On one hand, by the definition of derivations, Lemmas 4.1 and 4.3(3), we get the following equations:

$$\begin{aligned}
\sigma([H_i, F]) &= \sigma(F)^{(\lambda_1,1,0)} H_i(\lambda_1, 1, 0) + \delta_{\lambda_1 \neq 0} \sigma(H_i)^{(\lambda_1,0,1)} F(\lambda_1, 0, 1) \\
&\quad + \sigma(H_i)^{(\lambda_1-1,1,0)} F(\lambda_1 - 1, 1, 0) \\
&\overset{(3.5),(3.8),(3.12)}{=} \delta_{\lambda_1 \neq 0}((-1)^i \sigma(F)^{(\lambda_1,1,0)} + \sigma(H_i)^{(\lambda_1,0,1)})(\lambda_1, 1, 1) \\
&\quad + \delta_{\lambda_1 \neq p-1} \sigma(F)^{(\lambda_1,1,0)}(\lambda_1 + 1, 0, 0) + \delta_{\lambda_1 = p-1} \sigma(F)^{(\lambda_1,1,0)} \chi(f)^p (0, 0, 0), \\
\sigma([e, H_i]) &= \delta_{\lambda_1 \neq 0}(\sigma(H_i)^{(\lambda_1,0,1)} e(\lambda_1, 0, 1) - \sigma(e)^{(\lambda_1-2,1,1)} H_i(\lambda_1 - 2, 1, 1)) \\
&\quad + \sigma(H_i)^{(\lambda_1-1,1,0)} e(\lambda_1 - 1, 1, 0) - \sigma(e)^{(\lambda_1-1,0,0)} H_i(\lambda_1 - 1, 0, 0)
\end{aligned}$$

$$\begin{aligned}
&\overset{(3.7)-(3.9),(3.11),(3.12)}{=} \delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1,0,1)}\lambda_1(\lambda_1+1)+2\sigma(H_i)^{(\lambda_1-1,1,0)})(\lambda_1-1,0,1)\\
&\quad+\delta_{\lambda_1\neq 0}(-\delta_{\lambda_1\neq 1}\sigma(e)^{(\lambda_1-2,1,1)}+(-1)^i\sigma(e)^{(\lambda_1-1,0,0)})(\lambda_1-1,0,1)\\
&\quad+\delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1-1,1,0)}\lambda_1(\lambda_1-1)+\sigma(e)^{(\lambda_1-2,1,1)}\lambda_1)(\lambda_1-2,1,0)\\
&\quad+\delta_{\lambda_1\neq 0}(-(-1)^i\sigma(e)^{(\lambda_1-1,0,0)}(\lambda_1-1))(\lambda_1-2,1,0)\\
&\quad-\delta_{\lambda_1=1}\sigma(e)^{(\lambda_1-2,1,1)}\chi(f)^p(0,0,1)+\delta_{\lambda_1=0}(-1)^i\sigma(e)^{(\lambda_1-1,0,0)}(p-2,1,0),\\
\sigma([e,f]) &= \delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1,1,1)}e(\lambda_1,1,1)-\sigma(e)^{(\lambda_1-2,1,1)}f(\lambda_1-2,1,1))\\
&\quad+\sigma(f)^{(\lambda_1+1,0,0)}e(\lambda_1+1,0,0)-\sigma(e)^{(\lambda_1-1,0,0)}f(\lambda_1-1,0,0)\\
&\overset{(3.6),(3.7)}{=} \delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1,1,1)}\lambda_1(\lambda_1-1)-\delta_{\lambda_1\neq 1}\sigma(e)^{(\lambda_1-2,1,1)})(\lambda_1-1,1,1)\\
&\quad+\delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1+1,0,0)}\lambda_1(\lambda_1+1)-\sigma(e)^{(\lambda_1-1,0,0)})(\lambda_1,0,0)\\
&\quad-\delta_{\lambda_1=1}\sigma(e)^{(\lambda_1-2,1,1)}\chi(f)^p(0,1,1)-2\delta_{\lambda_1=0}\sigma(e)^{(\lambda_1-1,0,0)}\chi(f)^p(0,0,0),\\
\sigma([f,E]) &= \delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1,1,1)}E(\lambda_1,1,1)+\sigma(E)^{(\lambda_1-1,0,1)}f(\lambda_1-1,0,1))\\
&\quad-\sigma(f)^{(\lambda_1+1,0,0)}E(\lambda_1+1,0,0)+\sigma(E)^{(\lambda_1-2,1,0)}f(\lambda_1-2,1,0)\\
&\overset{(3.6),(3.10)}{=} \delta_{\lambda_1\neq 0}(\sigma(E)^{(\lambda_1-1,0,1)}-2\sigma(f)^{(\lambda_1+1,0,0)}(\lambda_1+1))(\lambda_1,0,1)\\
&\quad+\delta_{\lambda_1\neq 0}(\delta_{\lambda_1\neq 1}\sigma(E)^{(\lambda_1-2,1,0)}+\sigma(f)^{(\lambda_1+1,0,0)}\lambda_1(\lambda_1+1))(\lambda_1-1,1,0)\\
&\quad+\delta_{\lambda_1=1}\sigma(E)^{(\lambda_1-2,1,0)}\chi(f)^p(0,1,0)+\delta_{\lambda_1=0}\sigma(E)^{(\lambda_1-2,1,0)}(p-1,1,0),\\
\sigma([H_i,H_i]) &= 2\delta_{\lambda_1\neq 0}\sigma(H_i)^{(\lambda_1,0,1)}H_i(\lambda_1,0,1)+2\sigma(H_i)^{(\lambda_1-1,1,0)}H_i(\lambda_1-1,1,0)\\
&\overset{(3.8)-(3.9),(3.12)}{=} 2\delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1,0,1)}\lambda_1+\sigma(H_i)^{(\lambda_1-1,1,0)})((-1)^i(\lambda_1-1,1,1)\\
&\quad+(\lambda_1,0,0))+2\delta_{\lambda_1=0}\sigma(H_i)^{(\lambda_1-1,1,0)}\chi(f)^p(0,0,0).
\end{aligned}$$

On the other hand, from the multiplication of $\mathfrak{g}$ and Remark 3.1, we have the following equations:

$$\begin{aligned}
\sigma([H_i,H_i]) &= 2\sigma(h_i)=0,\quad \sigma([e,f])=\sigma(h_1-h_2)=0,\\
\sigma([H_i,F]) &= \sigma(f)=\delta_{\lambda_1\neq 0}\sigma(f)^{(\lambda_1,1,1)}(\lambda_1,1,1)+\sigma(f)^{(\lambda_1+1,0,0)}(\lambda_1+1,0,0),\\
\sigma([e,H_i]) &= (-1)^i\sigma(E)=(-1)^i\delta_{\lambda_1\neq 0}\sigma(E)^{(\lambda_1-1,0,1)}(\lambda_1-1,0,1)\\
&\quad+(-1)^i\sigma(E)^{(\lambda_1-2,1,0)}(\lambda_1-2,1,0),\\
\sigma([f,E]) &= \sigma(H_2-H_1)=\delta_{\lambda_1\neq 0}(\sigma(H_2)^{(\lambda_1,0,1)}-\sigma(H_1)^{(\lambda_1,0,1)})(\lambda_1,0,1)\\
&\quad+(\sigma(H_2)^{(\lambda_1-1,1,0)}-\sigma(H_1)^{(\lambda_1-1,1,0)})(\lambda_1-1,1,0),
\end{aligned}$$

where the first two equations are true from Lemma 4.3(2). Then each even weight-derivation is in the right set of the above claim.

Let $\sigma\in \mathrm{Der}(\mathfrak{g},Z_\chi(\lambda))_{(0),\bar{1}}$. On one hand, by the definition of derivations and Lemma 4.1, we get the following equations:

$$\begin{aligned}
-\sigma([H_i,F]) &= \delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1,1,1)}H_i(\lambda_1,1,1)+\sigma(H_i)^{(\lambda_1,0,0)}F(\lambda_1,0,0)\\
&\quad+\delta_{\lambda_1\neq 0}\sigma(H_i)^{(\lambda_1-1,1,1)}F(\lambda_1-1,1,1)\\
&\overset{(3.5),(3.8)}{=} \delta_{\lambda_1\neq 0}(-\lambda_1\sigma(F)^{(\lambda_1,1,1)}+\sigma(H_i)^{(\lambda_1,0,0)})(\lambda_1,1,0)
\end{aligned}$$

$$
\begin{aligned}
&+ \delta_{\lambda_1\neq 0}\delta_{\lambda_1\neq p-1}\sigma(F)^{(\lambda_1,1,1)}(\lambda_1+1,0,1)\\
&+ \delta_{\lambda_1=0}\sigma(H_i)^{(0,0,0)}(0,1,0) + \delta_{\lambda_1=p-1}\sigma(F)^{(p-1,1,1)}\chi(f)^p(0,0,1),\\
\sigma([e,H_i]) &= \delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1-1,1,1)}e(\lambda_1-1,1,1) + \sigma(e)^{(\lambda_1-1,0,1)}H_i(\lambda_1-1,0,1))\\
&+ \sigma(H_i)^{(\lambda_1,0,0)}e(\lambda_1,0,0) + \sigma(e)^{(\lambda_1-2,1,0)}H_i(\lambda_1-2,1,0)\\
&\overset{(3.7)-(3.9),(3.11)-(3.12)}{=} \delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1-1,1,1)}\lambda_1(\lambda_1-1)\\
&+ (-1)^i\sigma(e)^{(\lambda_1-2,1,0)})(\lambda_1-2,1,1)\\
&+ \delta_{\lambda_1\neq 0}(-1)^i(\lambda_1-1)\sigma(e)^{(\lambda_1-1,0,1)}(\lambda_1-2,1,1)\\
&+ \delta_{\lambda_1\neq 0}(\sigma(e)^{(\lambda_1-1,0,1)}\lambda_1 + \sigma(H_i)^{(\lambda_1,0,0)}\lambda_1(\lambda_1+1))(\lambda_1-1,0,0)\\
&+ \delta_{\lambda_1\neq 0}\delta_{\lambda_1\neq 1}\sigma(e)^{(\lambda_1-2,1,0)}(\lambda_1-1,0,0)\\
&+ \delta_{\lambda_1=1}\sigma(e)^{(p-1,1,0)}\chi(f)^p(0,0,0) + \delta_{\lambda_1=0}\sigma(e)^{(p-2,1,0)}(p-1,0,0),\\
\sigma([e,f]) &= \delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1+1,0,1)}e(\lambda_1+1,0,1) - \sigma(e)^{(\lambda_1-1,0,1)}f(\lambda_1-1,0,1))\\
&+ \sigma(f)^{(\lambda_1,1,0)}e(\lambda_1,1,0) - \sigma(e)^{(\lambda_1-2,1,0)}f(\lambda_1-2,1,0)\\
&\overset{(3.6)-(3.7),(3.11)}{=} \delta_{\lambda_1\neq 0}(\sigma(f)^{(\lambda_1+1,0,1)}\lambda_1(\lambda_1+1) + 2\sigma(f)^{(\lambda_1,1,0)}\\
&- \sigma(e)^{(\lambda_1-1,0,1)})(\lambda_1,0,1) - \delta_{\lambda_1\neq 1}\sigma(e)^{(\lambda_1-2,1,0)}(\lambda_1-1,1,0)\\
&- \delta_{\lambda_1=1}\sigma(e)^{(p-1,1,0)}\chi(f)^p(0,1,0)\\
&+ \delta_{\lambda_1\neq 0}\sigma(f)^{(\lambda_1,1,0)}\lambda_1(\lambda_1-1)(\lambda_1-1,1,0),\\
\sigma([f,H_i]) &= \delta_{\lambda_1\neq 0}(\sigma(H_i)^{(\lambda_1-1,1,1)}f(\lambda_1-1,1,1) + \sigma(f)^{(\lambda_1+1,0,1)}H_i(\lambda_1+-1,0,1))\\
&+ \sigma(H_i)^{(\lambda_1,0,0)}f(\lambda_1,0,0) + \sigma(f)^{(\lambda_1,1,0)}H_i(\lambda_1,1,0)\\
&\overset{(3.6)-(3.9),(3.12)}{=} \delta_{\lambda_1\neq 0}((\lambda_1+1)\sigma(f)^{(\lambda_1+1,0,1)} + \sigma(f)^{(\lambda_1,1,0)})(-1)^i(\lambda_1,1,1)\\
&+ \delta_{\lambda_1\neq 0}\lambda_1\sigma(f)^{(\lambda_1+1,0,1)}(\lambda_1+1,0,0) + \delta_{\lambda_1\neq 0}\sigma(H_i)^{(\lambda_1-1,1,1)}(\lambda_1,1,1)\\
&+ \delta_{\lambda_1\neq p-1}(\sigma(f)^{(\lambda_1,1,0)} + \sigma(H_i)^{(\lambda_1,0,0)})(\lambda_1+1,0,0)\\
&+ \delta_{\lambda_1=p-1}(\sigma(H_i)^{(p-1,0,0)} + \sigma(f)^{(p-1,1,0)})\chi(f)^p(0,0,0),
\end{aligned}
$$

where the first equation is from Lemma 4.3(3). On the other hand, from Lemmas 4.1 and 4.3(2), we have the following equations:

$$
\begin{aligned}
\sigma([e,f]) &= \sigma(h_1-h_2) = 0,\\
\sigma([e,H_i]) &= (-1)^i\sigma(E) = (-1)^i\delta_{\lambda_1\neq 0}\sigma(E)^{(\lambda_1-2,1,1)}(\lambda_1-2,1,1),\\
\sigma([f,H_i]) &= (-1)^{i+1}\sigma(F) = (-1)^{i+1}\delta_{\lambda_1\neq 0}\sigma(F)^{(\lambda_1,1,1)}(\lambda_1,1,1),\\
\sigma([H_i,F]) &= \sigma(f) = \delta_{\lambda_1\neq 0}\sigma(f)^{(\lambda_1+1,0,1)}(\lambda_1+1,0,1) + \sigma(f)^{(\lambda_1,1,0)}(\lambda_1,1,0),
\end{aligned}
$$

where the first three equations are true from Lemma 4.3. Then each even weight-derivation is in the right set of the above claim.

It follows that the above claim is true. As a result formula (1.2) holds.

For convenience, define some linear maps from $\mathfrak{g}$ to $L_\chi(\lambda)$ as follows:

(1) For $i = 1, 2$, define $\varphi_i \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L_\chi(\lambda))$ by

$$\varphi_i(h_i) = (0,0,0), \quad \varphi_i(x) = 0,$$

where $x = e, E, F, f, H_1, H_2, \delta_{i=1}h_2, \delta_{i=2}h_1$.

(2) For $i = 1, 2$, define $\psi_i \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L_\chi(\lambda))$ by

$$\psi_i(H_i) = (0,0,0), \quad \psi_i(x) = 0,$$

where $x = e, E, F, f, h_1, h_2, \delta_{i=1}H_2, \delta_{i=2}H_1$.

(3) Define $\varphi_3 \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L_\chi(\lambda))$ by

$$\varphi_3(f) = -(0,0,0), \quad \varphi_3(F) = (0,0,1), \quad \varphi_3(x) = 0,$$

where $x = e, E, h_1, h_2, H_1, H_2$.

(4) Define $\varphi_4 \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L_\chi(\lambda))$ by

$$\varphi_4(e) = (p-2,0,0), \quad \varphi_4(E) = (p-3,1,0), \quad \varphi_4(x) = 0,$$

where $x = f, F, h_1, h_2, H_1, H_2$.

(5) Define $\psi_3 \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L_\chi(\lambda))$ by

$$\psi_3(E) = -2(0,0,0), \quad \psi_3(F) = (2,0,0),$$

$$\psi_3(H_1) = (1,0,0), \quad \psi_3(H_2) = -(1,0,0), \quad \psi_3(x) = 0,$$

where $x = e, f, h_1, h_2$.

In the following, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2** Claim that

$$\mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0)} = \begin{cases} \langle \mathfrak{D}_{(\lambda_1+1,0,0)} \mid \mathfrak{D}_{(\lambda_1-1,1,0)} \rangle, & \chi = 0,\ \lambda = (\lambda_1, -\lambda_1) \in \mathbb{F}_p^2 \text{ with } \lambda_1 \neq 0, \pm 1, \\ \langle \varphi_1, \varphi_2 \mid \psi_1, \psi_2 \rangle, & (\lambda, \chi) = (0,0), \\ \langle \mathfrak{D}_{(1,0,0)} \mid \mathfrak{D}_{(0,1,0)}, \psi_3 \rangle, & \lambda = (1, p-1),\ \chi = 0, \\ \langle \varphi_3, \varphi_4 \mid 0 \rangle, & \lambda = (p-1, 1),\ \chi = 0, \\ \langle \mathfrak{D}_{(\lambda_1,0,0)} \mid \mathfrak{D}_{(\lambda_1-1,1,0)} \rangle, & \chi \text{ is nilpotent}, \\ 0, & \text{otherwise}. \end{cases}$$

It is routine to show that $\varphi_i, \psi_j$ are weight-derivations under the corresponding condition for $1 \le i \le 4, 1 \le j \le 3$. It is a standard fact that the space $\mathrm{Ider}(\mathfrak{g}, L_\chi(\lambda))_{(0)} = 0$ in case $(\lambda, \chi) = (0,0)$ or $\lambda = (p-1, 1), \chi = 0$, which implies that the derivations $\varphi_i, \psi_j$ are not inner, where $i = 1,2,3,4, j = 1,2$. In addition, $\mathrm{Ider}(\mathfrak{g}, L_\chi(\lambda))_{(0),\overline{1}} = \mathbb{F}\mathfrak{D}_{(0,1,0)}$ in case $\lambda = (1, p-1), \chi = 0$, which implies that the derivation $\psi_3$ is also not inner. It is true that the elements in the right sets of the above claim are linearly independent, the proof of which is omitted.

In the following, we prove that weight-derivations must be in the right sets of the above claim.

For convenience, denote by $P_2$ the proposition that $1 \le \lambda_1 \le \frac{p-1}{2}$; $P_3$ the one that $1 \le \lambda_1 \le \frac{p-1}{2}$ or $\lambda_1 = p-1$; $P_5$ the one that $2 \le \lambda_1 \le \frac{p-1}{2}$ or $\lambda_1 = p-1$.

Let $\sigma \in \mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0),\bar{0}}$. On one hand, by the definition of derivations and Lemma 4.2, we get the following equations:

$$\begin{aligned}
\sigma([H_i,F]) &= \sigma(F)^{(\lambda_1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})H_i(\lambda_1,1,0)\\
&\quad+\sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})F(\lambda_1-1,1,0)\\
&\quad+\delta_{(\chi,\lambda_1)=(0,p-1)}\sigma(F)^{(0,0,1)}H_i(0,0,1)\\
&\overset{(4.1)-(4.3)}{=}\sigma(F)^{(\lambda_1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1}\delta_{\lambda_1\neq p-1})(\lambda_1+1,0,0)\\
&\quad+\sigma(F)^{(\lambda_1,1,0)}\delta_{P_1}\delta_{\lambda_1=p-1}\chi(f)^p(0,0,0)\\
&\quad-\delta_{(\chi,\lambda_1)=(0,p-1)}\sigma(F)^{(0,0,1)}(0,0,0),\\
\sigma([H_i,E]) &= \sigma(E)^{(\lambda_1-2,1,0)}(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1})H_i(\lambda_1-2,1,0)\\
&\quad+\sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})E(\lambda_1-1,1,0)\\
&\quad+\delta_{(\chi,\lambda_1)=(0,1)}\sigma(E)^{(0,0,1)}H_i(0,0,1)\\
&\overset{(4.1)-(4.3),(4.5)}{=}\sigma(E)^{(\lambda_1-2,1,0)}(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1}\delta_{\lambda_1\neq 1})(\lambda_1-1,0,0)\\
&\quad+\sigma(E)^{(\lambda_1-2,1,0)}\delta_{P_1}\delta_{\lambda_1=1}\chi(f)^p(0,0,0)\\
&\quad+\delta_{(\chi,\lambda_1)=(0,1)}\sigma(E)^{(0,0,1)}(0,0,0),\\
\sigma([e,H_i]) &= \sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})e(\lambda_1-1,1,0)\\
&\quad-\sigma(e)^{(\lambda_1-1,0,0)}(\delta_{\chi=0}\delta_{P_3}+\delta_{P_1})H_i(\lambda_1-1,0,0)\\
&\overset{(4.4),(4.7)}{=}\lambda_1(\lambda_1+1)\sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_4}+\delta_{P_1})(\lambda_1-2,1,0)\\
&\quad+\delta_{(\chi,\lambda_1)=(0,1)}(2\sigma(H_i)^{(\lambda_1-1,1,0)}+(-1)^i\sigma(e)^{(\lambda_1-1,0,0)})(0,0,1)\\
&\quad+(-1)^i\sigma(e)^{(\lambda_1-1,0,0)}(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1})(\lambda_1-2,1,0),\\
\frac{1}{2}\sigma([H_i,H_i]) &= \sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})H_i(\lambda_1-1,1,0)\\
&\overset{(4.3)}{=}\sigma(H_i)^{(\lambda_1-1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})(\lambda_1,0,0).
\end{aligned}$$

On the other hand, from the multiplication of $\mathfrak{g}$, Lemmas 4.2–4.3, we have the following equations:

$$\begin{aligned}
\sigma([H_i,H_i]) &= 2\sigma(h_i)=0,\\
\sigma([H_i,F]) &= \sigma(f)=\sigma(f)^{(\lambda_1+1,0,1)}(\delta_{\chi=0}\delta_{P_3}+\delta_{P_1})(\lambda_1+1,0,1),\\
\sigma([H_i,E]) &= \sigma(e)=\sigma(e)^{(\lambda_1-1,0,0)}(\delta_{\chi=0}\delta_{P_3}+\delta_{P_1})(\lambda_1-1,0,0),\\
\sigma([e,H_i]) &= (-1)^i\sigma(E)=(-1)^i(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1})\sigma(E)^{(\lambda_1-2,1,0)}(\lambda_1-2,1,0)\\
&\quad+(-1)^i\delta_{(\chi,\lambda_1)=(0,1)}\sigma(E)^{(0,0,1)}(0,0,1).
\end{aligned}$$

Then each even weight-derivation is in the right sets of the above claim.

Let $\tau \in \mathrm{Der}(\mathfrak{g}, L_\chi(\lambda))_{(0),\bar{1}}$. On one hand, by the definition of derivations and Lemma 4.2, we get the following equations:

$$\tau([e,H_i]) = \tau(e)^{(\lambda_1-2,1,0)}(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1})H_i(\lambda_1-2,1,0)$$

$$
\begin{aligned}
&+\tau(H_i)^{(\lambda_1,0,,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})e(\lambda_1,0,0)\\
&+\delta_{(\chi,\lambda_1)=(0,1)}\tau(e)^{(0,0,1)}H_i(0,0,1)\\
&\overset{(4.1)-(4.3),(4.7)}{=}\tau(H_i)^{(\lambda_1,0,0)}\lambda_1(\lambda_1+1)(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})(\lambda_1-1,0,0)\\
&+\tau(e)^{(\lambda_1-2,1,0)}(\delta_{\chi=0}\delta_{P_5}+\delta_{P_1}\delta_{\lambda_1\neq 1})(\lambda_1-1,0,0)\\
&+\tau(e)^{(\lambda_1-2,1,0)}(\delta_{(\chi,\lambda_1)=(0,1)}+\delta_{P_1}\delta_{\lambda_1=1})(0,0,0),\\
\tau([f,H_i]) &= \tau(f)^{(\lambda_1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})H_i(\lambda_1,1,0)\\
&+\tau(H_i)^{(\lambda_1,0,,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1})f(\lambda_1,0,0)\\
&+\delta_{(\chi,\lambda_1)=(0,p-1)}\tau(f)^{(0,0,1)}H_i(0,0,1)\\
&\overset{(4.1)-(4.3),(4.6)}{=}\tau(H_i)^{(\lambda_1,0,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1}\delta_{\lambda_1\neq p-1})(\lambda_1+1,0,0)\\
&+\tau(f)^{(\lambda_1,1,0)}(\delta_{\chi=0}\delta_{P_2}+\delta_{P_1}\delta_{\lambda_1\neq p-1})(\lambda_1+1,0,0)\\
&+\tau(f)^{(\lambda_1,1,0)}(\delta_{P_1}\delta_{\lambda_1=p-1}\chi(f)^p-\delta_{(\chi,\lambda_1)=(0,p-1)})(0,0,0)\\
&+\tau(H_i)^{(\lambda_1,0,0)}\delta_{P_1}\delta_{\lambda_1=p-1}\chi(f)^p(0,0,0).
\end{aligned}
$$

On the other hand, from Lemma 4.3, we have the following equations:

$$
\begin{aligned}
\tau([e,H_i]) &= (-1)^i\tau(E) = (-1)^i\delta_{(\chi,\lambda_1)=(0,1)}\tau(E)^{(0,0,0)}(0,0,0),\\
\tau([f,H_i]) &= (-1)^{i+1}\tau(F) = (-1)^{i+1}\delta_{(\chi,\lambda_1)=(0,1)}\tau(F)^{(2,0,0)}(2,0,0).
\end{aligned}
$$

Then each odd weight-derivation is in the right sets of the above claim.

It follows that the above claim is true. As a result Theorem 1.2 holds.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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