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ON THE PUTNAM-FUGLEDE THEOREM OF NON-NORMAL OPERATORS

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Abstract

In this paper we have extended the Putnam–Fuglede Theorem of nomal operators and discussed the condition for the Putnam–Fuglede Theorem holding. We have proved that if A and B^* are hyponomal operators and $AX=XB$, then $A^*X=XB^*$; that if A and B^* are semi-hyponomal operators and X is invertible operator such that $AXB=X$, then $A^*XB^*=X$; that if T is a contraction and P is a positive compact opertor such that $T^*PT=P$, then $\overline{R(P)}$ reduces T to unifary. In the meantime, we have proved that $AXB=X$ and $A^*XB^*=X$ both are true if and only if 1° $N(X)^\perp$, $\overline{R(X)}$ reduce B, A to invertible operators, respectively; 2° Let $X=WP_0$ be polar decomposition, then we have that $B^{-1}|_{N(X)^\perp}$ and $A|_{\overline{R(X)}}$ are unitary equivalent by W which is unitary from $N(X)^\perp$ to $R(X)$, and P_0 and B commute.

In the operator theory of Hilbert space the normal operators have an importat property: If A is intwining between normal operators N_1 and N_2, i. e. A satisfies

$$N_1A=AN_2,$$

then A is intwining between N_1^* and N_2^* too, i. e.

$$N_1^*A=AN_2^*.$$

This theorem is called the Putnam–Fuglede theorem. In recent years there have been many extensions of this theorem.

(1) In [1] it has been proved that if A and B^* are hyponormal operators and X is Hilbert–Schmidt operator, then we can obtain $A^*X=XB^*$ from $AX=XB$.

(2) In [4] it has been proved that if A and B^* are hyponormal operators and X is injective operator with dense range such that $AX=XB$, then we have that $A^*X=XB^*$ and A and B are normal operators.

(3) In [2] it has been proved that if A and B^* are subnormal operators, then we can obtain $A^*X=XB^*$ from $AX=XB$.

(4) In [6] Yan has proved another form of this theorem: If N_1 and N_2 are normal operators and X satisfies $N_1XN_2=X$, then we have $N_1^*XN_2^*=X$ too. Note that this form is more gereral, since we can deduce the above form from it.

(5) In [7] we have proved that if A and B^* are semi-hyponormal operators and $0\overline{\in}\sigma_P(B)$ and X is injective operator with dense range such that $AX=XB$, then we

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have $A^*X=XB^*$ and A and B are normal operators.

This paper divides into two parts. The first part proceeds to extend the putnam–Fuglede theorem and second part discuses the conditions for the Putnam–Fuglede theorem holding.

Theorem 1. *If A and B^* are hyponormal operators on Hilbert spaces H and H' respectively (and X is an operator from H' to H such that $AX=XB$, then $A^*X=XB^*$ and $N(X)^\perp$ and $\overline{R(X)}$ reduce B and A to normal operators respectively.*

Proof From $AX=XB$ we know that $N(X)^\perp$ and $\overline{R(X)}$ are invariant subspace of B^* and A respectively. Hence $A|_{\overline{R(X)}}$ and $B^*|_{N(X)^\perp}$ are hyponormal operators too. By the decompositions $H=\overline{R(X)}\oplus\overline{R(X)}^\perp$ and $H'=N(X)^\perp\oplus N(X)$, we have

$$A=\begin{pmatrix} A_1 & * \\ 0 & A_2\end{pmatrix},\quad B=\begin{pmatrix} B_1 & 0 \\ * & B_2\end{pmatrix},\quad X=\begin{pmatrix} X_1 & 0 \\ 0 & 0\end{pmatrix}.$$

Here $A_1=A|_{\overline{R(X)}}$ and $B_1^*=B^*|_{N(X)^\perp}$ are hyponormal operators and X_1 is injective operator with dense range. We can obtain $A_1X_1=X_1B_1$ from $AX=XB$. Hence we have $A_1^*X_1=X_1B_1^*$ too and A_1 and B_1 are normal. By the properties of the hyponormal operators we obtain that $N(X)^\perp$ and $\overline{R(X)}$ reduce B and A to normal operators respectively. Therefore we have

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2\end{pmatrix},\quad B=\begin{pmatrix} B_1 & 0 \\ 0 & B_2\end{pmatrix},\quad X=\begin{pmatrix} X_1 & 0 \\ 0 & 0\end{pmatrix}.$$

Hence we can obtain $A^*X=XB^*$ through calculation.

Theorem 2. *If A and B^* are hyponormal operators and X is invertible operator such that $AXB=X$, then $A^*XB^*=X$ and A and B are invertible normal operators.*

Proof From $AXB=X$ and X is invertible we know that $R(A)=H$ and $N(A^*)=\{0\}$, but A is hyponormal and so we have $N(A)\subset N(A^*)$, hence A is bijective i. e. A is invertibe. Likewise we have that B is invertible. Hence we can write $AXB=X$ as $A^{-1}X=XB$ and A^{-1} and B^* are hyponormal operators. Hence $A^{-1*}X=XB^*$ and A^{-1} and B are normal operators, therefore we obtain that $A^*XB^*=X$ and A and B are invertibe normal operators.

Theorem 3. *If A and B^* are semi-hyponormal operators and X is Hilbert–Schmidt operator such that $AXB=X$, then $A^*XB^*=X$.*

Proof All Hilbert–Schmidt operators compose a Hilbert space σ_2 with inner product

$$(X, Y)=\mathrm{tr}\ (XY^*).$$

We define an operator on σ_2 as follow

$$\mathscr{T}Y=AYB.$$

It is obvious that $\mathscr{T}\geqslant 0$ when $A\geqslant 0$ and $B\geqslant 0$ and

$$\mathscr{T}^*\mathscr{T}Y=A^*AYBB^*.$$

And hence we know that $\mathscr{T}$ is a semi-hyponormal operator as well as A and B^* are semi-hyponormal operators. But we have known that $\mathscr{T}X=X$ from the condition $AXB=X$. Therefore we have $\mathscr{T}^*X=X$ too by the properties of the semi-hyponormal operator, i. e. $A^*XB^*=X$.

Similar to Theorem 2, we have

Theorem 4. *If A and B^* are semi-hyponormal operators and X is invertible operator such that $AXB=X$, then we have that $A^*XB^*=X$ and A and B are invertible normal operators.*

Theorem 5. *If A is a semi-hyponormal operator, B is a normal operator, and X is injective operator with dense range such that $AXB=X$, then we have that $A^*XB^*=X$ and A is a invertible normal operator.*

Proof From $AXB=X$ we know $A^nXB^n=X$. For $\delta<\dfrac{1}{\|A\|+1}$ we denote $C_\delta=\{|z|<\delta\}$. If $B=\int zdE_z$ is the spectral decomposition of B, when $y\in E(C_\delta)H$ we have

$$\|Xy\|=\|A^nXB^ny\|\leqslant\|A\|^n\cdot\|X\|\delta^n\cdot\|y\|\leqslant\left(\frac{\|A\|}{1+\|A\|}\right)^n\cdot\|X\|\,\|y\|.$$

Let $n\to\infty$, we obtain $Xy=0$ and since X is injective we have $y=0$. Hence B is invertible. So that we can write $AXB=X$ as $AX=XB^{-1}$, and we have $A^*X=XB^{-1*}$ by (5), i. e. $A^*XB^*=X$. Thus, A is a normal operator.

Note. In Theorem 5 if A is a hyponormal operator, then we have that if X satisfies $AXB=X$ then $A^*XB^*=X$, and $N(X)^\perp$ and $\overline{R(X)}$ reduce B and A to normal operators respectively just as in Theorem 1. Here we only need to prove that $N(X)$ and $R(X)^\perp$ are invariant subspaces of B and A respectively from the condition $AXB=X$. In fact, from $AXB=X$ we know $R(X)\subset R(A)$ and since A is hyponormal so that $R(A)\subset\overline{R(A^*)}=N(A)^\perp$, hence if $x\in N(X)$, we have $AXBx=Xx=0$ and so that $XBx=0$, i. e. $Bx\in N(X)$. Similarly we can prove that $\overline{R(X)}$ is invariant under A.

Theorem 6. *Let A and B be operators with polar decompositions $A=UP$ and $B=P'V^*$ on Hilbert space H and H' respectively, and let X be an operator from H' to H with polar decomposition $X=WP_0$. Then $AXB=X$ and $A^*XB^*=X$ both hold if and only if following conditions are satisfied.*

1. $\overline{R(X)}$ and $N(X)^\perp$ reduce A and B to invertible operators respectively.

2. W, as unitary operator from $N(X)^\perp$ to $\overline{R(X)}$, transforms $B^{-1}|_{N(X)^\perp}$ to $A|_{\overline{R(X)}}$, and P_0 is commute with B.

Proof From $AXB=X$ and $A^*XB^*=X$, we can obtain $N(X)$ and $\overline{R(X)}$ reduce B and A respectively. In fact, if $x\in N(X)$, then $AXBx=Xx=0$. But $R(X)\subset R(A^*)=N(X)^\perp$, hence we have $XBx=0$ from $AXBx=0$, i. e. $Bx\in N(X)$. Similaly from $A^*XB^*x=Xx=0$, we have $XB^*x=0$, i. e. $B^*x\in N(X)$, so that $N(X)$ reduces B. Likewise, we know that $\overline{R(X)}$ reduces A.

Therefore we can suppose that X is injective operator with dense range, since we can disscus on $N(X)^{\perp}$ and $\overline{R(X)}$.

Now we prove that A and B are invertible. From $AXB=X$ and $A^*XB^*=X$ we can obtain $P^2X(P')^2=X$. Let $P'=\int_0^l \lambda dE'_\lambda$ when $\delta<\frac{1}{1+\|P\|}$ and $y\in E'_\delta H'$ we have

$$\|Xy\|=\|P^{2k}X(P')^{2k}y\|\leqslant\left(\frac{\|P\|}{1+\|P\|}\right)^{2k}\|X\|\|y\|.$$

Letting $K\to\infty$, we obtain $Xy=0$ and so that $y=0$. Hence P' and B are invertible. Likewise, we can prove that A is invertible.

Since P' is invertible, we can write $P^2X(P')^2=X$ as $P^2X=X(P')^{-2}$, hence we have $PX=X(P')^{-1}$, therefore $PXP'=X$.

From $PXP'=X$ we have $W^*PWP_0P'=P_0$, i. e. $\bar{P}P_0=P_0(P')^{-1}$ where $\bar{P}=W^*PW$. Taking adjoint we have $P_0\bar{P}=(P')^{-1}P_0$, so that $\bar{P}P_0^2=P_0(P')^{-1}P_0=P_0^2\bar{P}$ and we obtain $\bar{P}P_0=P_0\bar{P}$ and $P_0(\bar{P}-(\bar{P}')^{-1})=0$. Since X is injective, we have obtained $\bar{P}=(P')^{-1}$, i. e. $(P')^{-1}=W^*PW$. Thus, we have $(P')^{-1}P_0=P_0(P')^{-1}$, i. e. $P_0P'=P'P_0$.

From $AXB=X$ and $PXP'=X$, we can obtain $UXV^*=X$. From $A^*XB^*=X$ and $P^{-1}X(P')^{-1}=X$, we can obtain $U^*XV=X$. So that we have $X^*X=VX^*XV^*$, hence $P_0^2V=VP_0^2$ and $P_0V=VP_0$. From $UXV^*=X$, we obtain $(UWV^*-W)P_0=0$, but range of P_0 is dense, so that $UWV^*=W$, i. e. $V=W^*UW$. And so we have $B^*=W^*AW$ and P_0 is commute with B.

If A, B and X satisfy the conditions 1 and 2, we can obtain $AXB=X$ and $A^*XB^*=X$ directly through calculation.

Theorem 7. *Let A and B be operators on Hilbert space H and H' respectively, and let X be an operator from H' to H with polar decomposition $X=WP_0$. Then $AX=XB$ and $A^*X=XB^*$ both hold if and only if following conditions are satisfied.*

1. *$\overline{R(X)}$ and $N(X)^{\perp}$ reduce A and B respectively,*
2. *W as unitary operator from $N(X)^{\perp}$ to $\overline{R(X)}$ transforms $B|_{N(X)^{\perp}}$ to $A|_{\overline{R(X)}}$ and P_0 is commute with B.*

Proof From $AX=XB$ and $A^*X=XB^*$, we know that $(A+\lambda)X(B+\lambda)^{-1}=X$ and $(A+\lambda)^*X(B+\lambda)^{-1*}=X$ for $|\lambda|>\max(\|A\|, \|B\|)$. Hence we know $N(X)^{\perp}$ and $\overline{R(X)}$ reduce $B+\lambda$ and $A+\lambda$ respectively, and so they reduce B and A respectively too. And we also have that P_0 is commute with $(B+\lambda)^{-1}$ and so P_0 is commute with B too. On the other hand, from $AX=XB$ and $A^*X=XB^*$, we obtain that

$$AWP_0=WP_0B=WBP_0,\quad A^*WP_0=WP_0B^*=WB^*P_0.$$

Hence on $N(X)^{\perp}$ following relations hold

$$AW=WB,\quad A^*W=WB^*,$$

i. e. W, as unitary operator from $N(X)^{\perp}$ to $\overline{R(X)}$, transforms $B|_{N(X)^{\perp}}$ to $A|_{\overline{R(X)}}$. The inverse statement is known directely through calculation.

Corollary 1. *If $AXB=X$, $A^*XB^*=X$, and A is φ-hyponormal operator, then*

$B|_{N(X)^\perp}$ *is* $\bar{\varphi}$*-hyponormal, where* $\varphi(t)=1\Big/\varphi\left(\frac{1}{t}\right)$. *Conversely, if* B *is* φ*-hyponormal operator, then* $A|_{\overline{R(X)}}$ *is* $\bar{\varphi}$*-hyponormal operator*

Corollary 2. *If* $AX=XB$, *and* $A^*X=XB^*$, *and* A *is* φ*-hyponormal operator, then* $B|_{N(X)^\perp}$ *is* φ*-hyponormal operator. Couversely, if* B *is* φ*-hyponormal operator then* $A|_{\overline{R(X)}}$ *is* φ*-hyponormal operator.*

Note. If an operator and its adjoint are both semi-hyponormal, then it is normal operator. Hence in Theorem 1, 2, 4, and 5, the normality is a direct result of the Putnam-Fuglede theorem.

At last, we disscus the operator equation $T^*XT=X$.

Theorem 8. *If* T^* *is a hyponormal operator and* X *is a positive operator such that* $T^*XT=X$. *Then* $\overline{R(X)}$ *reduce* T *to a unitary operator.*

Proof From $X\geqslant 0$ and $T^*XT=X$, we know

$$\|X^{1/2}Tx\|^2=\|X^{1/2}x\|, \text{ for } x\in H.$$

Hence there is a isometric operator V from $\overline{R(X)}$ to $\overline{R(X^{1/2}T)}\subset\overline{R(X)}$ such that

$$VX^{1/2}=X^{1/2}T.$$

Since V is a hyponormal operator on $\overline{R(X)}$, T^* is hyponormal operator. From Theorem 1 we have that $N(X^{1/2})^\perp=\overline{R(X)}$ reduces T to a normal operator and $TXT^*=X$ by (4). By Theorem 6 and $X\geqslant 0$ we know that X is commute with T^*, i. e.

$$TT^*X=X, \quad T^*TX=X.$$

Hence on $N(X)^\perp=\overline{R(X)}$ we have $T^*T=TT^*=I$, i. e. $T_{\overline{R(X)}}$ is a unitary operator.

Corollary 3. *Let* T^* *be hyponormal operator. Then* $T^*XT=X$ *has a solution* $X>0$ (i. e. $X\geqslant 0$ *and* $0\overline{\in}\sigma_P(X)$) *if and only if* T *is a unitary operator.*

Proof From $X>0$ we have $\overline{R(X)}=H$. By the theorem T is a unitary operator. For unitary operator T, $T^*XT=X$ has a solution $X=I>0$.

Note. The condition that T^* is hyponormal can not change to that T is hypo[normal. Take T unilateral shift, $T^*XT=X$ has solution $X=I$, but T is not normal.

Theorem 9. *Let* T *be a contraction operator and* P *be a positive compact operator such that* $T^*PT=P$. *Then* $N(P)^\perp$ *reduces* T *to a unitary operator.*

Proof Let $\sigma(P)=\{\lambda_n:n=1, 2, \cdots, \}$, $\lambda_1>\lambda_2>\cdots>\lambda_n>\cdots$ such that $\lambda_n\to 0$ and $E_n=\ker(P-\lambda_n)$. We prove that E_n reduce T.

By spectral decomposition of P we know $E_1=\{x|\,\|P^nx\|=\lambda_1^n\|x\|, n=1, 2, \cdots\}$. For $x\in E_1$, from $T^*PT=P$ and $\|T\|\leqslant 1$, we have

$$\lambda_1\|X\|\geqslant\|PTx\|\geqslant\|T^*PTx\|=\|Px\|=\lambda_1\|x\|,$$

$$\lambda_1^2\|x\|\geqslant\|P^2Tx\|\geqslant\|T^*P^2Tx\|\geqslant\|T^*PTT^*PTx\|=\|P^2x\|=\lambda_1^2\|x\|.$$

Likewise we can obtain

$$\|P^{2^n}Tx\|=\lambda_1^{2^n}\|x\|.$$

For m, we take n such that $2^n \geqslant m$, and so

$$\lambda_1^m \|x\| \geqslant \|P^m Tx\| \geqslant \lambda_1^{m-2^n} \|P^{2^n} Tx\| = \lambda_1^m \|x\|.$$

Hence we have $Tx \in E_1$, i. e. E_1 is invariant under T and from

$$T^* PTx = Px = \lambda_1 x = \lambda_1 T^* Tx,$$

we know that T is isometric on E_1. But E_1 is finite dimension space, so that E_1 reduce T.

Continuing this process we can obtain that $\overline{R(P)} = N(P)^\perp$ reduce T to a unitary operator.

Note. If we take T as unilateral shift, we know that the condition of the compactness of P can not be omitted.

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