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SUBMANIFOLDS OF A HIGHER DIMENSIONAL SPHERE

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Abstract

Let M be an m-dimensional manifold immersed in $S^{m+k}(r)$. Then $\Delta X=\mu H-\frac{m}{r^2}X$, where X is the position vector of M and H is a unit normal vector field which is orthogonal to X everywhere.

If M is a compact connected manifold with parallel mean curvature vector field ξ immersed in $S^{m+k}(r)$, and the sectional curvature of M is not less than $\frac{1}{2}\left(\frac{1}{r^2}+|\xi|^2\right)$, then M is a small sphere.

For a compact connected hypersurface M in $S^{m+1}(r)$, if the sectional curvature is non-negative and the scalar curvature is proportional to the mean curvature everywhere, then M is a totally umbilical hypersurface or the multiplication of two totally umbilical submanifolds.

Introduction

In § 1, We improve the formula of Tsunero Takahashi (see[1]) and set up a necessary and sufficient condition for submanifolds of a higher dimensional sphere, then we consider the compact connected submanifold and get Theorem 1.

In § 2, using the method of kentaro yano and Bang-yen chen (cf. [2]), we prove two simple results. For the compact hypersurface in a sphere, when the scalar curvature is proportional to the mean curvature everywhere. We have Theorem 4.

In § 3, we calculate the Laplacian of the square norm of the second fundamental form for the submanifolds of a sphere. Then we get a formula of Simons' type.

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Preliminaries

Let M be an m-dimensional manifold immersed in an $(m+k+1)$-dimensional Euclidean space E^{m+k+1}. We choose a local vector field of orthonormal frames E_1, $\cdots$,

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E_{m+k+1} in E^{m+k+1} such that, restricted to M, the vectors $E_1, \cdots, E_m$ are tangent to M; and consequently. the remaining vectors $E_{m+1}, \cdots, E_{m+k+1}$ are normal to M. Let x be a position vector in E^{m+k+1}, restricted to M. We make use of X. We shall agree the following convention on the ranges of indices:

$$1\leqslant A, B, C\cdots\leqslant m+k+1, \quad 1\leqslant i, j, l, p, \cdots\leqslant m,$$
$$m+1\leqslant \alpha, \beta, \gamma, \cdots\leqslant m+k+1 \quad (k\geqslant 1).$$

It's well-known that the structure equations and fundamental formulas are given by

$$dx=\sum_A \omega^A E_A, \qquad dE_A=\sum_B \omega_A^B E_B,$$
$$d\omega^A=\sum_B \omega^B\wedge\omega_B^A, \quad d\omega_A^B=\sum_C \omega_A^C\wedge\omega_C^B,$$
$$\omega_A^B+\omega_B^A=0. \tag{1}$$

We restrict these forms to M. Then we have

$$dX=\sum_i \omega^i E_i, \quad \omega^\alpha=0, \quad d\omega^i=\sum_j \omega^j\wedge\omega_j^i,$$
$$\omega_j^i+\omega_i^j=0, \quad \omega_j^\alpha=\sum_i h_{ji}^\alpha\omega^i, \quad h_{ji}^\alpha=h_{ij}^\alpha,$$
$$d\omega_i^j=\sum_l \omega_i^l\wedge\omega_l^j+\frac{1}{2}\sum_{p,l} R_{i\,pl}^{j}\omega^p\wedge\omega^l,$$
$$R_{i\,pl}^{j}=\sum_\alpha (h_{il}^\alpha h_{jp}^\alpha-h_{ip}^\alpha h_{jl}^\alpha). \tag{2}$$

We call h_{ij}^α the components of the second fundamental form. If, for any α fixed, matrix (h_{ij}^α) has a unique characteristic root, then we call M pseudo-umbilical along E_α, We call $H_1=\frac{1}{m}\sum_{\alpha,i} h_{ii}^\alpha E_\alpha$ the mean curvature vector of M in E^{m+k+1}. We know

$$\Delta X=mH_1, \tag{3}$$

where Δ is a Laplacian of M in E^{m+k+1}.

Let $S^{m+k}(r)$ be a sphere which has a radius r with centre at the origin. Suppose $M\subset S^{m+k}(r)$. We ste $E_{m+k+1}=-\frac{1}{r}X$. Clearly

$$\omega_{m+k+1}^\alpha=0, \quad h_{ij}^{m+k+1}=\frac{1}{r}\delta_{ij}. \tag{4}$$

Let $\langle\ ,\ \rangle$ be inner product in E^{m+k+1}. As [1], we set $g_{ij}=\langle\nabla_i X, \nabla_j X\rangle$, g^{ij} is the element of inverse matrix, ∇_i is the covariant derivative with respect to the i-th variable ξ^i in a chart of M, where M is equipped with the induced metric from E^{m+k+1}.

§ 1. Necessary and sufficient condition for submanifolds in a sphere

If $M\subset S^{m+k}(r)$, Set $\sum_i h_{ii}^\alpha=h_\alpha$. From (3), we can see

$$\Delta X=\sum_{\alpha=m+1}^{m+k} h_\alpha E_\alpha-\frac{m}{r^2}X. \tag{5}$$

From a Lemma in [1], we know that $\frac{1}{m}\cdot\sum_{\alpha=m+1}^{m+k} h_\alpha E_\alpha$ is the mean curvature vector of M in $S^{m+k}(r)$. By H, we denote the unit vector. Then (5) becomes

$$\Delta X = \mu H - \frac{m}{r^2} X, \tag{6}$$

where $\langle H, X\rangle = 0$, $\mu = \left(\sum_{\alpha=m+1}^{m+k} h_\alpha^2\right)^{1/2}$.

Conversely. suppose $M \subset E^{m+k+1}$, and satisfies

$$\Delta X = \mu H - \lambda X, \tag{7}$$

where λ is a non-zero constant and μ is a function of M. H is a unit normal vector field which is orthogonal to X everywhere.

Firstly. We have

$$\langle \Delta X, \Delta X\rangle = \mu^2 + \lambda^2 \langle X, X\rangle. \tag{8}$$

Set

$$\Delta X = \sum_\alpha h_\alpha E_\alpha, \quad H = \sum_\alpha r_\alpha E_\alpha.$$

Then we have

$$\sum_\alpha h_\alpha^2 = \mu^2 + \lambda^2\langle X, X\rangle, \quad \sum_\alpha r_\alpha^2 = 1. \tag{9}$$

Because

$$\lambda X = \sum_\alpha (\mu r_\alpha - h_\alpha) E_\alpha, \tag{10}$$

we can see

$$\begin{aligned}\lambda\langle \nabla_j X, \nabla_i X\rangle &= \sum_\alpha (\mu r_\alpha - h_\alpha)\langle \nabla_j E_\alpha, \nabla_i X\rangle \\ &= -\sum_\alpha (\mu r_\alpha - h_\alpha)\langle E_\alpha, \nabla_j \nabla_i X\rangle,\end{aligned} \tag{11}$$

we get at once

$$\lambda m = -\sum_\alpha (\mu r_\alpha - h_\alpha) h_\alpha. \tag{12}$$

From (9), (10) and (12). we have

$$\langle X, X\rangle = \frac{m}{\lambda}. \tag{13}$$

Thus we have proved the following lemma:

Lemma. *Let M be an m-dimensional manifold immersed in a $S^{m+k}(r) \subset E^{m+k+1}$ $(k \geqslant 1)$. Then $\Delta X = \mu H - \frac{m}{r^2} X$, where H is a unit normal vector field which is orthogonal to X everywhere. $\frac{\mu}{m}$ is the length of the mean curvature vector of M in $S^{m+k}(r)$. X is the position vector of M in E^{m+k+1}.*

Conversely, let M be an m-dimensional manifold immersed in E^{m+k+1} $(k \geqslant 1)$, if $\Delta X = \mu H - \lambda X$, where λ is a non-zero constant. H is a unit normal vector field which is orthogonal to X everywhere. Then we have $\lambda > 0$, and $M \subset S^{m+k}\left(\sqrt{\frac{m}{\lambda}}\right)$. Moreover, $\frac{|\mu|}{m}$ is the length of it's mean curvature vector in $S^{m+k}\left(\sqrt{\frac{m}{\lambda}}\right)$.

Corollary. *If $\mu = 0$ in M, M is a minimal submanifold in $S^{m+k}(r)$. We can see that the necessary and sufficient condition is $\Delta X = -\frac{m}{r^2} X$. This is a result of Tsunero*

Takahashi (*cf*. [1]).

Secondly, we have the following theorem:

Theorem 1. *Suppose M is a compact connected and oriented m-dimensional submanifold in E^{m+k+1} ($k\geqslant 1$). M doesn't contain the origin of E^{m+k+1}. If the position vector field X parallels to $\Delta X-\mu H$ everywhere, where H is a unit normal vector field which is or thogonal to X, μ is a function of M.* Then M is contained in a hypersphere. and the length of the mean curvature vector of M in this hypersphere is $\frac{|\mu|}{m}$.*

Proof first that $\Delta X-\mu H\equiv 0$ on M we point out is impossible. If it were true, then we should have

$$0=\int_M \langle \Delta X-\mu H,\ X\rangle dV=\int_M \langle \Delta X,\ X\rangle dV$$
$$=\int_M \nabla_j\langle g^{ij}\nabla_i X,\ X\rangle dV-\int_M \langle g^{ij}\nabla_i X,\ \nabla_j X\rangle dV=-m \text{ vol } M.$$

This is a contradiction.

We define set U

$$U=\{P\in M\,|\,\Delta X(P)-\mu(P)H(P)\neq 0\}. \tag{14}$$

Because of continuity, we know that U is a non-empty open set. By $N(P)$ we denote the unit vector of $\Delta X(P)-\mu(P)H(P)$. Then

$$Y(P)=f(P)N(P), \quad \text{on } U. \tag{15}$$

Y_p is an arbitrary tangent vector of M at P. $\tilde{\nabla}$ is covariant derivative in E^{m+k+1}. Then

$$\tilde{\nabla}_{Y_p}X=(Y_pf)N(P)+f(P)\tilde{\nabla}_{Y_p}N. \tag{16}$$

Because $\langle\tilde{\nabla}_{Y_p}X,\ N(P)\rangle=\langle\tilde{\nabla}_{Y_p}N,\ N(P)\rangle=0$, we can see

$$Y_pf=0. \tag{17}$$

By U_i we denote the connected components of U, then

$$f=f_i \quad (\text{constant}), \quad \text{on } U_i. \tag{18}$$

From the condition of Theorem 1, we know $f_i\neq 0$, and $U_i\subset S^{m+k}(|f_i|)$. Since M is compact, there is a $d<\infty$ such that for arbitrary i, $|f_i|\leqslant d$. We make use of the lemma and get

$$\Delta X-\mu H=-\frac{m}{f_i^2}X, \quad \text{on } U_i. \tag{19}$$

From $\langle\Delta X-\mu H,\ \Delta X-\mu H\rangle=\frac{m^2}{f_i^2}\geqslant\frac{m^2}{d^2}>0$, we know at once that U is a closed set. M is connected, explicitly $U=M$, and all of f_i are equal to a constant f. Then

$$M\subset S^{m+k}(|f|).$$

Corollary. *If $\mu\equiv 0$ on M. From X parallels with ΔX. we know $\Delta X=\lambda X$, where λ is a negative constant. M is a minimal submanifold in a hypersphere. This is a result of Bang-yen chen* (*cf*. [3]).

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* in this paper, μ is a differentiable function.

§ 2. Some application

Suppose $M\subset S^{m+k}(r)$. In (6) we set $E_{m+k}=H$ and $E_{m+k+1}=-\frac{1}{r}X$. Then

$$\sum_i h_{ii}^{m+k}=\mu,\quad \sum_i h_{ii}^{\beta}=0\quad (m+1\leqslant\beta\leqslant m+k-1). \tag{20}$$

Because (h_{ij}^{m+k}) is a symmetric matrices, we choose $E_1, \cdots, E_m$ at a point of M such that $h_{ij}^{m+k}=k_i\delta_{ij}$, $\sum_i R_i=\mu$. The Ricci curvature

$$R(E_i,\ E_j)=\sum_p R_{ipj}^p=\left[k_i(\mu-k_j)+\frac{m-1}{r^2}\right]\delta_{ij}-\sum_{\alpha=m+1}^{m+k-1}\sum_p h_{ip}^{\alpha}h_{pj}^{\alpha}. \tag{21}$$

The scalar curvature

$$R=\sum_i R(E_i,\ E_i)=\frac{m(m-1)}{r^2}+\mu^2-\sum_i k_i^2-\sum_{\alpha=m+1}^{m+k-1}\sum_{i,p}(h_{ip}^{\alpha})^2. \tag{22}$$

Theorem 2. *M is an m-dimensional connected submanifold in E^{m+k+1} $(k\geqslant 1)$. If μ is a function of M. H is a unit normal vector field which is orthogonal to X everywhere. If $\Delta X-\mu H$ is a non-zero vector field which is perpendicular to H, this field (i. e. $\Delta X-\mu H$) is parallel in the normal bundle, and M is pseudo-umbilical along this direction. Then M is a submanifold in a sphere and owns $\frac{|\mu|}{m}$ as the length of mean curvature vector in this sphere.*

Proof Set $\Delta X-\mu H=fE_{m+1}$. Suppose $f>0$ and E_{m+1} is a unit vector field. Because fE_{m+1} is parallel in the normal bundle, so is E_{m+1}. f is a constant. From $dE_{m+1}=\sum_j\omega_{m+1}^jE_j$, and we choose $E_1, \cdots, E_m$ such that, $h_{ij}^{m+1}=\alpha\delta_{ij}$, then $\omega_{m+1}^j=-\alpha\omega^j$. We can see $m\alpha=\sum_i h_{ii}^{m+1}=f$, α is a constant. $\forall P\in M$, we define $\psi(P)=X(P)+\frac{mE_{m+1}(P)}{f}$. We know $d\psi=0$, ψ is a constant vector. Set $X-\psi=X_1$. Then $\langle X_1,\ X_1\rangle=\frac{m^2}{f^2}$, $\Delta X_1-\mu H=-\frac{f^2}{m}X_1$, $\langle X_1,\ H\rangle=0$. So we have Theorem 2.

Theorem 3. *Let M be an m-dimensional connected submanifold in $S^{m+k}(r)$. M has non-zero constant mean curvature vector field H_2.* Then M is a totally-umbilical hypersurface in a $S^{m+1}(r)\subset S^{m+k}(r)$ if and only if the scalar curvature*

$$R=\left(m^2|H_2|^2+\frac{m^2}{r^2}\right)\left(1-\frac{1}{m}\right).$$

Proof Because of $H_2=\frac{1}{m}\mu H$, then $\mu=m|H_2|$ (constant). We know

$$\left(\frac{1}{m}\sum_i k_i^2\right)^{1/2}\geqslant\frac{1}{m}\sum_i|k_i|\geqslant\frac{1}{m}\mu=|H_2|>0. \tag{23}$$

Then from (22). we get

$$R\leqslant\left(m^2|H_2|^2+\frac{m^2}{r^2}\right)\left(1-\frac{1}{m}\right). \tag{24}$$

* It means that $|H_2|$ is a constant, where $k\geqslant 1$.

If the equality holds. Then $h^\alpha_{ip}=0$ $(m+1\leqslant\alpha\leqslant m+k-1)$, $k_i=\frac{1}{m}\mu=|H_2|$ (constant) >0. So $\omega^i_\alpha=0$. By exterior differential, we have $\omega^{m+k}_\alpha\wedge\omega^i_{m+k}=0$. Because $\omega^{m+k}_i=k_i\omega^i$, then we have $\omega^{m+k}_\alpha=0$. Then we get

$$\omega^i_\alpha=\omega^{m+k}_\alpha=\omega^{m+k+1}_\alpha=0 \quad (m+1\leqslant\alpha\leqslant m+k-1).$$

From the proof of Theorem 1 in [4] or Theorem 10 in [5], we can see that M is contained in $(m+2)$-dimensional linear space in E^{m+k+1}. But $\langle X, X\rangle=r^2$, then $M\subset S^{m+1}(r)$, and M is a totally-umbilical hypersurface in this $S^{m+1}(r)$.

Conversely, it is trivial.

Theorem 4. *Let M be an m-dimensional compact connected and oriented hypersurface with non-negative sectional curvature immersed in $S^{m+1}(r)$. Suppose that the length of the mean curvature vector ξ in $S^{m+1}(r)$ doesn't vanish everywhere and $R=b|\xi|$, where R is the scalar curvature. b is a constant and $b>\frac{2}{r}m\sqrt{m(m-1)}$. Then M is a totally-umbilical hypersurface, or the product of two totally-umbilical curved submanifolds.*

Proof From the Lemma, we know $\mu H=m\xi$ and $\mu=m|\xi|$. From (9.3) in [5], we can see

$$\frac{1}{2}\Delta\Big[\sum_{i,j}(h^{m+1}_{ij})^2\Big]=\sum_{i<j}K_{ij}(k_j-k_i)^2+\sum_{i,j}h^{m+1}_{ij}\mu_{ij}+\sum_{i,j,l}(h^{m+1}_{ijl})^2, \tag{25}$$

where $H=E_{m+1}$, $-\frac{1}{r}X=E_{m+2}$, K_{ij} is the sectional curvature defined by E_i and E_j.

Integrating (25) over M and using stokes theorem, we have

$$\int_M\sum_{i<j}K_{ij}(k_j-k_i)^2dV+\int_M\sum_{i,j}h^{m+1}_{ij}\mu_{ij}dV+\int_M\sum_{i,j,l}(h^{m+1}_{ijl})^2dV=0. \tag{26}$$

For arbitrary j, we define a linear operator $*$ by

$$*(\omega^j)=(-1)^{j-1}\omega^1\wedge\cdots\wedge\omega^{j-1}\wedge\omega^{j+1}\wedge\cdots\wedge\omega^m. \tag{27}$$

By calculation we get

$$\sum_{i,j,l}d[h^{m+1}_{ij}h^{m+1}_{lli}*(\omega^j)]=\sum_{i,j}h^{m+1}_{ij}\mu_{ij}dV+\sum_i\Big(\sum_l h^{m+1}_{lli}\Big)^2dV. \tag{28}$$

Substituting (28) into (26), then we obtain

$$\int_M\sum_{i<j}K_{ij}(k_j-k_i)^2dV+\int_M\Big[\sum_{i,j,l}(h^{m+1}_{ijl})^2-\sum_i\Big(\sum_l h^{m+1}_{lli}\Big)^2\Big]dV=0. \tag{29}$$

Set $N=\mu^2-\sum_{i,j}(h^{m+1}_{ij})^2$. Because of Schwarz inequality, we can see

$$\sum_{i,j}(h^{m+1}_{ij})^2\sum_{i,j,l}(h^{m+1}_{ijl})^2\geqslant\sum_l\Big[\sum_{i,j}h^{m+1}_{ij}h^{m+1}_{ijl}\Big]^2. \tag{30}$$

From $R=\frac{m(m-1)}{r^2}+N$, then $\sum_l h^{m+1}_{ll}=\mu=\frac{m^2(m-1)}{r^2b}+\frac{mN}{b}$. We make use of above equation and obtain

$$\begin{aligned}\sum_{i,j}(h^{m+1}_{ij})^2\Big[\sum_{i,j,l}(h^{m+1}_{ijl})^2-\sum_i\Big(\sum_l h^{m+1}_{lli}\Big)^2\Big]&\geqslant N\sum_p\Big(\sum_l h^{m+1}_{llp}\Big)^2-\mu\sum_{l,p}h^{m+1}_{llp}N_p+\frac{1}{4}\sum_p N_p^2\\&=\Big[\frac{1}{4}-\frac{m^3(m-1)}{r^2b^2}\Big]\sum_p N_p^2\geqslant0,\end{aligned} \tag{31}$$

where $N_p=2\sum_l h_{ll}^{m+1}\sum_i h_{iip}^{m+1}-2\sum_{i,j}h_{ij}^{m+1}h_{ijp}^{m+1}$. We see at once

$$\sum_{i<j}K_{ij}(k_j-k_i)^2=0,\quad \sum_{ij,l}(h_{ijl}^{m+1})^2=\sum_i(\sum_l h_{lli}^{m+1})^2. \tag{32}$$

Then $\sum_p N_p^2=0$, i. e. N is a constant and μ is a constant, too. $\sum_l h_{lli}^{m+1}=0$. From (31), $h_{ijl}^{m+1}=0$. We obtain that k_i are constants. And from $K_{ij}(k_j-k_i)^2=0$, $K_{ij}=\frac{1}{r^2}+k_ik_j$, we can see that there are at most two distinct eigenvalues. According to [6]. Theorem 4 is proved.

§ 3. A Simon's formula

Suppose M is a compact connected m-dimensional manifold immersed in $S^{m+k}(r)\subset E^{m+k+1}$ $(k\geqslant 1)$. Let S be the square norm of the second fundamental form of M in E^{m+k+1}. We make use of the Lemma in § 1 and the method of Udo Simon (cf. [7]). By a long calculation, we obtain a Simon's formula when μ is a constant*

$$\frac{1}{2}\Delta S=\sum_A\sum_{i<j}\left(2K_{ij}^A-\frac{1}{r^2}\right)(\sigma_j^A-\sigma_i^A)^2+\sum_{i,j,l}\langle\nabla_lY_{ji},\ \nabla^lY^{ji}\rangle$$
$$+\mu\sum_{i,j}\langle\nabla_i\nabla_jH,\ Y^{ji}\rangle, \tag{33}$$

where $\nabla^l=\sum_j g^{lj}\nabla_j$, $Y_{ji}=\nabla_i\nabla_jX-\frac{1}{m}g_{ji}\Delta X$, $Y^{ji}=\sum_{l,p}g^{lj}g^{pi}Y_{lp}$. If x^A is A-th coordinate function of position vector X. For fixed A, $m\times m$ matrix $(\nabla_i\nabla_jx^A)$ is real symmetric matrix, when we choose the orthogonal basis, this matrix is also real symmetric matrix. By $\sigma_1^A,\cdots,\sigma_m^A$ we denote the eigenvalues and by $E_1^A,\cdots,E_m^A$ the unit eigenvector. K_{ij}^A denotes the sectional curvature defined by E_i^A, E_j^A.

Based on (33), we have the following theorem.

Theorem 5. *Let M be a compact connected m-dimensional submanifold in $S^{m+k}(r)\subset E^{m+k+1}$ $(k\geqslant 1)$. M has parallel mean curvature vector field ξ in $S^{m+k}(r)$.*

(1) If $\xi=0$ on M, and the sectional curvature of M is not less than $\frac{1}{2r^2}$, then M is a totally geodesic submanifold $S^m(r)$ (Udo Simon in 1977);

(2) if $\xi\neq 0$ on M, and the sectional curvature of M is not less than $\frac{1}{2}\left(\frac{1}{r^2}+|\xi|^2\right)$, then M is a small sphere (i. e. totally-umbilical hypersurface in $S^{m+1}(r)$).

Proof (1) Because $\mu=m|\xi|=0$, the third term on right hand side in (33) vanishes. (2) Because the sectional curvature of M is positive, from theorem g in [5], we know that M is a pseudo-umbilical submanifold, i. e. $k_i=\frac{\mu}{m}=|\xi|$ (constant). By calculation, we obtain

* Where $\Delta X=\mu H-\frac{m}{r^2}X$

$$\sum_{i,j}\langle\nabla_i\nabla_j H, Y^{ji}\rangle = -\frac{\mu}{m^2}\sum_A\sum_{i<j}(\sigma_i^A-\sigma_j^A)^2. \tag{34}$$

Then in (1) and (2), we have

$$\frac{1}{2}\Delta S = \sum_A\sum_{i<j}\left(2K_{ij}^A - \frac{1}{r^2} - |\xi|^2\right)(\sigma_j^A-\sigma_i^A)^2 + \sum_{i,j,l}\langle\nabla_l Y_{ji}, \nabla^l Y^{ji}\rangle. \tag{35}$$

By the hypothesis of the theorem, we can see $\langle\nabla_l Y_{ji}, \nabla^l Y^{ji}\rangle = 0$. Then $\nabla_l Y_{ji}=0$. Because M is irreducible, $Y_{ji}=\lambda g_{ji}$. But $\sum_{j,i} g^{ji}Y_{ij}\neq 0$, then $\lambda=0$, $Y_{ji}=0$, we make use of Gauss equation and Theorem 3. Theorem 5 is proved.

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