Chin. Ann. of Math.
4B (1) 1983

OPTIMAL CONTROL FOR LINEAR SYSTEM WITH QUADRATIC INDEFINITE CRITERION ON HILBERT SPACES

YOU YUNCHENG (尤云程)
(*Institute of Mathematics, Fudan University*)

Abstract

In this paper we consider optimal control problems for linear system on real separable Hilbert spaces with quadratic criterion, in which the state weighted operators are indefinite. Wellposedness and solvability, existence and uniqueness of optimal control are discussed. We prove that the closed-loop syntheses of optimal control are state linear feedback. Existence of solutions of related operator Riccati equations is investigated.

Starting from the contribution given by kalman[1], the optimal control theory of linear system with quadratic criterion has been developed to some extent in finite dimensional case[2] and it has been generalized to the infinite dimensional case[3–5]. Moreover, the case with singular criterion has also been discussed[6]. This theory has already been applied to the analytic design of computer control system[7].

We consider the optimal control problem of linear system described by a time-invariant evolution equation on real separable Hilbert spaces X, U with a quadratic indefinite criterion

$$\frac{dx}{dt}=Ax+Bu, \quad x(t)\in X,\ u(t)\in U,\ t\geqslant 0, \tag{1}$$

$$[0,\ t_1]:\ \inf_{u\in\mathscr{U}_{ad}}\left\{J(u)=\langle Q_1x(t_1),\ x(t_1)\rangle+\int_0^{t_1}[\langle Qx(t),\ x(t)\rangle+\langle Ru(t),\ u(t)\rangle]dt\right\}, \tag{2}$$

$$[0,\ \infty);\ \inf_{u\in\mathscr{U}_{ad}}\left\{J(u)=\int_0^{\infty}[\langle Qx(t),\ x(t)\rangle+\langle Ru(t),\ u(t)\rangle]dt\right\}. \tag{3}$$

In this paper, we assume that A is an infinitesimal generator of C_0-semigroup $T(t)$ on X, a closed operator with dense domain $D(A)$ and its range in X. Let $\mathscr{L}(\cdot,\cdot)$ denote a Banach space consisting of bounded linear operators from the former Banach space to the latter with operator norm. We assume that $B\in\mathscr{L}(U,X)$, Q, $Q_1\in\mathscr{L}(X,X)$, $R\in\mathscr{L}(U,U)$, Q, moreover Q_1 and R are self-adjoint.

For any $x(0)=x_0\in X$, $u\in L^{loc}([0,\ \infty),\ U)$, the mild solution of (1) is

$$x(t)=T(t)x_0+\int_0^t T(t-s)Bu(s)ds, \quad t\geqslant 0. \tag{4}$$

Manuscript received January 28, 1981.

We denote the problem (1)—(2) on finite time interval $[0, t_1]$ and the problem (1)—(3) on the infinite time interval $[0, \infty)$ simply by (FTP) and (ITP) respectively. Assume that

$$\mathscr{U}_{ad}=\mathscr{U}=\begin{cases} L^2([0, t_1], U), \\ L^2([0, \infty), U), \end{cases} \quad \text{and} \quad \mathscr{X}=\begin{cases} L^2([0, t_1], X), \ (FTP), \\ L^2([0, \infty), X), \ (ITP). \end{cases} \tag{5}$$

If a self-adjoint operator $K\geqslant\delta I$, $\delta>0$ is a constant, then K is called coercively positive definite and denoted by $K>0$. If K is nonnegative, we denote it by $K\geqslant 0$. In the present paper, all integrals involving operator-valued functions are Bochner integrals in strong sense.

Almost all of the works concerning the linear-quadratic optimal control problem, whether finite or infinite dimensional case, always made a priori hypothesis that

$$Q\geqslant 0, \ Q_1\geqslant 0, \ \text{and} \ R>0.$$

We refer to this case as the standard case. Molinari[8] pointed out that Q will be of negativity in the linear-quadratic problem relating to network analysis, and in his paper survey was made for the finite dimensional case. On the other hand, with a view to generalizing the frequency theorem of the absolute stability of nonlinear regulation systems, Yakubovitz[9-11] obtained some results about the problem on infinite time interval.

Under the conditions that A is an unbounded operator and Q, Q_1 are sign-indefinite, we investigate the infinite dimensional problem and discuss mainly its well-posedness and solvability, closed-loop optimal controls and the related operator Riccati equations.

§ 1. Well-posedness and Solvability of Optimal Control Problems

Definition 1. *Given an optimal control problem (FTP) or (ITP), if for any given initial value $x(0)=x_0\in X$*

$$V(x_0)\equiv\inf_{u\in\mathscr{U}_{ad}}\{J(u)\,|\,x(0)=x_0\}>-\infty,$$

then the problem is called well-posed; if in addition there exists $u^*\in\mathscr{U}_{ad}$ such that $J(u^*)=V(x_0)>-\infty$, then it is called solvable.

We denote the optimal control (corresponding to a given initial value), the corresponding optimal state trajectory and the optimal process by u^*, x^* and $\{x^*, u^*\}$ respectively. $V(x_0)$ is the optimum of criterion. The adjoint of a bounded linear operator K is denoted by K^*.

Theorem 1. *The necessary condition for (FTP) as well as (ITP) to be well-posed is $R\geqslant 0$.*

Proof For the contrary, we can choose an admissible control function $u(\cdot)$ such that $J(u)|_{x(0)=0}<0$. Thus $J(ku)|_{x(0)=0}=k^2J(u)|_{x(0)=0}\to-\infty(k\to\infty)$. This is a contradiction to wellposedness.

We allow Q and Q_1 to be sign-indefinite so that quadratic criterion concerned is non-convex. Using the extremum lemma for real quadratic functional defined on Hilbert space (see[10]), we attain the conditions characterizing well-posedness and solvabilty.

Theorem 2. *(FTP) is wellposed and solvable if and only if the following conditions are satisfied*

1) $H\equiv R+L^*QL+L_1^*Q_1L_1\geqslant 0$, *and*

2) *there exists an admissible process* $\{x, u\}$ *satisfying*

$$Ru(t)+B^*[T^*(t_1-t)Q_1x(t_1)+\int_t^{t_1}T^*(\sigma-t)Qx(\sigma)d\sigma]=0, \quad t\in[0, t_1], \tag{6}$$

where the operators L *and* L_1 *are defined by* $(Lu)(t)=\int_0^t T(t-s)Bu(t)ds$, *and* $L_1u=(Lu)(t_1)$ *respectively. Furthermore, if* $H>0$, *then for any* $x(0)=x_0\in X$, u^* *exists uniquely.*

Proof We denot e $Z=\mathscr{X}\times\mathscr{U}\times X$ and $\mathscr{M}_{x_0}$ is the set of all augmented processes $\{x(\cdot), u(\cdot), x(t_1)\}$ which have the same initial state x_0.

$$\mathscr{M}_{x_0}=\mathscr{M}_0+\{T(\cdot)x_0, 0, T(t_1)x_0\}.$$

According to the extremum lemma mentioned above, the necessay and sufficient conditions for this optimal control problem to be well-posed and solvable are

i) diag $(Q, R, Q_1)|\mathscr{U}_0\geqslant 0$, i. e., $\langle(R+L^*QL+L_1^*Q_1L_1)u, u\rangle_{\mathscr{U}}\geqslant 0$, $\forall u\in\mathscr{U}$;

ii) $\exists u^0\in\mathscr{U}$, such that $\left\langle \operatorname{diag}(Q, R, Q_1)\begin{pmatrix} Lu^0+T(\cdot)x_0 \\ u^0 \\ L_1u^0+T(t_1)x_0\end{pmatrix}, \begin{pmatrix} Lu \\ u \\ L_1u\end{pmatrix}\right\rangle_z=0$, $\forall u\in\mathscr{U}$.

From ii), there exists an admissible control process $\{x^0, u^0\}$ which satisfies

$$Ru+L^*Qx+L_1^*Q_1x(t_1)=0. \tag{7}$$

Noting that

$$(L^*\varphi)(t)=\int_t^{t_1}B^*T^*(\sigma-t)\varphi(\sigma)d\sigma, \quad (L_1^*x_1)(t)=B^*T^*(t_1-t)x_1, \tag{8}$$

we then obtain (6) by substitution of (8) into (7). As a consequence of the extremum lemma, $H>0$ implies that 2) holds and u^* exists uniquely.

Definition 2. *Operator* A *is called having the spectrum isolation property, if its spectrum* $\sigma(A)$ *and the imaginary axis are disjoint, and*

$$\exists C>0 \text{ (a positive constant)}, \ \|(i\omega I-A)^{-1}\|\leqslant C, \ \forall\omega\in\mathbf{R}.$$

Definition 3. *Linear system* $\{A, B\}$ *is called* L^2*-stabilizable (resp. exponentially stabilizable), if there exists an operator* $K\in\mathscr{L}(X, U)$ *such that the perturbed semigroup* $G(t)$ *generated by* $A+BK$ *satisfies*

$$\int_0^\infty \|G(t)x_0\|^2 dt < \infty, \ \forall x_0 \in X.$$

(resp. $\|G(t)\| \leqslant Me^{-\omega t}$, $\forall t \geqslant 0$, *and* M、ω *are two constants*).

Theorem 3. *Assume that A has the spectrvm isolation property and that $\{A, B\}$ is L^2-stabilizable.*

1) *If the following condition in frequency domain holds,*

$$\exists \delta_0 > 0, \ \Phi(i\omega) \equiv R + B^*(i\omega I - A)^{-1*} Q (i\omega I - A)^{-1} B \geqslant \delta_0 I_U, \ \forall \omega \in \mathbf{R}, \tag{9}$$

then (ITP) is well-posed and solvable, and $\forall x(0) = x_0 \in X$, u^* exists uniquely.

2) $\Phi(i\omega) \geqslant 0$ *for any $\omega \in \mathbf{R}$, is a necessary condition for (ITP) to be well-posed.*

Proof As $\{A, B\}$ is L^2-stabilizable, there is $K \in \mathscr{L}(X, U)$ such that $A+BK$ generates an exponentially stable semigroup $G(t)$. Let $Z = \mathscr{X} \times \mathscr{U}$, denote the set of those processes corresponding to the same initial state x_0 by $\mathscr{M}_{x_0}$, then $\mathscr{M}_{x_0} = \mathscr{M}_0 + \{G(\cdot)x_0, KG(\cdot)x_0\}$ is non-empty.

1) Conduct L^2-Fourier transform to $u = Kx + v$, $v \in \mathscr{U}$, and $x(t) = \int_0^t G(t-s)Bv(s)\,ds$. We obtain that the Fourier transforms $\tilde{x}$、$\tilde{u}$ of x、u satisfy the relation $\tilde{x}(\omega) = (i\omega I - A - BK)^{-1} B\tilde{v}(\omega) = (i\omega I - A)^{-1} B\tilde{u}(\omega)$, and

$$\begin{aligned}
\alpha &\equiv \inf_{u \in \mathscr{U} \setminus \{0\}} \frac{J(u)}{\|x\|_{\mathscr{X}}^2 + \|u\|_{\mathscr{U}}^2} \\
&= \inf_{u \in \mathscr{U} \setminus \{0\}} \frac{1}{\|\tilde{x}\|_{L^2}^2 + \|u\|_{L^2}^2} \int_{-\infty}^{+\infty} \langle \Phi(i\omega)\tilde{u}(\omega), \tilde{u}(\omega) \rangle d\omega \\
&\geqslant \delta_0 [\sup_{\omega \in \mathbf{R}} (\|(i\omega I - A)^{-1}\|^2 \|B\|^2) + 1]^{-1} = \alpha_0 > 0.
\end{aligned} \tag{10}$$

By the extremum lemma, the first part of this theorem is proved.

2) If there are $\omega_0 \in \mathbf{R}$, $u_0 \in U$, such that $\langle \Phi(i\omega_0)u_0, u_0 \rangle < 0$, then

$$x_0 = (i\omega_0 I - A)^{-1} B u_0 \in D(A).$$

consider the natural complex extensions of X, U, and the corresponding complex extensions of the related operators. For $t_1 > 0$, apply the control

$$\hat{u}(t) = \begin{cases} e^{i\omega_0 t} u_0 - KG(t)x_0, & t \in [0, t_1], \\ KG(t-t_1)(e^{i\omega_0 t_1} x_0 - G(t_1)x_0), & t \in (t_1, \infty). \end{cases} \tag{11}$$

Let $\hat{x}(t) = (L\hat{u})(t)$. On account of the exponential stability of $G(t)$, it follows that

$$J(\hat{u})|_{\hat{x}(0)=0} = t_1 \langle \Phi(i\omega_0)u_0, u_0 \rangle + O(\sqrt{t_1}), \quad t_1 \to +\infty. \tag{12}$$

If t_1 is chosen large sufficiently,

$$\operatorname{Re} J(\hat{u})|_{\hat{x}(0)=0} = J(u_1)|_{x_1(0)=0} + J(u_2)|_{x_2(0)=0} < 0, \quad \hat{u} = u_1 + iu_2, \quad \hat{x} = x_1 + ix_2.$$

Thus there must be a real process with zero initial state, either $\{x_1, u_1\}$ or $\{x_2, u_2\}$, such that the corresponding value of criterion is negative. The second part of this theorem is proved.

When A does not have the spectrum isolation property, the condition (9) may be reformulated, but omitted here.

Remark 1. The results we obtained show that in order to assure the well-posedness of (FTP) and (ITP), $J(u)$ must be nonnegative on the subspace $\mathscr{M}_0$ in Z.

Remark 2. The standard case Q, $Q_1 \geqslant 0$ and $R > 0$ becomes now a special case which satisfies the sufficient conditions $H > 0$ in (FTP) and (9) in (ITP).

§ 2. Open-loop Determination of Optimal Control

Theorem 4. *Assume that (FTP) is well-posed and solvable, any optimal control u^* must satisfy the following open-loop equation*

$$Ru(t) + \int_0^{t_1} W(t, \sigma) u(\sigma) d\sigma = -B^* [T^*(t_1 - t) Q_1 T(t_1) x_0 + \int_t^{t_1} T^*(\sigma - t) Q T(\sigma) x_0 d\sigma], \tag{13}$$

where

$$W(t, \sigma) = B^* T^*(t_1 - t) Q_1 T(t_1 - \sigma) B + \int_{\max(t, \sigma)}^{t_1} B^* T^*(s - t) Q T(s - \sigma) B ds, \ t, \ \sigma \in [0, t_1]. \tag{14}$$

If, in addition, $R > 0$, then optimal state trajectory x^* must satisfy

$$x(t) = T(t) x_0 - \int_0^t T(t-s) B R^{-1} B^* T^*(t_1 - s) Q_1 x(t_1) ds + \int_0^{t_1} K(t, \sigma) x(\sigma) d\sigma, \tag{15}$$

where

$$K(t, \sigma) = -\int_0^{\min(t, \sigma)} T(t-s) B R^{-1} B^* T^*(\sigma - s) Q ds, \ t, \ \sigma \in [0, t_1]. \tag{16}$$

Proof By the approach of operator transposition and by nterchanging the order of integration suitably according to the Fubini theorem for Bochner integrals ([12], p. 84), it follows that

$$(L^* Q L u)(t) = \int_0^{t_1} \left(\int_{\max(t, \sigma)}^{t_1} B^* T^*(s - t) Q T(s - \sigma) B u(\sigma) ds \right) d\sigma,$$

$$(L_1^* Q_1 L_1 u)(t) = \int_0^{t_1} B^* T^*(t_1 - t) Q_1 T(t_1 - \sigma) B u(\sigma) d\sigma.$$

Substitute them into (7) and take notice of (8), then (13), (14) are obtained. Besides, substitute (6) into the mild solution, we obtain (15), (16).

Theorem 5. *Assume that (ITP) is well-posed and solvable, and that $T(t)$ is exponentially stable, any optimal control u^* must satisfy the following open-loop equation,*

$$Ru(t) + \int_0^\infty \Pi(t, \sigma) u(\sigma) d\sigma = -\int_t^\infty B^* T^*(\sigma - t) Q T(\sigma) x_0 d\sigma, \quad t \in [0, \infty), \tag{17}$$

where

$$\Pi(t, \sigma) = \int_{\max(t, \sigma)}^\infty B^* T^*(s - t) Q T(s - \sigma) B ds, \quad t, \ \sigma \in [0, \infty). \tag{18}$$

If, in addition, $R > 0$, then optimal state trajectory x^* must satisfy

$$x(t) = T(t) x_0 + \int_0^\infty \Gamma(t, \sigma) x(\sigma) d\sigma, \quad t \in [0, \infty), \tag{19}$$

where

$$\Gamma(t, \sigma) = -\int_0^{\min(t,\sigma)} T(t-s)BR^{-1}B^*T^*(\sigma-s)Qds, \quad t, \sigma \in [0, \infty). \tag{20}$$

Proof Using the Hausdorff-Young inequality, we deduce from the exponential stability of $T(t)$ that $L \in \mathscr{L}(L^2([0, \infty), U), L^2([0, \infty), X))$. The rest of the argument is similar to that of Theorem 4.

§ 3. Closed-loop Syntheses of Optimal Control

Under the hypotheses that A is an unbounded operator and Q, Q_1 are indefinite, the syntheses of closed-loop optimal controls of (FTP) and (ITP) are key results but difficult to prove. Through the manipulation in integral form we obtain an equality by which the synthesis of (FTP) is achieved. Similarly the synthesis of (ITP) is achieved.

Lemma 1. *$\forall x(0) = x_0 \in X$, $\forall u \in \mathscr{U}$, and assume that $N \in \mathscr{L}(X, X)$ is an arbitrary self-adjoint operator, then it holds that $\forall 0 \le t \le \sigma < +\infty$,*

$$\langle NT(\sigma-t)x(t), T(\sigma-t)x(t)\rangle = \langle Nx(\sigma), x(\sigma)\rangle - 2\int_t^\sigma \langle NT(\sigma-s)x(s), T(\sigma-s)Bu(s)\rangle ds. \tag{21}$$

Its proof is simply a verification.

Lemma 2. *$\forall x(0) = x_0 \in X$, $\forall u \in \mathscr{U}$, and assume that $P(t)$ is the solution of the operator Riccati integral equation*

$$P(t) = T^*(t_1-t)Q_1T(t_1-t) + \int_t^{t_1} T^*(\sigma-t)[Q - P(\sigma)BR^{-1}B^*P(\sigma)]T(\sigma-t)d\sigma, \quad t \in [0, t_1] \tag{22}$$

(we always mean the solution as a strongly continuous, self-adjoint bounded operator-valued function). Then it holds that

$$\langle P(t)x(t), x(t)\rangle = \langle Q_1x(t_1), x(t_1)\rangle - 2\int_t^{t_1} \langle x(s), P(s)Bu(s)\rangle ds + \int_t^{t_1} \langle (Q - P(s)BR^{-1}B^*P(s))x(s), x(s)\rangle ds, \quad \forall t \in [0, t_1]. \tag{23}$$

Proof Using Lemma 1, we obtain the following two relations.

$$\langle Q_1x(t_1), x(t_1)\rangle - 2\int_t^{t_1} \langle Q_1T(t_1-s)x(s), T(t_1-s)Bu(s)\rangle ds = \langle Q_1T(t_1-t)x(t), T(t_1-t)x(t)\rangle. \tag{24}$$

$$\int_t^{t_1} \langle (Q - P(s)BR^{-1}B^*P(s))x(s), x(s)\rangle ds - 2\int_t^{t_1} \left\langle x(s), \left[\int_s^{t_1} T^*(\sigma-s)(Q - P(\sigma)BR^{-1}B^*P(\sigma))T(\sigma-s)d\sigma\right] Bu(s)\right\rangle ds = \left\langle \int_t^{t_1} T^*(\sigma-t)(Q - P(\sigma)BR^{-1}B^*P(\sigma))T(\sigma-t)d\sigma \cdot x(t), x(t)\right\rangle. \tag{25}$$

Summing up (24) and (25) leads immediately to (23).

Theorem 6. *Assume that $R > 0$. If the operator Riccati integral equation (22)* *admits a solution $P(\cdot)$, then for any $x(0)=x_0\in X$,*

$$u^*(t)=-R^{-1}B^*P(t)x^*(t),\quad t\in[0,\ t_1] \tag{26}$$

is closed-loop optimal control to (ETP). Moreover,

$$V(x_0)=\langle P(0)x_0,\ x_0\rangle,\quad \forall x_0\in X. \tag{27}$$

Proof By lemma 2, let $t=0$ in (23), it turns out that

$$\begin{aligned}\langle P(0)x_0,\ x_0\rangle&=\langle Q_1x(t_1),\ x(t_1)\rangle-2\int_0^{t_1}\langle u(t),\ B^*P(t)x(t)\rangle dt\\&+\int_0^{t_1}\langle(Q-P(t)BR^{-1}B^*P(t))x(t),\ x(t)\rangle dt\\&=J(u)-\int_0^{t_1}\langle R(u(t)+R^{-1}B^*P(t)x(t)),\ u(t)+R^{-1}B^*P(t)x(t)\rangle dt,\quad \forall u\in\mathscr{U}.\end{aligned} \tag{28}$$

On the other hand, the feedback control (26) yields the state trajectory being the mild solution of perturbed evolution equation

$$\frac{dx(t)}{dt}=(A-BR^{-1}B^*P(t))x(t),\quad x(0)=x_0. \tag{29}$$

Thus we know (26) represents an admissible control and it must be optimal.

Theorem 7. *Assume that $\{A,\ B\}$ is L^2-stabilizable, $R>0$. If the operator Riccati algebraic equation*

$$\langle Px,\ Ay\rangle+\langle Ax,\ Py\rangle+\langle Qx,\ y\rangle-\langle PBR^{-1}B^*Px,\ y\rangle=0,\quad \forall x,\ y\in D(A) \tag{30}$$

has a solution $P=P^*\in\mathscr{L}(X,\ X)$ such that $A-BR^{-1}B^*P$ generates an exponentially stable semigroup $G(t)$, then for any $x(0)=x_0\in X$,

$$u^*(t)=-R^{-1}B^*Px^*(t),\quad t\geqslant 0 \tag{31}$$

is closed-loop optimal control to (ITP), the optimal trajectory $x^*(t)=G(t)x_0$ and

$$V(x_0)=\langle Px_0,\ x_0\rangle,\quad \forall x_0\in X. \tag{32}$$

Proof Suppose that $K\in\mathscr{L}(X,\ U)$ is such an operator that $A+BK$ generates an exponentially stable semigroup $W(t)$. By the transform $u=Kx+v$, (ITP) is shifted to a new problem equivalently:

$$\begin{cases}\dfrac{dx}{dt}=(A+BK)x+Bv,\\ \inf\limits_{v\in\mathscr{U}}\left\{\hat{J}(v)=\displaystyle\int_0^\infty\left\langle\begin{pmatrix}Q_1 & S_1^*\\ S_1 & R\end{pmatrix}\begin{pmatrix}x(t)\\ v(t)\end{pmatrix},\ \begin{pmatrix}x(t)\\ v(t)\end{pmatrix}\right\rangle_{X\times U}dt\right\},\end{cases} \tag{33}$$

where $Q_1=Q+K^*RK$, $S_1=RK$. It is easy to verify that (30) is equivalent to the following equation

$$P=\int_0^\infty W^*(\eta)\left[Q_1-(S_1^*+PB)R^{-1}(B^*P+S_1)\right]W(\eta)d\eta. \tag{34}$$

Similarly to (23) and (28) we have

$$\langle Px_0, x_0\rangle=\int_0^\infty\langle(Q-PBK-K^*B^*P-PBR^{-1}B^*P)x(\eta), x(\eta)\rangle d\eta$$
$$-2\int_0^\infty\langle x(\eta), PBv(\eta)\rangle d\eta$$
$$=\hat{J}(v)-\int_0^\infty\{\langle Rv(t), v(t)\rangle+2\langle RKx(t), v(t)\rangle+\langle K^*RKx(t), x(t)\rangle$$
$$+2\langle B^*Px(t), Kx(t)+v(t)\rangle+\langle PBR^{-1}B^*Px(t), x(t)\rangle\}dt$$
$$=J(u)-\int_0^\infty\langle R(u(t)+R^{-1}B^*Px(t)), u(t)+R^{-1}B^*Px(t)\rangle dt$$
$$\leqslant J(u), \quad \forall u\in\mathscr{U}. \tag{35}$$

On the other hand, the state feedback control (31) is admissible and makes

$$J(u^*)=\langle Px_0, x_0\rangle.$$

§ 4. Existence of Solutions of Operator Riccati Equation

Theorem 8. *For (FTP) and its associated operator Riccati integral equation (22), if $R>0$ and $H>0$, then*

1) *there exists a unique solution $P(\cdot)$ of equation (22);*

2) *for any $x(0)=x_0\in X$, there exists a unique optimal control $u^*(\cdot)$ given by (26), where $P(\cdot)$ is the solution of (22) just mentioned;*

3) $x^*(t)=G(t, s)x^*(s)=G(t, 0)x_0$, $0\leqslant s\leqslant t\leqslant t_1$, where *$G(t, s)$ is the evolution operator determined by the perturbed evolution equation*

$$\frac{dx}{dt}=(A-BR^{-1}B^*P(t))x; \tag{36}$$

4) *the optimum of criterion $V(x_0)=\langle P(0)x_0, x_0\rangle$, $\forall x_0\in X$.*

Proof We give the proof of this theorem via the following Lemma 3 to Lemma 5. Denote $cx_0=T(\cdot)x_0$, $Dx_0=T(t_1)x_0$. Then we can write that

$$u^*(t)=-H^{-1}(L^*QC+L_1^*Q_1D)(t)x_0\equiv N(t)x_0, \quad t\in[0, t_1]. \tag{37}$$

Lemma 3. $\forall t\in[0, t_1]$, $N(t)\in\mathscr{L}(X, U)$. *Moreover,* $N\in\mathscr{L}(X, C([0, t_1], U))$.

Proof We see $L^*QC+L_1^*Q_1D\in\mathscr{L}(X, C([0, t_1], U))$. As the equation $Hu=f$ is

$$u(t)=-R^{-1}\int_0^{t_1}W(t, \sigma)u(\sigma)d\sigma+R^{-1}f(t), \ t\in[0, t_1], \ f\in C([0, t_1], U),$$

where $W(t, \sigma)$ is shown by (14), it follows that $u=H^{-1}f\in C([0, t_1], U)$ and H is a bijection on $c([0, t_1], U)$. Thus $H^{-1}\in\mathscr{L}(C([0, t_1], U), C([0, t_1], U))$.

Lemma 4. *Under the hypotheses in Theorem 8, the optimal control problem of the same linear system on time interval $[s, t_1]$ $(0\leqslant s\leqslant t_1)$ with the corresponding quadratic criterion*

$$\inf_{u\in L^2([s,t_1],U)}\Big\{J(u)=\langle Q_1x(t_1), x(t_1)\rangle$$
$$+\int_s^{t_1}[\langle Qx(t), x(t)\rangle+\langle Ru(t), u(t)\rangle]dt\,|\,x(s)=x_0\Big\} \tag{38}$$

has a uniqve optimal control for any given $x(s)=x_0\in X$, and its optimal trajectory is $x_s^*(t)=G(t,s)x_0$, $0\leqslant s\leqslant t\leqslant t_1$, where $\{G(t,s);\ 0\leqslant s\leqslant t\leqslant t_1\}$ is an evolution operator whose operator-norm is uniformly bounded.

Proof $H>0$ implies that for any given $x(s)=x_0\in X$, the optimal solution of this problem exists uniquely.

Similarly as in Lemma 3 we have

$$u_s^*(t)=N_s(t)x_0;\ \|N_s(t)\|_{\mathscr{L}(X,U)}\leqslant M,\quad \forall 0\leqslant s\leqslant t\leqslant t_1. \tag{39}$$

Substituting $u_s^*(t)=N_s(t)x_0$ into the mild solution of (1), we define $G(t,s)$ to be

$$\begin{aligned} G(t,s)x&=T(t-s)x+\int_s^t T(t-\eta)Bu_s^*(\eta)d\eta\\ &=T(t-s)x+\int_s^t T(t-\eta)BN_s(\eta)x d\eta,\quad \forall x\in X. \end{aligned}\tag{40}$$

From (39) we obtain those properties of $G(t,s)$.

According to the optimality principle, substitution of $x^*(t)=x_s^*(t)=G(t,s)x^*(s)$ into (6) implies

$$u^*(t)=-R^{-1}B^*P(t)x^*(t),\quad t\in[0,t_1], \tag{41}$$

where

$$P(t)=T^*(t_1-t)Q_1G(t_1,t)+\int_t^{t_1}T^*(\sigma-t)QG(\sigma,t)d\sigma,\quad t\in[0,t_1]. \tag{42}$$

From (40) and (41) it follows that

$$G(t,s)=T(t-s)+\int_s^t T(t-\eta)(-BR^{-1}B^*)P(\eta)G(\eta,s)d\eta,\quad 0\leqslant s\leqslant t\leqslant t_1. \tag{43}$$

Lemma 5. *$P(t)$, given by (42), is the unique solution of operator Riccati integral equation* (22).

Proof That $P(t)$ is strongly continuous can be easily verified from (42) and the properties of $G(t,s)$.

Substituting (43) into (42), we obtain

$$\begin{aligned} P(t)=&T^*(t_1-t)\Big[Q_1[T(t_1-t)+\int_t^{t_1}T(t_1-\eta)(-BR^{-1}B^*)P(\eta)G(\eta,t)d\eta\Big]\\ &+\int_t^{t_1}T^*(\sigma-t)Q[T(\sigma-t)+\int_t^{\sigma}T(\sigma-\eta)(-BR^{-1}B^*)P(\eta)G(\eta,t)d\eta]d\sigma. \end{aligned}\tag{44}$$

According to the duality theorem on the perturbation of evolution operator (see [14]), (43) is equivalent to the following

$$G(t,s)=T(t-s)+\int_s^t G(t,\eta)(-BR^{-1}B^*)P(\eta)T(\eta-s)d\eta,\quad 0\leqslant s\leqslant t\leqslant t_1. \tag{45}$$

Then we obtain

$$\begin{aligned} P(t)=&T^*(t_1-t)Q_1T(t_1-t)+\int_t^{t_1}T^*(\sigma-t)QT(\sigma-t)d\sigma\\ &-\int_t^{t_1}T^*(\eta-t)[T^*(t_1-\eta)Q_1G(t_1,\eta)]BR^{-1}B^*P(\eta)T(\eta-t)d\eta\\ &-\int_t^{t_1}T^*(\eta-t)\Big[\int_\eta^{t_1}T^*(\sigma-\eta)QG(\sigma,\eta)d\sigma\Big]BR^{-1}B^*P(\eta)T(\eta-t)d\eta\\ =&T^*(t_1-t)Q_1T(t_1-t)+\int_t^{t_1}T^*(\eta-t)[Q-P(\eta)BR^{-1}B^*P(\eta)]T(\eta-t)d\eta. \end{aligned}\tag{46}$$

In view of the uniqueness of the strongly continuous solution of (22), see [4] for its proof and it isn't influenced by the sign-indefinite property of Q and Q_1, it must occur that $P(t)=P^*(t)$, $t\in[0, t_1]$.

Thus, we have proved all the conclusions stated in Theorem 8. The similar approach can be used to discuss the solution of operator Riccati algebraic equation (30).

Theorem 9. *To (ITP) and its associated operator Riccati algebraic equation* (30), *assume that A has the spectrum isolation property and $\{A, B\}$ is L^2-stabilizable. If $R>0$ and the condition* (9) *is satisfied, then*

1) *there exists a solution $P=P^*\in\mathscr{L}(X, X)$ of equation* (30) *such that $A-BR^{-1}B^*P$ generates an exponentially stable semigroup $G(t)$, $t\geqslant 0$;*

2) *for any $x(0)=x_0\in X$, there exists a unique optimal control $u^*(\cdot)$ given by* (31), *where P is the solution of* (30) *just mentioned;*

3) *$x^*(t)=G(t)x_0$, $t\geqslant 0$, where $G(t)$ is the semigroup generated by $A-BR^{-1}B^*P$;*

4) *the optimum of criterion $V(x_0)=\langle Px_0, x_0\rangle$, $\forall x_0\in X$.*

Proof Since $\{A, B\}$ is L^2-stabilizable, it is sufficient to consider only the case when $T(t)$ itself is exponentially stable.

Now we assume, for simplicity, that $S=0$, otherwise, taking S into account, it will not influence our argument below.

According to Theorem 3, (9) implies that (ITP) is well-posed and solvable, what is more, the optimal control exists uniquely. From the open-loop relations which u^* satisfies, i. e., (17), (18) and $u^*(t)=-R^{-1}(L^*Qx^*)(t)=-R^{-1}B^*\int_t^\infty T^*(\sigma-t)Qx^*(\sigma)\,d\sigma$, using the following estimates,

$$\|\Pi(t, \sigma)\|_{\mathscr{L}(U,U)}\leqslant\frac{c}{2\omega}e^{-\omega|t-\sigma|}, \quad \omega>0;$$

$$\|[(L^*QL)\psi](t)\|_U\leqslant\int_0^\infty\|\Pi(t, \sigma)\psi(\sigma)\|_U\,d\sigma$$

$$\leqslant c\|\psi\|_{L^2([0,\infty),U)}, \quad \forall\psi\in L^2([0, \infty), U);$$

$$\|(L^*Q\varphi)(t)\|_U\leqslant c\|\varphi\|_{L^2([0,\infty),X)}, \quad \forall\varphi\in L^2([0, \infty), X);$$

$$\|L^*QT(\cdot)x_0\|_{L^2([0,\infty),U)}\leqslant c\|x_0\|, \quad \forall x_0\in X, \tag{47}$$

we obtain as in Lemma 3 that

$$u^*(t)=N(t)x_0, \quad t\in[0, \infty); \quad \|N(t)\|_{\mathscr{L}(X,U)}\leqslant M\,(\text{constant}), \quad \forall t\in[0, \infty).$$

Similarly, as in proving Lemma 4, we obtain $x^*(t)=G(t, s)x^*(s)=G(t, 0)x_0$, $0\leqslant s\leqslant t<\infty$. But here

$$G(t, s)=G(t-s, 0), \quad \forall 0\leqslant s\leqslant t<\infty. \tag{48}$$

Rewrite $G(t, 0)$ as $G(t)$, then $G(t)$ is a C_0-semigroup.

Similar to the proof of Lemma 5, we have

$$u^*(t)=-R^{-1}B^*Px^*(t), \quad P=\int_0^\infty T^*(t)QG(t)\,dt, \quad x^*(t)=G(t)x_0. \tag{49}$$

For arbitrary x, $y \in D(A)$, differentiating

$$\langle Px, y\rangle = \langle \int_t^\infty T^*(\eta - t) QG(\eta - t) x d\eta, y\rangle, \tag{50}$$

it follows that P given by (49) is a solution of (30), $P = P^* \in \mathscr{L}(X, X)$.

Now that $\{G(\cdot)x_0;\ \forall x_0 \in X\} \subset L^2([0, \infty), X)$, we know that $G(t)$ is L^2-stable, thus it is exponentially stable. From Theorem 7 it follows that $V(x_0) = \langle Px_0, x_0\rangle$, $\forall x_0 \in X$.

Theorem 10. *Assume that A has the spectrum isolation property, B^* is compact, $\{A, B\}$ is L^2*-stabilizable, and $R>0$. *If (ITP) is well-posed, then there is a solution of the equation* (30).

Proof We exert a positive perturbation to $F(x, u) = \langle Qx, x\rangle + \langle Ru, u\rangle$,

$$(0<\delta\leqslant 1), \quad F_\delta(x, u) = F(x, u) + \delta\|u\|^2. \tag{51}$$

Suppose that P_δ is the solution of equation $(30)_\delta$ corresponding to the perturbed optimal control problem $(\text{ITP})_\delta$. According to the spectral resolution, $P_\delta = P_\delta^+ - P_\delta^-$, where $P_\delta^+ \geqslant 0$ and $P_\delta^- > 0$. For any $x \in X$, we write $x = x^+ + x^-$. The well-posedness condition implies that

$$-\infty < V(x) = \inf_{\substack{u\in\mathscr{U}\\ x(0)=x}} J(u) \leqslant \inf_{\substack{u\in\mathscr{U}\\ x(0)=x}} \{J(u) + \delta\|u\|^2_{\mathscr{U}}\} = \langle P_\delta x, x\rangle \leqslant \langle P_1 x, x\rangle.$$

Moreover, we have

$$\begin{aligned} \|\sqrt{P_\delta^+}x\|^2 &= \langle P_\delta^+ x, x\rangle = \langle P_\delta^+ x^+, x^+\rangle = \langle P_\delta x^+, x^+\rangle \leqslant \langle P_1 x^+, x^+\rangle \leqslant \|P_1\|\|x^+\|^2, \\ \|\sqrt{P_\delta^-}x\|^2 &= \langle P_\delta^- x, x\rangle = \langle P_\delta^- x^-, x^-\rangle = -\langle P_\delta x^-, x^-\rangle \leqslant -\min(0, V(x^-)) \\ &= \max(0, V(x^-)). \end{aligned} \tag{52}$$

Thus

$$\|P_\delta\| \leqslant \text{const}, \quad \text{for all} \quad 0<\delta\leqslant 1.$$

There exists a sequence $\{\delta_n\}$ such that $\{P_{\delta_n}\}$ converges weakly to an operator $P = P^* \in \mathscr{L}(X, X)$ as $\delta_n \to +0$. It follows that

$$\lim_{n\to\infty} R_{\delta_n}^{-1} = \lim_{n\to\infty} (R + \delta_n I)^{-1} = R^{-1} \text{ (convergence in operator-norm)}.$$

The compactness of B^* implies further that

$$\lim_{n\to\infty} [-R_{\delta_n}^{-1} B^* P_{\delta_n}] = -R^{-1}B^*P \quad \text{(convergence in strong sense)}.$$

Let (30) δ_n satisfied by P_{δ_n} pass to the limit, it follows that P satisfies the equation (30).

Remark 3. In the standard case, where Q, $Q_1 \geqslant 0$ and $R>0$, the operator Riccati equations (22) and (30) are solvable respectively.

The author wishes to acknowledge his thanks for the direction given by Professor Jin Fulin and Associate Professor Li Xunjing.

References

[1] Kalman, R. E., *Bol. Soc. Mat. Mex.*, (1960), 102—119.

[2] Anderson, B. D. O., and Moore, J. B., Linear Optimal Control, Prentice-Hall, 1971.

[3] Balakrishnan, A. V., Applied Functional Analysis, Springer-Verlag, 1976.

[4] Curtain, R. F., and Pritchard, A. J., Infinite Dimensional Linear System Theory, Springer-Vrlag, 1978.

[5] Lions, J. L., Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, 1971.

[6] Clements, D. J., and Anderson, B. D. O., Singular Optimal Control——The Linear Quadratic Problem, Springer-Verlag, 1978.

[7] Li Xunjing, *Journal of Fudan University (Natural Science)*, **17**: 2 (1978), 38—48.

[8] Molinari, B. P., *Automatica*, **13** (1977), 347—357.

[9] Якубович, В. А., *Сиб. Мат. Жур.*, **15** (1974), 639—668.

[10] Якубович, В. А., *ibid.*, **16** (1975), 1081—1102.

[11] Лихтарников, А. Л., И. Якубович, В. А., *ibid*, **17** (1976), 1069—1085.

[12] Hille, E., and Phillips, R. S., Functional Analysis and Semigroups, Colloquium Publications, Amer. Math. Soc., **31** (1957).

[13] Kwan Chaochi, Lectures on Functional Analysis, Advaneed Education Publishing House, Beijing, 1958.

[14] Curtain, R. F. and Pritchard, A. J., *SIAM J. Control and Optimization*, **14** (1976), 951—983.