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ON A THEOREM CONCERNING CARLESON MEASURE AND ITS APPLICATIONS

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Abstract

A measure μ is called Carleson measure, iff the condition of Carleson type $\mu(Q^*)\leqslant C|Q|^\alpha(\alpha\geqslant 1)$ is satisfied, where C is a constant independent of the cube Q with edge length $q>0$ in R^n and $Q^*=\{(y, t)\in R_+^{n+1}|y\in Q, 0<t<q\}$. In this paper the following theorem is established: "Suppose that μ is a Carleson measure, $\Phi(y, t)$ is continuous in R_+^{n+1} and $\Phi^*(x)=\sup\limits_{|y-x|<t,(y,t)\in R_+^{n+1}}|\Phi(y,t)|$. Then the following inequalities hold: (1) $\mu(\{|\Phi(y,t)|>s\})\leqslant C|\{\Phi^*(x)>s\}|^\alpha(\forall s>0)$, (2) $\int_{R_+^{n+1}}|\Phi(y,t)|^\alpha d\mu\leqslant C\left[\int_{R^n}\Phi^*(x)dx\right]^\alpha$, (3) $\int_{Q^*}|\Phi(y, t)|^\alpha d\mu\leqslant C\left[\int_{3Q}\Phi^*(x)dx\right]^\alpha$, where $3Q$ denotes the cube with the same center as Q but of edge length $3q$". In virtue of this theorem, the proof of three propositions in the paper of C. Fefferman and E. M. Stein (Acta Math., 1972) is simplified.

§ 1. Introduction

The following three propositions, which played an important role in the theory of H^p fuctions, were proved by C. Fefferman and E. M. Stein[1] in 1972.

Proposition 1. *Suppose* $1<P_0<r<\infty$ *and* $r/P_0=1+\lambda$. *If* u *is the Poisson integral of* $f\in L^{P_0}(R^n)$, *then*

$$\left\{\int_{R_+^{n+1}}t^{\lambda n}|u(x, t)|^r\frac{dx\,dt}{t}\right\}^{1/r}\leqslant C\|f\|_{P_0}, \tag{1.1}$$

where C *is a constant independent of* f *and* R_+^{n+1} *denotes the upper half space of* R^{n+1}.

Proposition 2. *Suppose* $1<P_0<r$ *and* $r/P_0=1+\lambda$. *Let* $f\in L^{P_0}(R^n)$, u *be its Poisson integral and*

$$T(x, h)=\{(y, t)\in R_+^{n+1}|\,|y-x|<h,\ 0<t<h\},$$

then the mapping

$$f\to\sup_{h>0}\left\{\frac{\int_{T(x,h)}t^{\lambda n}|u(y, t)|^r\frac{dy\,dt}{t}}{\int_{T(x,h)}t^{\lambda n}\frac{dy\,dt}{t}}\right\}^{1/r}=K_\lambda^r(f)$$

is of weak type (P_0, P_0) *and strong type* (P, P) *for* $P>P_0$, *i. e., the following two inequalities hold respectively*

$$|\{x\in R^n|K_\lambda^r(f)(x)>\alpha\}|\leqslant C\|f\|_{P_0}/\alpha\}^{P_0}, \tag{1.2}$$

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where for any measurable set E, $|E|$ denotes its Lebesgue measure, and

$$\|K_\lambda^*(f)\|_P \leqslant C\|f\|_P \quad (P>P_0). \tag{1.3}$$

Proposition 3. *For any $u=u(x, t)$ harmonic in R_+^{n+1}, let*

$$M_\lambda(u)(x_0)=\sup_{0<h<\infty}\left\{\frac{\int_{T(x_0,h)} t^{\lambda n}|u(y, t)|\frac{dy\,dt}{t}}{\int_{T(x_0,h)} t^{\lambda n}\frac{dy\,dt}{t}}\right\}.$$

Suppose now $0<P_0<1$ and $P_0^{-1}=1+\lambda$, then the mapping $u\to M_\lambda(u)$ is of weak type (P_0, P_0) and strong type (P, P) for $P_0<P<\infty$, i. e.

$$|\{x\in R^n \mid M_\lambda(u)(x)>\alpha\}|\leqslant C\{\|u\|_{H^{P_0}}/\alpha\}^{P_0}; \tag{1.4}$$

$$\|M_\lambda(u)\|_P\leqslant C\|u\|_{H^P}, \quad P_0<P<\infty. \tag{1.5}$$

In this note we shall establish a result concerning the measure of Carleson type. By making use of this result, the above mentioned three propositions will be easily deduced. thus we obtain a quite simple proof of the inequalities (1.1)–(1.5) concerning harmonic fuctions in R_+^{n+1}. It seems to us that our result may have further applications to the theory of H^P fuctions. Before we state our main result, we have to give some definitions:

Definition 1.1. *A positive measure μ on R_+^{n+1} satisfying the condition of Carleson type*

$$\mu(Q^*)\leqslant C|Q|^\alpha \quad (\alpha\geqslant 1) \tag{1.6}$$

is called measure of Carleson type or simply Carleson measure. Here Q is an arbitrary cube in R^n with sides parallel to the axes.

$$Q^*=\{(y, t)\in R_+^{n+1} \mid y\in Q,\ 0<t<q=\text{length of the side of } Q\}$$

and the constant C is independent of Q.

Definition 1.2. *Let $\psi(y, t)$ be a function defined on R_+^{n+1} and*

$$\Gamma(x)=\{(y, t)\in R_+^{n+1} \mid |y-x|<t\}$$

be the cone with vertex at $x\in R^n$. We denote by

$$\psi^*(x)=\sup_{(y,t)\in\Gamma(x)}|\psi(y, t)| \quad (x\in R^n)$$

the nontangential maximal function of ψ.

Our main result runs as follows:

Theorem. *Let μ be a Carleson measure with $\alpha\geqslant 1$ and $\psi(y, t)$ be a continuous function on R_+^{n+1}. Then we have*

a) $$\mu(\{|\psi(y, t)|>s\})\leqslant C|\{\psi^*(x)>s\}|^\alpha \tag{1.7}$$

for each $s>0$;

b) $$\int_{R_+^{n+1}}|\psi(y, t)|^\alpha d\mu\leqslant C\left\{\int_{R^n}\psi^*(x)\,dx\right\}^\alpha; \tag{1.8}$$

c) $$\int_{Q^*}|\psi(y, t)|^\alpha d\mu\leqslant C\left\{\int_{3Q}\psi^*(x)\,dx\right\}^\alpha \tag{1.9}$$

for each cube Q, where $3Q$ denotes the cube with the same center as Q but with the length *of side equal to* $3q$, *later on*, $KQ(K>0)$ *will have the same meaning.*

§ 2. Proof of the theorem

a) For each $s>0$, applying Whitney's theorem[2] to the open set $\{\psi^*(x)>s\}$, we obtain a pairwise disjoint family of cubes $\{Q_j\}$ with $\{\psi^*(x)>s\}=\bigcup_j Q_j$ and $\sum_j |Q_j| = 1\{x: \psi^*(x)>s\}|$. Moreover, these cubes have the following property: For an appropriate choise of the positive number K, actually we may choose $K=10\sqrt{n}$, each cube KQ_j intersects the set $\{x: \psi^*(x)\leqslant s\}$. We now have

$$\{(y, t)\in R_+^{n+1} \mid |\psi(y, t)|>s\} \subseteq \bigcup_j (10\sqrt{n}\,Q_j)\times[0, 10\sqrt{n}\,q_j]$$

$$=\bigcup_j (10\sqrt{n}\,Q_j)^* \quad (q_j=\text{length of the side of } Q_j).$$

To see this, note that if $(x, r)\in\{(y, t)\in R_+^{n+1} \mid |\psi(y, t)|>s\}$, then $\psi^*(x)>s$. Hence x belongs to some Q_j. It suffices to prove $r\in[0, 10\sqrt{n}\,q_j]$. Suppose $r>10\sqrt{n}\,q_j$, then we have $\psi^*(\tilde{x})>s$ as long as $\tilde{x}\in R^n$ and $|\tilde{x}-x|<r$. This contradicts the Whitney's theorem. So

$$\mu(\{(y, t)\in R_+^{n+1} \mid |\psi(y, t)|>s\})\leqslant\mu(\bigcup_j (10\sqrt{n}\,Q_j)^*\leqslant\sum_j \mu[(10\sqrt{n}\,Q_j)^*]$$

$$\leqslant C_1\sum_j |10\sqrt{n}\,Q_j|^\alpha\leqslant C\sum_j |Q_j|^\alpha\leqslant C\{\sum_j |Q_j|\}^\alpha=C|\{x: \psi^*(x)>s\}|^\alpha,$$

which proves a).

b) We have

$$\iint_{R_+^{n+1}} |\psi(y, t)|^\alpha d\mu=C_1\int_0^\infty s^{\alpha-1}\mu(\{(y, t) \mid |\psi(y, t)|>s\})ds$$

$$\leqslant C_2\int_0^\infty s^{\alpha-1}|\{x: \psi^*(x)>s\}|^\alpha ds$$

$$\leqslant C_2\{\sup_{s>0} s|\{x: \psi^*(x)>s\}|\}^{\alpha-1}\left(\int_0^\infty |\{x: \psi^*(x)>s\}|ds\right)$$

$$\leqslant C\left\{\int_{R^n}\psi^*(x)dx\right\}^{\alpha-1}\left\{\int_{R^n}\psi^*(x)dx\right\}=C\left\{\int_{R^n}\psi^*(x)dx\right\}^\alpha,$$

which proves b).

c) Let χ_{Q^*} be the characteristic function of Q^*, then

$$\iint_{R_+^{n+1}} |\psi(y, t)|^\alpha\chi_{Q^*}d\mu\leqslant C\left\{\int_{R^n}\tilde{\psi}^*(x)dx\right\}^\alpha,$$

where

$$\tilde{\psi}^*(x)=\sup_{(y,t)\in\Gamma(x)}|\psi(y, t)\chi_{Q^*}|.$$

It is clear that

$$\iint_{Q^*}|\psi(y, t)|^\alpha d\mu\leqslant C\left\{\int_{R^n}\tilde{\psi}^*(x)dx\right\}^\alpha$$

and

$$\tilde{\psi}^*(x)\leqslant\psi^*(x)=\sup_{(y,t)\in\Gamma(x)}|\psi(y, t)|.$$

It suffices to prove that $\tilde{\psi}^*(x)=0$ as long as $x \overline{\in} 3Q$. In fact, if $x \overline{\in} 3Q$ and $(y, t) \in Q^*$, then $|x-y| \geqslant 3q$. From this, $(y, t) \overline{\in} \Gamma(x)$. Therefore if $x \overline{\in} 3Q$ and $(y, t) \in \Gamma(x)$, then $(y, t) \overline{\in} Q^*$. It follows that $\tilde{\psi}^*(x)=0$.

Remark. We could use the ball S instead of Q and $S^*=\{(y, t) \in R_+^{n+1} \mid |y-x| < h, 0<t<h\}$ instead of Q^*, where x is the center of S and h is the radius of S.

§ 3. Applications—simpler proof of Propositions 1—3.

We require the following well-known definitions and lemmas:

Definition 3.1. *Let f be locally integrable on R^n. We define Hardy-Littlewood maximal fuction $M(f)$ by*

$$M(f)(x)=\sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all finite Q in R^n.

Lemma 3.2. *$M(f)$ is of weak type* (1.1) *and strong type (P, P) for $1<P\leqslant\infty$.*

The proof was given in [2].

Lemma 3.3. *Suppose u is the Poisson integral of an $f \in L^P(R^n)$ $(P \geqslant 1)$, then there is a constant C independent of f such that*

$$u^*(x) \leqslant CM(f)(x).$$

The proof was given in [4].

Lemma 3.4. (harmonic majorant) *Suppose $u(x, t) \in H^P$, $0<P<1$. Then there is a P_k, $0<P_k<P$, such that*

$$|u(x, t)| \leqslant \{v(x, t)\}^{1/P_k},$$

where $v(x, t)$ is the Poisson integral of $f \in L^{P/P_k}(R^n)$. Moreover

$$\|u\|_{H^P}=\|f\|_{P/P_k}^{1/P_k}.$$

The proof was given in [1] and [3].

Now we are going to give the apllications of our theorom.

i) Proof of Proposition 1. The measure $d\mu=t^{\lambda n-1} dy\, dt$ satisfies the condition of the Carleson measure with $\alpha=1+\lambda$. Since

$$\iint_{Q^*} t^{\lambda n-1} dy\, dt=|Q|^{1+\lambda}=|Q|^\alpha.$$

Applying the result b) with $\psi(y, t)=|u(y, t)|^{\frac{r}{1+\lambda}}$, we obtain

$$\left\{\iint_{R_+^{n+1}} t^{\lambda n}|u(y, t)|^r \frac{dy\, dt}{t}\right\}^{1/r}=\left\{\iint_{R_+^{n+1}} (|u(y, t)|^{\frac{r}{1+\lambda}})^{1+\lambda} d\mu\right\}^{1/r}$$

$$\leqslant C\left\{\int_{R^n} (|u(y, t)|^{\frac{r}{1+\lambda}})^*(x) dx\right\}^{(1+\lambda)/r} \leqslant C\left\{\int_{R^n} [u^*(x)]^{\frac{r}{1+\lambda}} dx\right\}^{(1+\lambda)/r}$$

$$\leqslant C\left\{\int_{R^n} [M(f)(x)]^{\frac{r}{1+\lambda}} dx\right\}^{(1+\lambda)/r}=C\|M(f)\|_{P_0} \leqslant C\|f\|_{P_0}.$$

ii) Proof of Proposition 2. When $P=\infty$, the proposition is obvious. By the Marcinkiewicz interpolation theorem, it suffices to prove that $K_\lambda^r(f)$ is of weak type (P_0, P_0). Now $d\mu=t^{\lambda n-1}dy\,dt$ satisfies the condition of the Carleson measure with $\alpha=1+\lambda$. Let $S=\{y\in R^n \mid |y-x|<h\}$ and $T(x, h)=S^*$. In virtue of the remark of our theorem, we have

$$\iint_{T(x,h)} t^{\lambda n}|u(y, t)|^r \frac{dy\,dt}{t}=\iint_{S^*}(|u(y, t)|^{\frac{r}{1+\lambda}})^{1+\lambda}\,d\mu$$

$$\leqslant C_1\left\{\int_{3S}(|u(y, t)|^{\frac{r}{1+\lambda}})^*(x)\,dx\right\}^{1+\lambda}.$$

Thus

$$\left\{\frac{\iint_{T(x,h)} t^{\lambda n}|u(y, t)|^r \frac{dy\,dt}{t}}{\iint_{T(x,h)} t^{\lambda n}\frac{dy\,dt}{t}}\right\}^{1/r}\leqslant C_2\left\{\frac{1}{|S|^{1+\lambda}}\left[\int_{3S}(u^*(x))^{\frac{r}{1+\lambda}}dx\right]^{1+\lambda}\right\}^{1/r}$$

$$\leqslant C_3\left\{\frac{1}{|3S|}\int_{3S}(u^*(x))^{\frac{r}{1+\lambda}}dx\right\}^{(1+\lambda)/r}\leqslant C_4\left\{\frac{1}{|3S|}\int_{3S}[M(f)(x)]^{\frac{r}{1+\lambda}}dx\right\}^{(1+\lambda)/r}$$

$$\leqslant C_4\{M([M(f)]^{\frac{r}{1+\lambda}})\}^{(1+\lambda)/r}=C_4\{M([M(f)]^{P_0})(x)\}^{1/P_0}.$$

For each $\alpha>0$ applying Lemma 3.2, we obtain

$$|\{x: K_\lambda^r(f)(x)>\alpha\}|\leqslant|\{x: C_4\{M([M(f)]^{P_0})(x)\}^{1/P_0}>\alpha\}|$$

$$=1\{x: C_4^{P_0}M([M(f)]^{P_0})(x)>\alpha^{P_0}\}|\leqslant C_5\alpha^{-P_0}\|(M(f)]^{P_0}\|_1$$

$$=C_5\alpha^{-P_0}\|M(f)\|_{P_0}^{P_0}\leqslant C\alpha^{-P_0}\|f\|_{P_0}^{P_0}.$$

iii) Proof of Proposition 3. In virtue of Lemma 3.4, we know that there is a P_k satisfying $0<P_k<P_0=(1+\lambda)^{-1}$, the function $v(x, t)$ being the Poisson integral of $f\in L^{P_0/P_k}(R^n)$, such that

$$|u(x, t)|\leqslant\{v(x, t)\}^{1/P_k}.$$

Moreover

$$\|u\|_{H^{P_0}}=\|f\|_{P_0/P_k}^{1/P_k}.$$

Let $\tilde{r}=P_k^{-1}$, $\tilde{P}_0=P_0/P_k$ and $\tilde{r}/\tilde{P}_0=1+\lambda$. From Proposition 2, the mapping

$$f\to\sup_{h>0}\left\{\frac{\iint_{T(x,h)} t^{\lambda n}|v(y, t)|^{\tilde{r}}\frac{dy\,dt}{t}}{\iint_{T(x,h)} t^{\lambda n}\frac{dy\,dt}{t}}\right\}^{1/\tilde{r}}$$

is of weak type $(\tilde{P}_0, \tilde{P}_0)$ and strong type (P, P) for $P>\tilde{P}_0$. Note that

$$\{M_\lambda(u)(x)\}^{1/\tilde{r}}\leqslant\sup_{h>0}\left\{\frac{\iint_{T(x,h)} t^{n\lambda}|v(y, t)|^{\tilde{r}}\frac{dy\,dt}{t}}{\iint_{T(x,h)} t^{\lambda n}\frac{dy\,dt}{t}}\right\}. \tag{3.1}$$

Thus we obtain for each $\alpha>0$

$$|\{x: M_\lambda(u)(x)>\alpha\}| = |\{x: (M_\lambda(u)(x))^{1/\tilde{r}}>\alpha^{1/\tilde{r}}\}|$$

$$\leqslant \left|\left\{x: \left(\sup_{h>0} \frac{\iint_{T(x,h)} t^{\lambda n}|v(y,t)|^{\tilde{r}}\frac{dy\,dt}{t}}{\iint_{T(x,h)} t^{\lambda n}\frac{dy\,dt}{t}}\right)^{1/\tilde{r}}>\alpha^{1/\tilde{r}}\right\}\right|$$

$$\leqslant C\alpha^{-\tilde{P}_0/\tilde{r}}\|f\|_{\tilde{P}_0}^{\tilde{P}_0} = C\alpha^{-P_0}\|u\|_{H^{P_0}}^{P_0}.$$

When $P>P_0$, it is clear that $P\tilde{r}>\tilde{P}_0$. Applying (3.1) and Propoistion 2, we see that $M_\lambda(u)(x)$ is of strong type (P, P) for $P>P_0$, i. e.

$$\|M_\lambda(u)\|_P \leqslant C\|u\|_{H^P}.$$

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