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THE FIRST BOUNDARY VALUE PROBLEM FOR QUASILINEAR DEGENERATE PARABOLIC EQUATIONS OF SECOND ORDER IN SEVERAL SPACE VARIABLES

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Abstract

In this paper, we study the first boundary value problem for quasilinear equations of the form

$$Lu\equiv\frac{\partial u}{\partial t}-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(a^{ij}(u, x, t)\frac{\partial u}{\partial x_j}\right)-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}f^i(u, x, t)=g(u, x, t)$$

with $a^{ij}=a^{ji}$ and

$$\sum_{i,j=1}^{m}a^{ij}(u, x, t)\xi_i\xi_j\geq 0,\quad \forall\xi=(\xi_1, \xi_2, \cdots, \xi_m)\in R^m.$$

Under certain conditions, the existence of generalized solutions in BV is proved by means of the method of parabolic regularization. To do this, we need some estimates on the family $\{u_\varepsilon\}$ of solutions of regularized problems and the most difficult step is to estimate $|\text{grad } u_\varepsilon|_{L^1}$. In addition, some results on the uniqueness and stability of generalized solutions are established.

§1. Introduction

The investigation of global solutions of the first boundary value problem for quasilinear degenerate parabolic equations of second order began at the fifties [1] and has been continued in recent years (e. g. [2, 3]). However most of the previous works were concerned with special equations. The paper [4] by A. I. Bol'pert and S. I. Hudjaev was the first devoted to the global solutions for general quasilinear degenerate parabolic equations; there the existence and the uniqueness of solutions in BV, a class of discontinuous functions, were established for the Cauchy problem. In a recent paper [5], one of the authors studied the global solutions of the first boundary value problem for such general equations in one space variable and some results on existence and uniqueness were obtained. In this paper, we wish to extend these results to the equations in several space variables.

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Let Ω be a bounded region in R^m with boundary Σ appropriately smooth. We will investigate the first boundary value problem for quasilinear degenerate parabolic equations of the form

$$\mathrm{Lu}\equiv\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a^{ij}(u,x,t)\frac{\partial u}{\partial x_j}\right)-\frac{\partial f^i(u,x,t)}{\partial x_i}=g(u,x,t) \tag{1.1}$$

in the cylinder $Q_T=\Omega\times(0,T)$. For simplicity, we consider only the homogeneous boundary value problem. Here $a^{ij}=a^{ji}$, f^i and g are appropriately smooth for $u\in(-\infty,\infty)$ and $(x,t)\in\bar{Q}_T$ and

$$a^{ij}(u,x,t)\xi_i\xi_j\geqslant 0,\quad \forall\xi=(\xi_1,\xi_2,\cdots,\xi_m)\in R^m. \tag{1.2}$$

Throughout this paper, pairs of equal indices imply a summation from 1 up to m. We will assume that $f^i_{x_iu}$ and g_u are bounded for $u\in(-\infty,\infty)$ and $(x,t)\in\bar{Q}_T$.

When $a^{ij}\equiv0\,(i,j=1,2,\cdots,m)$, the equation (1.1) degenerates to a first order equation. In this case, the same problem has been investigated in [6].

As in [5], we will investigate the solvability in BV (Q_T), the class of all integrable functions on Q_T whose generalized derivatives are measures with bounded variation. Similar to [5], the existence will be proved by means of the method of parabolic regularization, namely, the solution of our problem will be obtained as a limit point of the family $\{u_\varepsilon\}$ of solutions of regularized problems

$$L_\varepsilon u\equiv\frac{\partial u}{\partial t}-\varepsilon\Delta u-\frac{\partial}{\partial x_i}\left(a^{ij}(u,x,t)\frac{\partial u}{\partial x_j}\right)-\frac{\partial f^i(u,x,t)}{\partial x_i}=g(u,x,t),\quad(\varepsilon>0) \tag{1.3}$$

$$u(x,0)=u_0(x), \tag{1.4}$$

$$u|_{\Sigma\times[0,T]}=0. \tag{1.5}$$

In order to prove the compactness of $\{u_\varepsilon\}$ we need to establish some estimates on $\{u_\varepsilon\}$. However, in present case of several space variables, it is more difficult to estimate $|\mathrm{grad}\,u_\varepsilon|_{L^1(\Omega)}$. In addition, since for the limit function u of a certain subsequence of $\{u_\varepsilon\}$, it is possible that the trace $\gamma\left(\widehat{a^{ij}(u},x,t)\frac{\partial u}{\partial x_j}\right)$ on Σ does not exist $\left(\widehat{a^{ij}(u},x,t)\frac{\partial u}{\partial x_j}\right.$ need not have the trace $\left.\gamma\left(\widehat{a^{ij}(u},x,t)\frac{\partial u}{\partial x_j}\right)\text{ on }\Sigma\right)$, we have to make a detour to avoid $\gamma\left(\widehat{a^{ij}(u},x,t)\frac{\partial u}{\partial x_j}\right)$ in defining solutions.

Let

$$\begin{aligned}S_1&=\{(x,t)\in\Sigma\times[0,T];\quad a^{ij}(0,x,t)n_in_j=0\},\\ S_2&=\{(x,t)\in\Sigma\times[0,T];\quad a^{ij}(0,x,t)n_in_j>0\},\end{aligned} \tag{1.6}$$

where $n=(n_1,n_2,\cdots,n_m)$ denotes the inner normal on Σ. We will assume that

$$S_1\cap\bar{S}_2=\phi; \tag{1.7}$$

in other words, S_1 ann S_2 can be expressed by

$$S_1=\Sigma_1\times[0,T]\quad\text{and}\quad S_2=\Sigma_2\times[0,T]$$

with

$$\Sigma_1\cup\Sigma_2=\Sigma,\quad \Sigma_1\cap\Sigma_2=\phi.$$

The paper is constructed as follows. We first define solutions of the first boundary value problem for (1.1) in § 2. Subsequently, in § 3, we establish the required estimates of solutions u_ε of regularized problems (1.3), (1.4), (1.5). On the basis of these estimates, we then prove the existence of solutions of our problem in § 4. The last section (§ 5) is devoted to the uniqueness and stability of solutions.

We begin with some preliminary discussions.

Let $\delta_0>0$ be small enough such that

$$E^{\delta_0}=\{x\in\bar{\Omega};\ \mathrm{dist}(x,\ \Sigma)\leqslant\delta_0\}\subset\bigcup_{\tau=1}^{N}V_\tau,$$

where V_τ is a region, on which one can introduce local coordinates

$$y_k=F_\tau^k(x)\quad (k=1,\ 2,\ \cdots,\ m),\quad y_m|_\Sigma=0 \tag{1.8}$$

with F_τ^k appropriately smooth and $F_\tau^m=F_l^m$ on $V_\tau\cap V_l$ such that the y_m-axies coincides with the inner normal.

Let $\delta(\sigma)\in C^\infty(R)$, $\delta(\sigma)\geqslant 0$, $\delta(\sigma)=0(|\sigma|\geqslant 1)$, $\int_{-\infty}^{\infty}\delta(\sigma)d\sigma=1$. Denote

$$\delta_h(\sigma)=\frac{1}{h}\,\delta\Big(\frac{\sigma}{h}\Big).$$

Obviously, $\delta_h(\sigma)\in C^\infty(R)$, $\delta_h(\sigma)\geqslant 0$, $\delta_h(\sigma)=0(|\sigma|\geqslant h)$, $\int_{-\infty}^{\infty}\delta_h(\sigma)d\sigma=1$ and $|\delta_h(\sigma)|\leqslant\frac{c}{h}$ for some constant c.

By a theorem on partition of unity, there exist nonnegative functions $\varphi_\tau(x)\in C_0^2(V_\tau)$ such that $\sum_{\tau=1}^{N}\varphi_\tau=1$ for $x\in E^{\delta_0/2}$.

Let

$$p^h(\sigma)=1-\int_{-\infty}^{\sigma-2h}\delta_h(\tau)d\tau.$$

We will frequently make use of the function

$$\rho_h(x)=\sum_{\tau=1}^{N}\varphi_\tau(x)p_\tau^h(x), \tag{1.9}$$

where

$$p_\tau^h(x)=p^h(F_\tau^m(x)). \tag{1.10}$$

Since supp $\varphi_\tau\subset V_\tau$, we may think $\rho_h(x)$ to be defined on $\bar{\Omega}$. Obviously $\rho_h(x)\in C^2(\bar{\Omega})$ and

$$\left.\begin{aligned}&\rho_h(x)=1,\ \text{near }\Sigma,\\&\lim_{h\to 0^+}\rho_h(x)=0,\ \text{for any }x\in\Omega,\\&0\leqslant\rho_h(x)\leqslant 1,\quad |\mathrm{grad}\ \rho_h|\leqslant\frac{c}{h},\end{aligned}\right\} \tag{1.11}$$

where c is a constant independent of h.

We need the following auxiliary propositions concerning the trace of BV functions.

Lemma 1. *Let $u\in BV(Q_T)\cap L^\infty(Q_T)$. Then u (more precisely, certain function equivalent to u) has its trace γu at almost all of the boundary points of Q_T and $\gamma u\in L^\infty$.*

By the trace γu at a boundary point we mean the limit of u at this point taking along the normal.

Proof Since $u\in BV(Q_T)$, for almost all $x\in\Omega$, as a function of t, $u(x,t)\in BV(0,T)$. Hence $u(x,t)$ has the right limit at every point of $[0,T)$. In particular, the limit $\lim\limits_{t\to0^+}u(x,t)$ exists. Obviously, $\gamma u\in L^\infty(\Omega)$.

Now we consider the trace on $\Sigma\times(0,T)$. Introduce local coordinates (1.8) on V_τ. Since $u\in BV(Q_T)$, for almost all $t\in[0,T]$, as a function of y, $\bar{u}(y,t)=u(y_1, y_2,\cdots,y_m,t)=u(x,t)\in BV(V_\tau)$. Hence $\bar{u}(y,t)$ has the right limit on $\Sigma^\tau=\Sigma\cap V_\tau$ almost everywhere as $y_m\to0^+$, namely, $u(x,t)$ has the limit taking along the normal. Obviously. $\gamma u\in L^\infty(\Sigma^\tau\times(0,T))$.

Lemma 2. *Let* $u\in BV(\Omega)\cap L^\infty(\Omega)$. *Then*

$$\lim_{h\to0^+}\int_\Omega u(x)\frac{\partial\rho_h}{\partial x_i}dx=-\int_\Sigma\gamma u\cdot n_i d\sigma,$$

where ρ_h *is the function in* (1.9).

Proof It suffices to prove

$$\lim_{h\to0^+}\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau p_\tau^h}{\partial x_i}dx=-\int_{\Sigma\cap V_\tau}\gamma u\varphi_\tau n_i d\sigma.$$

From the definition of p_τ^h we have

$$\begin{aligned}\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau p_\tau^h}{\partial x_i}dx&=\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau}{\partial x_i}p_\tau^h dx+\int_{\Omega\cap V_\tau}u(x)\varphi_\tau\frac{\partial p_\tau^h}{\partial x_i}dx\\&=\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau}{\partial x_i}p_\tau^h dx-\int_{\Omega\cap V_\tau}u(x)\varphi_\tau\delta_h(F_\tau^m-2h)\frac{\partial F_\tau^m}{\partial x_i}dx\\&=\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau}{\partial x_i}p_\tau^h dx\\&\quad-\int_{-1}^1\delta(z)\Big(\int_{S_\tau(zh+2h)}\bar{u}(y_1,\cdots,y_{m-1},zh+2h)\cdot\varphi_\tau\frac{\partial F_\tau^m}{\partial x_i}Kdy_1\cdots dy_{m-1}\Big)dz,\end{aligned}$$

where $S_\tau(y_0)=\{y\in\bar{V}_\tau;\ y=y_0\}$, $\bar{V}_\tau$ being the region obtained from V by local coordinates transfor mation, $\bar{u}(y)=u(x)$. By Lebesgue's dominated convergence theorem, we deduce

$$\lim_{h\to0^+}\int_{\Omega\cap V_\tau}u(x)\frac{\partial\varphi_\tau F_\tau^m}{\partial x_i}dx=-\int_{S_\tau(0)}\gamma\bar{u}|_{y_m=0}\ \varphi_\tau\frac{\partial F_\tau^m}{\partial x_i}Kdy_1\cdots dy_{m-1}=-\int_{\Sigma\cap V_\tau}\gamma u\cdot\varphi_\tau n_i d\sigma.$$

Lemma 3. *Let* $u(x)\in BV(\Omega)\cap L^\infty(\Omega)$. *Then*

$$\int_\Omega\frac{\partial u}{\partial x_i}dx=-\int_\Omega\gamma u\cdot n_i d\sigma.$$

Proof Since $1-\rho_h\in C_0^2(\Omega)$, we have

$$\int_\Omega\frac{\partial u}{\partial x_i}(1-\rho_h)dx=-\int_\Omega u\frac{\partial(1-\rho_h)}{\partial x_i}dx$$

and hence

$$\int_\Omega\frac{\partial u}{\partial x_i}dx=\int_\Omega\frac{\partial u}{\partial x_i}\rho_h dx+\int_\Omega u\frac{\partial\rho_h}{\partial x_i}dx.$$

Letting $h\to0$, by Lemma 2 and Lebesgue's theorem, we obtain the desired result.

§ 2. Definition of generalized solutions

Let Γ_u be the set of all jump points of $u\in BV(Q_T)$, ν the normal of Γ_u at $X=(x, t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X\in\Gamma_u$ with respect to $(\nu, Y-X)>0$ and $(\nu, Y-X)<0$ respectively. For continuous function $p(u, x, t)$ and $u=u(x, t)\in BV(Q_T)$, denote

$$\widehat{p(u}, x, t)=\int_0^1 p(\tau u^+ + (1-\tau)u^+, x, t)d\tau,$$

which is called the composite mean value of $p(u, x, t)$ and $u=u(x, t)$.

Definition. *A function $u\in BV(Q_T)\cap L^\infty(Q_T)$ is said to be the generalized solution of the first boundary value problem for (1.1) with homogeneous boundary value and initial value $u_0(x)$ if the following conditions are fulfilled:*

1. *There exist functions $g^i\in L^2(Q_T)$ $(i=1, 2, \cdots, m)$ such that*

$$\iint_{Q_T}\varphi(x, t)g^i(x, t)dx\,dt=\iint_{Q_T}\varphi(x, t)\widehat{r^{ij}(u}, x, t)\frac{\partial u}{\partial x_j}dx\,dt, \quad \forall\varphi\in C_0^2(Q_T), \tag{2.1}$$

where (r^{ij}) is the square root of the matrix (a^{ij}).

2. *For almost all $x\in\Omega$*

$$\gamma u(x, 0)=u_0(x). \tag{2.2}$$

3. *For almost all $t\in[0, T]$*

$$A^{ij}(\gamma u, x, t)n_in_j=0 \text{ a. e. on}\Sigma, \tag{2.3}$$

where $A^{ij}(u, x, t)=\int_0^u a^{ij}(\tau, x, t)d\tau$.

4. *u satisfies*

$$\iint_{Q_T}\Big\{|u-k|\frac{\partial\varphi_1}{\partial t}-\operatorname{sgn}(u-k)\Big[\widehat{a^{ij}(u}, x, t)\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}+\Big(f^i(u, x, t)-f^i(k, x, t)\Big)\frac{\partial\varphi_1}{\partial x_i}$$
$$-(f^i_{x_i}(k, x, t)+g(u, x, t)\varphi_1)\Big]\Big\}dx\,dt+\operatorname{sgn}k\iint_{Q_T}\Big\{u\frac{\partial\varphi_2}{\partial t}$$
$$-\widehat{a^{ij}(u}, x, t)\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}-(f^i(u, x, t)-f^i(k, x, t))\frac{\partial\varphi_2}{\partial x_i}$$
$$+(f^i_{x_i}(k, x, t)+g(u, x, t))\varphi_2\Big\}dx\,dt\geqslant 0, \tag{2.4}$$

where $\varphi_1, \varphi_2\in C^2(\bar{Q}_T)$, $\varphi_1\geqslant 0$, $\varphi_1|_{\Sigma\times(0,T)}=\varphi_2|_{\Sigma\times(0,T)}$, supp φ_1, supp $\varphi_2\subset\bar\Omega\times(0, T)$ and $k\in R$.

Proposition 1. *Conditino 3 in the above definition is equivalent to the fact that*

$$\gamma u=0 \quad a. e. \text{ on } S_2 \tag{2.5}$$

and for almost all points of S_1 with $\gamma u\neq 0$

$$a^{ij}(s, x, t)n_j=\gamma^{ij}(s, x, t)n_j=0, \quad \forall s\in I(0, \gamma u), \tag{2.6}$$

where $I(\alpha, \beta)$ denotes the closed interval with endpoints α and β. If for any $s_0\neq 0$

$$\int_0^{s_0} a^{ij}(s, x, t) n_i n_j ds \neq 0 \quad a.\ e.\ on\ S_1, \tag{2.7}$$

then

$$\gamma u = 0 \quad a.\ e.\ on\ \Sigma\times[0, T]. \tag{2.8}$$

Proof The first conclusion follows from the definition of S_1 and S_2 (see (1.6)) and the assumption (1.2). The second conclusion is evident.

Proposition 2. *If $u\in \mathrm{BV}(Q_T)\cap L^\infty(Q_T)$ satisfies conditions 1 and 4 in the above definition, then*

4'. *u satisfies*

$$\iint_{Q_T}\Big\{|u-k|\frac{\partial\varphi}{\partial t}-\mathrm{sgn}(u-k)\Big[\widehat{a^{ij}(u}, x, t)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + (f^i(u, x, t)-f^i(k, x, t))\frac{\partial\varphi}{\partial x_i}-(f^i_{x_i}(k, x, t)+g(u, x, t))\varphi\Big]\Big\}dx\,dt\geqslant 0 \tag{2.9}$$

for any nonnegative function $\varphi\in C_0^2(Q_T)$ and $k\in R$.

Conversely, if $u\in \mathrm{BV}(Q_T)\cap L^\infty(Q_T)$ satisfies conditions 1, 3 and 4', and $\gamma u=0$ a. e. on $\Sigma\times[0, T]$, $g^i\in L^\infty(Q_T)$ $(i=1, 2, \cdots, m)$, then u satisfies 4.

The first part is evident. In order to prove the second part, we first prove

Lemma 4. *If $u\in \mathrm{BV}(Q_T)\cap L^\infty(Q_T)$ satisfies conditions 1, 3 and 4' with $g^i\in L^\infty(Q_T)$ $(i=1, 2, \cdots, m)$, then for any $\varphi_1, \varphi_2\in C^2(\bar Q_T)$ with $\varphi_1|_{\Sigma\times(0,T)}=\varphi_2|_{\Sigma\times(0,T)}$ and $0\neq k\in R$,*

$$\lim_{h\to 0^+}\iint_{Q_T}\widehat{a^{ij}(u}, x, t)\frac{\partial u}{\partial x_j}(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\frac{\partial\rho_h}{\partial x_i}\,dx\,dt=0,$$

where ρ_h is the function in (1.9).

Proof We have

$$\iint_{Q_T}\widehat{a^{ij}}\frac{\partial u}{\partial x_j}(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\frac{\partial\rho_h}{\partial x_i}\,dx\,dt$$
$$=\sum_{\tau=1}^N\int_0^T\int_{\Omega\cap V_\tau} r^{ij}g^j(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\Big(\frac{\partial\varphi_\tau}{\partial x_i}\rho_\tau^h+\varphi_\tau\delta_h(F_\tau^m-2h)\frac{\partial F_\tau^m}{\partial x_i}\Big)dx\,dt.$$

It is enough to prove that for almost all $t\in[0, T]$

$$\lim_{h\to 0^+}\int_{\Omega\cap V_\tau} r^{ij}g^j(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\varphi_\tau\delta_h(F_\tau^m-2h)\frac{\partial F_\tau^m}{\partial x_i}\,dx=0.$$

Since g^j are bounded,

$$\Big|\int_{\Omega\cap V_\tau} r^{ij}g^j(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\varphi_\tau\delta_h(F_\tau^m-2h)\frac{\partial F_\tau^m}{\partial x_i}\,dx\Big|$$
$$\leqslant\sum_{j=1}^m\frac{C_1}{h_1}\int_h^{3h}dy_m\int_{S_\tau(y_m)}\Big|r^{ij}(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\frac{\partial F^m}{\partial x_i}\Big|dy_1\cdots dy_{m-1}$$
$$=\sum_{j=1}^m C_1\int_{-1}^1 dz\int_{S_\tau(zh+2h)}\Big|r^{ij}(\varphi_1\,\mathrm{sgn}(u-k)+\varphi_2\,\mathrm{sgn}\,k)\frac{\partial F^m}{\partial x_i}\Big|_{y_m=zh+2h}dy_1\cdots dy_{m-1},$$

where $S_\tau(y_0)=\{y\in \bar V_\tau;\ y_m=y_0\}$, $\bar V_\tau$ being the region obtained from V_τ by local coordinates transformation.

From (2.5), (2.6)

$$\gamma u=0 \quad \text{a.e. on } \Sigma_2$$

and

$$r^{ij}(\gamma u,\ x,\ t)\frac{\partial F_\tau^m}{\partial x_i}=0 \quad \text{a. e. on } \Sigma_1$$

whenever

$$\gamma u\neq 0.$$

Therefore

$$\lim_{h\to 0^+} r^{ij}g^j\frac{\partial F_\tau^m}{\partial x_i}(\varphi_1 \operatorname{sgn}(u-k)+\varphi_2 \operatorname{sgn} k)\Big|_{y_m=zh+2h}=0.$$

By Lebesque's theorem this yields the desired result.

Now we return to the proof of the second part of proposition 2.

Let $\varphi_1,\ \varphi_2\in C^2(\overline{Q}_T)$, $\varphi_1\geqslant 0$, $\varphi_1|_{\Sigma\times(0,T)}=\varphi_2|_{\Sigma\times(0,T)}$, $\operatorname{supp}\varphi_1$, $\operatorname{supp}\varphi_2\subset\overline{\Omega}\times(0,\ T)$. Set

$$\varphi_1=\varphi_1'+\varphi_1'',\quad \varphi_2=\varphi_2'+\varphi_2''$$

with $\varphi_1'=\varphi_1(1-\rho_h)$, $\varphi_1''=\varphi_1\rho_h$, $\varphi_2'=\varphi_2(1-\rho_h)$, $\varphi_2''=\varphi_2\rho_h$, ρ_h being the function in (1.9), and substitute into the left hand side of (2.4), we obtain

$$\begin{aligned}&\iint\limits_{Q_T}\Big\{|u-k|\frac{\partial\varphi_1'}{\partial t}-\operatorname{sgn}(u-k)\Big[\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1'}{\partial x_i}+(f^i(u,\ x,\ t)-f^i(k,\ x,\ t)\frac{\partial\varphi_1'}{\partial x_i}\\&\quad-(f^i_{x_i}(k,\ x,\ t)+g(u,\ x,\ t))\varphi_1'\Big]\Big\}dx\,dt+\iint\limits_{Q_T}\Big\{|u-k|\frac{\partial\varphi_1''}{\partial t}-\operatorname{sgn}(u-k)\Big[\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1''}{\partial x_i}\\&\quad+(f^i(u,\ x,\ t)-f^i(k,\ x,\ t))\frac{\partial\varphi_1''}{\partial x_i}-(f^i_{x_i}(k,\ x,\ t)+g)\varphi_1''\Big]dx\,dt\\&\quad+\operatorname{sgn}k\iint\limits_{Q_T}\Big[u\frac{\partial\varphi_2'}{\partial t}-\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2'}{\partial x_i}-f^i(u,\ x,\ t)\frac{\partial\varphi_2'}{\partial x_i}+g(u,\ x,\ t)\varphi_2'\Big]dx\,dt\\&\quad+\operatorname{sgn}k\iint\limits_{Q_T}\Big[u\frac{\partial\varphi_2''}{\partial t}-\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2''}{\partial x_i}-(f^i(u,\ x,\ t)-f^i(k,\ x,\ t)\frac{\partial\varphi_2''}{\partial x_i}\\&\quad+(f^i_{x_i}(k,\ x,\ t)+g(u,\ x,\ t))\varphi_2''\Big]dx\,dt=I_1+I_2+I_3+I_4.\end{aligned}$$

implies $I_1\geqslant 0$ and $I_3=0$. Obviously (2.9)

$$\begin{aligned}I_2+I_4=&\iint\limits_{Q_T}\Big\{|u-k|\frac{\partial\varphi_1}{\partial t}\rho_h-\operatorname{sgn}(u-k)\Big[\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\rho_h+(f^i(u,\ x,\ t)\\&-f^i(k,\ x,\ t))\frac{\partial\varphi_1}{\partial x_i}\rho_h-(f^i_{x_i}(k,\ x,\ t)+g)\varphi_1\rho_h\Big]\Big\}dx\,dt\\&+\operatorname{sgn}k\iint\limits_{Q_T}\Big[u\frac{\partial\varphi_2}{\partial t}\rho_h-\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}\rho_h-(f^i(u,\ x,\ t)-f^i(k,\ x,\ t))\frac{\partial\varphi_2}{\partial x_i}\rho_h\\&+(f^i_{x_i}(k,\ x,\ t)+g)\varphi_2\rho_h\Big]dx\,dt-\iint\limits_{Q_T}\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\rho_h}{\partial x_i}(\varphi_1\operatorname{sgn}(u-k)\\&+\varphi_2\operatorname{sgn}k)dx\,dt-\iint\limits_{Q_T}(f^i(u,\ x,\ t)\\&-f^i(k,\ x,\ t))\frac{\partial\rho_h}{\partial x_i}(\varphi_1\operatorname{sgn}(u-k)+\varphi_2\operatorname{sgn}k)dx\,dt.\end{aligned}$$

The first two integrals of the right hand side tend to zero as $h\to 0$. By Lemma 4, so do the last two integrals, if $k\neq 0$. Thus u satisfies (2.4) for $k\neq 0$. By a limit process we can assert that (2.4) holds even for $k=0$.

§ 3. Estimates on Solutions of regularized Problems

Assume that $u_0(x)$ is appropriately smooth on $\bar{\Omega}$ and certain compatibility conditions on the boundary of the lower base of Q_T are fulfilled, for example, $u_0(x)$ itself and its derivatives up to the fourth order vanish on Σ and $f^i_{x_i}(0, x, t)+g(0, x, t)$ itself, its first order derivative with respect to t and its first and second order derivatives with respect to x vanish for $x\in\Sigma$, $t=0$. Then for any fixed $\varepsilon>0$, the problems (1.3), (1.4), (1.5) have a unique solution u_ε from $C^2(\bar{Q}_T)\cap C^3(Q_T)$.

The first estimate we need follows from the standard maximum principle, namely

$$|u_\varepsilon|\leqslant M, \tag{3.1}$$

where the constant M is independent of ε.

Proposition 3. *The solution u_ε of the problems (1.3), (1.4), (1.5) satisfies*

$$\varepsilon\int_\Sigma\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma+\int_\Sigma a^{ij}(0, x, t)n_in_j\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\leqslant C_1+C_2\left(\|\operatorname{grad} u_\varepsilon\|_{L^1(\Omega)}+\left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L^1(\Omega)}\right) \tag{3.2}$$

with constants c_1, c_2 independent of ε.

Proof Denote

$$f(x, t)=\varepsilon\Delta u_\varepsilon+\frac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon, x, t)\frac{\partial u_\varepsilon}{\partial x_j}\right),$$

$$f^+(x, t)=\begin{cases}f(x, t), & \text{if } f(x, t)>0,\\ 0, & \text{if } f(x, t)\leqslant 0,\end{cases}$$

$$f^-(x, t)=\begin{cases}-f(x, t), & \text{if } f(x, t)<0,\\ 0, & \text{if } f(x, t)\geqslant 0.\end{cases}$$

For any fixed t, consider

$$\begin{cases}\varepsilon\Delta u_1+\dfrac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon, x, t)\dfrac{\partial u_1}{\partial x_j}\right)=f^+(x, t),\\ u_1|_\Sigma=0.\end{cases}$$

From the nonpositivity of the solution of the porblem

$$\begin{cases}\varepsilon\Delta u_{1\eta}+\dfrac{\partial}{\partial x_i}(a^{ij}(u_\varepsilon, x, t)\dfrac{\partial u_{1\eta}}{\partial x_j}-\eta u_{1\eta}=f^+(x, t) \quad (\eta>0),\\ u_{1\eta}|_\Sigma=0,\end{cases}$$

we can assert $u_1\leqslant 0$ and hence $\dfrac{\partial u_1}{\partial n}\leqslant 0$ where n is the inner normal. Therefore

$$\int_\Omega f^+dx=-\varepsilon\int_\Sigma\frac{\partial u_1}{\partial n}d\sigma-\int_\Sigma a^{ij}(0, x, t)n_in_j\frac{\partial u_1}{\partial n}d\sigma$$
$$=\varepsilon\int_\Sigma\left|\frac{\partial u_1}{\partial n}\right|d\sigma+\int_\Sigma a^{ij}(0, x, t)n_in_j\left|\frac{\partial u_1}{\partial n}\right|d\sigma.$$

Similarly, for the solution u_2 of the problem

$$\begin{cases} \varepsilon \Delta u_2 + \dfrac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon, x, t)\dfrac{\partial u_2}{\partial x_j}\right) = f^-(x, t), \\ u_2|_\Sigma = 0, \end{cases}$$

we have

$$\int_\Omega f^- dx = \varepsilon \int_\Sigma \left|\frac{\partial u_2}{\partial n}\right| d\sigma + \int_\Sigma a^{ij}(0, x, t) n_i n_j \left|\frac{\partial u_2}{\partial n}\right| d\sigma.$$

Since the function $u_1 - u_2$ is a solution of the problem

$$\begin{cases} \varepsilon \Delta u + \dfrac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon, x, t)\dfrac{\partial u}{\partial x_j}\right) = f(x, t), \\ u|_\Sigma = 0, \end{cases}$$

by uniqueness of solutions, $u_1 - u_2 = u_\varepsilon$. Thus

$$\varepsilon \int_\Sigma \left|\frac{\partial u_\varepsilon}{\partial n}\right| d\sigma + \int_\Sigma a^{ij}(0, x, t) n_i n_j \left|\frac{\partial u_\varepsilon}{\partial n}\right| d\sigma \leqslant \varepsilon \int_\Sigma \left|\frac{\partial u_1}{\partial n}\right| d\sigma$$

$$+ \varepsilon \int_\Sigma \left|\frac{\partial u_2}{\partial n}\right| d\sigma + \int_\Sigma a^{ij}(0, x, t) n_i n_j \left|\frac{\partial u_1}{\partial n}\right| d\sigma + \int_\Sigma a^{ij}(0, x, t) n_i n_j \left|\frac{\partial u_2}{\partial n}\right| d\sigma$$

$$= \int_\Omega |f| dx \leqslant c_1 + c_2\left(|\operatorname{grad} u_\varepsilon|_{L'(\Omega)} + \left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L'(\Omega)}\right).$$

This completes the proof.

Let

$$\operatorname{sgn}_\eta \tau = \begin{cases} 1 & (\tau > \eta), \\ \dfrac{\tau}{\eta} & (|\tau| \leqslant \eta), \\ -1 & (\tau < -\eta). \end{cases}$$

For $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$, let $|\xi| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2)^{\frac{1}{2}}$

$$I_\eta(\xi) = \int_0^{|\xi|} \operatorname{sgn}_\eta \tau \, d\tau.$$

Clearly

$$\frac{\partial I_\eta}{\partial \xi_i} = \operatorname{sgn}_\eta |\xi| \cdot \frac{\xi_i}{|\xi|},$$

$$\frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p} = \begin{cases} \dfrac{(\operatorname{sgn}'_\eta |\xi|)|\xi| \xi_s \xi_p - \xi_p \xi_s \operatorname{sgn}_\eta |\xi|}{|\xi|^3} & (s \neq p), \\ \dfrac{(\operatorname{sgn}'_\eta |\xi|)|\xi| \xi_s \xi_p - \xi_p \xi_s \operatorname{sgn}_\eta |\xi|}{|\xi|^3} + \dfrac{\operatorname{sgn}_\eta |\xi|}{|\xi|} & (s = p). \end{cases} \tag{3.3}$$

Lemma 5. *Let (q^{sp}) be the square root of the matrix $\left(\dfrac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p}\right)$. Then there exists a constant $M_0 > 0$ such that*

$$|q^{sp}(\xi)| \leqslant \frac{M_0}{|\xi|^{\frac{1}{2}}}. \tag{3.4}$$

Proof Since (q^{sp}) is satisfies $(q^{sp})^2 = \left(\dfrac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p}\right)$, we have

$$\sum_p (q^{sp})^2 = \frac{\partial^2 I_\eta}{\partial \xi_2^s}.$$

(3.3)gives $\left|\dfrac{\partial^2 I_\eta}{\partial \xi_s^2}\right| \leqslant \dfrac{N}{|\xi|}$ for some constant $N > 0$, so (3.4) holds.

The main purpose of this section is to prove

Theorem 1. *Suppose* $S_1\cap\bar{S}_2=\phi$ *and* $a^{ij}(0, x, t)$ *can be smoothly extended to a neighborhood of* S_1 *such that in this neighborhood*

$$a^{ij}(0, x, t)\xi_i\xi_j\geqslant 0, \quad \forall\xi\in R^m. \tag{3.5}$$

Suppose there exists a constant $\delta>0$ *such that for* $|u|\leqslant M$ *(*M *is the constant in* (3.1)) $(x, t)\in\bar{Q}_T$

$$a^{ij}\xi_i\xi_j-\delta\sum_{s=1}^{m+1}\sum_{j=1}^{m}(a^{ij}_{x_s}\xi_i)^2\geqslant 0, \quad \forall\xi\in R^m \tag{3.6}$$

$(x_{m+1}=t)$. *Then the solutions* u_ε *of regularized problems* (1.3), (1.4), (1.5) *satisfy*

$$|\operatorname{grad} u_\varepsilon|_{L^1(\Omega)}\leqslant M_1, \tag{3.7}$$

where $\operatorname{grad}u=(u_{x_1}, u_{x_2}, \cdots, u_{x_m}, u_{x_{m+1}})=(u_{x_1}, u_{x_2}, \cdots, u_{x_m}, u_t)$ and *the connstant* M_1 *is independent of* ε.

Proof Differentiate (1.3) with respect to x_s $(s=1, 2, \cdots, m+1)$ and sum up for s after multiplying the resulting relation by $u_{x_s}\dfrac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}$. In what follows, we simply denote u_ε by u. Integrating over Ω yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega I_\eta(|\operatorname{grad} u|)dx-\varepsilon\int_\Omega(\Delta u_{x_s})u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\\&-\int_\Omega\frac{\partial}{\partial x_i}(a^{ij}_u u_{x_s}u_{x_j}+a^{ij}_{x_s}u_{x_j}+a^{ij}u_{x_sx_j})u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\\&-\int_\Omega\frac{\partial}{\partial x_i}(f^i_u u_{x_s}+f^i_{x_s})u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx=\int_\Omega\Big(\frac{\partial}{\partial x_s}g\Big)u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx.\end{aligned}$$

By integrating by parts, the last equality becomes

$$\begin{aligned}&\frac{d}{dt}\int_\Omega I_\eta(|\operatorname{grad} u|)dx+\varepsilon\int_\Omega\frac{\partial^2 I_\eta}{\partial\xi_s\,\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx\\&+\int_\Omega a^{ij}\frac{\partial^2 I_\eta}{\partial\xi_s\,\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx+\int_\Omega a^{ij}_{x_s}u_{x_j}\frac{\partial^2 I_\eta}{\partial\xi_s\,\partial\xi_p}u_{x_px_i}dx\\&-\int_\Omega\Big(\frac{\partial}{\partial x_i}a^{ij}_u\Big)u_{x_j}(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx\\&-\int_\Omega\Big(\frac{\partial}{\partial x_i}f^i_u\Big)(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx\\&-\int_\Omega a^{ij}_u u_{x_ix_j}(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx+\varepsilon\int_\Sigma\frac{\partial I_\eta}{\partial x_i}n_i d\sigma\\&+\int_\Sigma a^{ij}\frac{\partial I_\eta}{\partial x_j}n_i d\sigma+\int_\Sigma a^{ij}_u u_{x_j}I_\eta n_i d\sigma\\&+\int_\Sigma a^{ij}_{x_s}u_{x_j}u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}n_i d\sigma+\int_\Sigma f^i_u I_\eta n_i d\sigma\\&=\int_\Omega\Big(\frac{\partial}{\partial x_i}f^i_{x_s}+\frac{\partial}{\partial x_s}g\Big)u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{gran} u|}{|\operatorname{grad} u|}dx.\end{aligned}\tag{3.8}$$

Here pairs of equal indices of i and j imply the summation from 1 up to m, while pairs of equal indices of s and p imply the summation from 1 up to $m+1$.

We first estimate the third and fourth terms on the left side of (3.8).

Set

$$\begin{pmatrix} v_1^i \\ v_2^i \\ \vdots \\ v_{m+1}^i \end{pmatrix} = \begin{pmatrix} q^{11} q^{12} \cdots q^{1\ m+1} \\ q^{21} q^{22} \cdots q^{2\ m+1} \\ \vdots \qquad \vdots \\ q^{m+1\ 1} q^{m+1\ 2} \cdots q^{m+1\ m+1} \end{pmatrix} \begin{pmatrix} u_{x_1 x_i} \\ u_{x_2 x_i} \\ \vdots \\ u_{x_{m+1} x_i} \end{pmatrix}$$

where (q^{sp}) is the square root of $\left(\frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p}\right)$. Then

$$\left| a_{x_s}^{ij} u_{x_j} \frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p} u_{x_p x_i} \right| = \left| \sum_{i,j=1}^{m} (a_{x_1}^{ij} u_{x_j}, a_{x_2}^{ij} u_{x_j}, \cdots, a_{x_{m+1}}^{ij} u_{x_j})(q^{sp}) \begin{pmatrix} v_1^i \\ v_2^i \\ \vdots \\ v_{m+1}^i \end{pmatrix} \right|$$

$$= |a_{x_s}^{ij} u_{x_j} q^{sp} v_p^i| \leqslant \delta \sum_{s,p=1}^{m+1} \sum_{j=1}^{m} (a_{x_s}^{ij} v_p^i)^2 + \frac{1}{4\delta} \sum_{s,p=1}^{m+1} \sum_{j=1}^{m} (q^{sp} u_{x_j})^2,$$

$$a^{ij} \frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p} u_{x_s x_i} u_{x_p x_i} = \sum_{p=1}^{m+1} a^{ij} v_p^i v_p^j.$$

Thus by virtue of the assumption (3.6) and Lemma 5, there exists a constant $\beta > 0$ such that

$$\int_\Omega a^{ij} \frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p} u_{x_s x_j} u_{x_p x_i} dx + \int_\Omega a_{x_s}^{ij} u_{x_j} \frac{\partial^2 I_\eta}{\partial \xi_s \partial \xi_p} u_{x_p x_i} dx$$

$$\geqslant -\frac{1}{4\delta} \int_\Omega \sum_{s,p=1}^{m+1} \sum_{j=1}^{m} (q^{sp} u_{x_j})^2 dx \geqslant -\beta \int_\Omega |\operatorname{grad} u| dx. \tag{3.9}$$

Observe that on Σ we have

$$f_u^i \frac{\partial u}{\partial n} n_i = -\varepsilon \Delta u - \frac{\partial}{\partial x_i}\left(a^{ij} \frac{\partial u}{\partial x_j}\right) - g(0, x, t) - f_{x_i}^i(0, x, t).$$

Thus the five surface integrals in (3.8) can be rewritten as

$$-\int_\Sigma (g(0, x, t) + f_{x_i}^i(0, x, t)) \frac{I_\eta}{\frac{\partial u}{\partial n}} d\sigma + \varepsilon \int_\Sigma \left(\frac{\partial I_\eta}{\partial x_i} n_i - \Delta u \frac{I_\eta}{\frac{\partial u}{\partial n}}\right) d\sigma$$

$$+ \int_{\Sigma_1} a^{ij} \left(\frac{\partial I_\eta}{\partial x_j} n_i - u_{x_i x_j} \frac{I_\eta}{\frac{\partial u}{\partial n}}\right) d\sigma + \int_{\Sigma_2} a^{ij} \left(\frac{\partial I_\eta}{\partial x_j} n_i - u_{x_i x_j} \frac{I_\eta}{\frac{\partial u}{\partial n}}\right) d\sigma$$

$$- \int_{\Sigma_1} a_{x_i}^{ij} u_{x_j} \frac{I_\eta}{\frac{\partial u}{\partial n}} d\sigma - \int_{\Sigma_2} a_{x_i}^{ij} u_{x_j} \frac{I_\eta}{\frac{\partial u}{\partial n}} d\sigma$$

$$+ \int_{\Sigma_1} a_{x_s}^{ij} u_{x_j} u_{x_s} \frac{\operatorname{sgn}_\eta |\operatorname{grad} u|}{|\operatorname{grad} u|} n_i d\sigma + \int_{\Sigma_2} a_{x_s}^{ij} u_{x_j} u_{x_s} \frac{\operatorname{sgn}_\eta |\operatorname{grad} u|}{|\operatorname{grad} u|} n_i d\sigma$$

$$= I_1 + I_2 + I_{31} + I_{32} + I_{41} + I_{42} + I_{51} + I_{52}.$$

Obviously I_1 is bounded. From the assumption (3.5) and the definition of Σ_1, it is easy to see that for $(x, t) \in S_1 = \Sigma_1 \times [0, T]$

$$a_{x_i}^{ij} u_{x_j} n_i = a_{x_s}^{ij}(0, x, t) n_i n_j \frac{\partial u}{\partial n} = 0,$$

and hence $I_{51} = 0$. Similarly, for $(x, t) \in S_1$, $a^{ij} n_i = 0$ and

$$a^{ij}u_{x_ix_j}+a^{ij}_{x_i}u_{x_j}=(a^{ij}u_{x_j})_{x_i}-a^{ij}_{t}u_{x_i}u_{x_j}=\frac{\partial}{\partial n}(a^{ij}u_{x_j})n_i-a^{ij}_{t}u_{x_i}u_{x_j}$$

$$=\left(\frac{\partial}{\partial n}a^{ij}\right)u_{x_j}n_i+a^{ij}\left(\frac{\partial}{\partial n}u_{x_j}\right)n_i-a^{ij}_{t}u_{x_i}u_{x_j}=0.$$

Hence

$$I_{31}+I_{41}=\int_{\Sigma_1}a^{ij}\frac{\partial I_\eta}{\partial x_i}n_i d\sigma-\int_{\Sigma_1}(a^{ij}u_{x_ix_j}+a^{ij}_{x_i}u_{x_j})\frac{I_\eta}{\frac{\partial u}{\partial n}}d\sigma=0.$$

Since Σ_2 is a closed subset of Σ, it follows from Proposition 3 that I_{42} and I_{52} can be estimated by $|\mathrm{grad} u|_{L^1(\Omega)}$.

It remains to estimate I_2 and I_{32}. Since $u_{x_{m+1}}|_\Sigma=u_t|_\Sigma=0$, we have

$$I_2\to I_2^0=\varepsilon\int_\Sigma \mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)(u_{x_sx_i}n_in_s-\Delta u)d\sigma,$$

$$I_{32}\to I_{32}^0=\int_{\Sigma_2}\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)a^{ij}(u_{x_sx_i}n_sn_j-u_{x_ix_j})d\sigma$$

as $\eta\to 0$. Here pairs of equal indices of s imply a summation from 1 up to m.

As in § 1, we introduce Local coordinates on $V_\tau(\tau=1, 2, \cdots, N)$

$$y_k=F^k_\tau(x)\quad (k=1, 2, \cdots, m),\ y_m|_\Sigma=0.$$

By elementary computations we obtain on $\Sigma^\tau=\Sigma\cap V_\tau$

$$u_{x_ix_j}=\sum_{k=1}^m u_{y_my_k}F^m_{x_i}F^k_{x_j}+\sum_{k=1}^{m-1}u_{y_my_k}F^m_{x_j}F^k_{x_i}+u_{y_m}F^m_{x_ix_j}$$

$$u_{x_ix_s}n_sn_j=\sum_{k=1}^m u_{y_my_k}F^m_{x_i}F^k_{x_s}F^m_{x_s}F^m_{x_j}/|\mathrm{grad}\ F^m|^2$$

$$+\sum_{k=1}^{m-1}u_{x_my_k}F^k_{x_i}F^m_{x_j}+u_{y_m}F^m_{x_ix_s}F^m_{x_s}F^m_{x_j}/|\mathrm{grad}\ F^m|^2$$

in which $F^k=F^k_\tau$.

Hence

$$u_{x_sx_i}n_sn_i-\Delta u=u_{y_m}(F^m_{x_ix_s}F^m_{x_s}F^m_{x_i}/|\mathrm{grad}\ F^m|^2-F^m_{x_ix_i}).$$

From this and Proposition 3 we see that I_2^0 can be estimated by $|\mathrm{grad}\ u|_{L^1(\Omega)}$.

To estimate I_{32}^0, we choose nonnegative functions $\eta_\tau\in C^2(\Sigma_2)$ $(\tau=1, 2, \cdots, N)$ with supp $\eta_\tau\subset\Sigma_2^\tau=\Sigma_2\cap V_\tau$ and $\sum_{\tau=1}^N\eta_\tau=1$ on Σ_2. Write I_{32}^0 as

$$I_{32}^0=\sum_{\tau=1}^N\int_{\Sigma_2^\tau}\eta_\tau a^{ij}\,\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)(u_{x_sx_i}n_sn_j-u_{x_ix_j})d\sigma.$$

We have

$$\int_{\Sigma_2^\tau}\eta_\tau a^{ij}\,\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)(u_{x_sx_i}n_sn_j-u_{x_ix_j})d\sigma$$

$$=\sum_{k=1}^{m-1}\int_{\Sigma_2^\tau}\eta_\tau a^{ij}\,\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)u_{y_my_k}(F^k_{x_s}F^m_{x_s}F^m_{x_i}/|\mathrm{grad}\ F^m|^2-F^k_{x_i})F^m_{x_j}K\,dy_1\cdots dy_{m-1}$$

$$+\int_{\Sigma_2^\tau}\eta_\tau a^{ij}\,\mathrm{sgn}\left(\frac{\partial u}{\partial n}\right)u_{y_m}(F^m_{x_ix_s}F^m_{x_s}F^m_{x_j}/|\mathrm{grad}\ F^m|^2-F^m_{x_ix_j})K\,dy_1\cdots dy_{m-1}.$$

Integrating by parts and noting supp $\eta_\tau\subset\Sigma_2^\tau$, we obtain

$$\int_{\Sigma_2^\tau}\eta_\tau a^{ij}\,\mathrm{sgn}\Big(\frac{\partial u}{\partial n}\Big)u_{y_m y_k}(F^m_{x_s}F^k_{x_s}F^m_{x_i}/|\mathrm{grad}\,F^m|^2-F^k_{x_i})F^k_{x_j}K\,dy_1\cdots dy_{m-1}$$

$$=-\int_{\Sigma_2^\tau}\mathrm{sgn}\Big(\frac{\partial u}{\partial n}\Big)u_{y_m}\frac{\partial}{\partial y_k}[\eta_\tau a^{ij}(F^k_{x_s}F^m_{x_s}F^m_{x_i}/|\mathrm{grad}\,F^m|^2-F^k_{x_i})F^k_{x_j}K]\,dy_1\cdots dy_{m-1}.$$

Therefore
$$|I^0_{32}|\leqslant C_3\int_{\Sigma_2}\Big|\frac{\partial u}{\partial n}\Big|\,d\sigma.$$

Since for $(x,\ t)\in\Sigma_2$, $a^{ij}(0,\ x,\ t)n_i n_j\geqslant\alpha$ with some constant $\alpha>0$, it follows that

$$|I^0_{32}|\leqslant C_4\int_{\Sigma_2}a^{ij}(0,\ x,\ t)n_i n_j\Big|\frac{\partial u}{\partial n}\Big|\,d\sigma.$$

This and Gronwall's Lemma yield tne desired estimate and the proof is completed.

Using Theorem 1, we can easily obtain another estimate.

Proposition 4. *Under the assumptions of Theorem* 1, *the solutions* $u=u_\varepsilon$ *of regularized problems* (1.3), (1.4), (1.5) *satisfy*

$$\iint_{Q_T}a^{ij}(u,\ x,\ t)u_{x_i}u_{x_j}\,dx\,dt\leqslant M_2$$

for some constant M_2 *independent of* ε.

§ 4. Existence of generalized solutions

Theorem 2. *Under the conditions of Theorem* 1, *there exists a subsequence* $\{u_{\varepsilon_n}\}$ *of the family* $\{u_\varepsilon\}$ *of solutions of regularized problems* (1.3), (1.4), (1.5), *which converges strongly in* $L^1(Q_T)$ *and the limit function* u *is a generalized solution of the first boundary value problem for* (1.1).

Proof The existence of a strongly convergent subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ follows from (3.1), Theorem 1 and Kolmogoroff's Theorem. Let u be the limit function. Then $u\in\mathrm{BV}(Q_T)\cap L^\infty(Q_T)$.

By Proposition 4
$$\iint_{Q_T}\Big|r^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\Big|^2dx\,dt\leqslant M_2.$$

This means that $\Big\{r^{ij}\dfrac{\partial u_{\varepsilon_n}}{\partial x_j}\Big\}$ is weakly compact in $L^2(Q_T)$. Without loss of generality, we may assume that $\Big\{r^{ij}\dfrac{\partial u_{\varepsilon_n}}{\partial x_j}\Big\}$ itself converges weakly in $L^2(Q_T)$ to a function $g^i\in L^2(Q_T)$. Thus for any $\varphi\in C^2(\bar Q_T)$,

$$\iint_{Q_T}\varphi g^i\,dx\,dt=\lim_{\varepsilon_n\to0}\iint_{Q_T}\varphi r^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\,dx\,dt=\lim_{\varepsilon_n\to0}\iint_{Q_T}\varphi\Big(\int_0^{u_{\varepsilon_n}}r^{ij}(\tau,\ x,\ t)\,d\tau\Big)_{x_j}dx\,dt$$
$$-\iint_{Q_T}\varphi\int_0^u r^{ij}_{x_j}(\tau,\ x,\ t)\,d\tau\,dx\,dt=-\lim_{\varepsilon_n\to0}\iint_{Q_T}\frac{\partial\varphi}{\partial x_j}\int_0^{u_{\varepsilon_n}}r^{ij}(\tau,\ x,\ t)\,d\tau\,dx\,dt$$
$$-\iint_{Q_T}\varphi\int_0^u r^{ij}_{x_j}(\tau,\ x,\ t)\,d\tau\,dx\,dt=-\iint_{Q_T}\frac{\partial\varphi}{\partial x_j}\int_0^u r^{ij}(\tau,\ x,\ t)\,d\tau\,dx\,dt$$
$$-\iint_{Q_T}\varphi\int_0^u r^{ij}_{x_j}(\tau,\ x,\ t)\,d\tau\,dx\,dt,$$

and hence, by Lemma 3, we obtain

$$\iint_{Q_T} \varphi g^i \, dx\, dt = \int_0^T \int_\Sigma \left(\int_0^{\gamma u} r^{ij}(\tau, x, t)\, d\tau \right) \varphi n_i \, dx\, dt + \iint_{Q_T} \varphi \hat{r}^{ij} \frac{\partial u}{\partial x_j} dx\, dt. \tag{4.1}$$

In particular, for $\varphi \in C_0^2(Q_T)$

$$\iint_{Q_T} \varphi g^i \, dx\, dt = \iint_{Q_T} \varphi \hat{r}^{ij} \frac{\partial u}{\partial x_j} dx\, dt. \tag{4.2}$$

This means that u satisfies (2.1). By a limit process we can assert that (4.2) holds even for $\varphi \in C^2(\bar{Q}_T)$. Combining (4.1) with (4.2) gives

$$\int_0^T \int_\Sigma \left(\int_0^{\gamma u} r^{ij}(\tau, x, t) d\tau \right) \varphi n_i \, dx\, dt = 0$$

for any $\varphi \in C^2(\bar{Q}_T)$. Hence

$$\int_0^{\gamma u} r^{ij}(\tau, x, t) n_i n_j d\tau = 0, \quad \text{a. e. on } \Sigma \times [0, T].$$

This shows that u satisfies (2.3).

From (3.7) it is easy to see that u satisfies (2.2).

Now let $\varphi_1 \in C^2(\bar{Q}_T)$, $\varphi_1 \geqslant 0$, supp $\varphi_1 \subset \bar{\Omega} \times (0, T)$. Multiply (1.3) by $\varphi_1 \operatorname{sgn}_\eta (u_\varepsilon - k)$ and integrate over Q_T. Integrating by parts, we obtain

$$\begin{aligned}
&-\iint_{Q_T} I_\eta(u_\varepsilon - k) \frac{\partial \varphi_1}{\partial t} dx\, dt + \iint_{Q_T} \operatorname{sgn}_\eta (u_\varepsilon - k) \Big[\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi_1}{\partial x_i} + a^{ij} \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \varphi_1}{\partial x_i} \\
&\quad + (f^i(u_\varepsilon, x, t) - f^i(k, x, t)) \frac{\partial \varphi_1}{\partial x_i} \Big] dx\, dt + \iint_{Q_T} \operatorname{sgn}'_\eta (u_\varepsilon - k) \Big(\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_i} \\
&\quad + a^{ij} \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial u_\varepsilon}{\partial x_i} \Big) \varphi_1 \, dx\, dt + \iint_{Q_T} \operatorname{sgn}'_\eta (u_\varepsilon - k) (f^i(u_\varepsilon, x, t) - f^i(k, x, t)) \frac{\partial u_\varepsilon}{\partial x_i} \varphi_1 \, dx\, dt \\
&\quad + \iint_{Q_T} \operatorname{sgn}_\eta (u_\varepsilon - k) (f^i_{x_i}(k, x, t) - g(u_\varepsilon, x \text{ `} t)) \varphi_1 \, dx\, dt \\
&\quad - \operatorname{sgn}_\eta k \int_0^T \int_\Sigma \Big(\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} + a^{ij} \frac{\partial u_\varepsilon}{\partial x_j} \Big) \varphi_1 n_i \, d\sigma\, dt \\
&\quad - \operatorname{sgn}_\eta k \int_0^T \int_\Sigma (f^i(0, x, t) - f^i(k, x, t)) \varphi_1 n_i \, d\sigma\, dt = 0.
\end{aligned}$$

Since the third term is nonnegative and the fourth term tends to zero as $\eta \to 0$, we have

$$\begin{aligned}
&\iint_{Q_T} \Big\{ |u_\varepsilon - k| \frac{\partial \varphi_1}{\partial t} - \operatorname{sgn} (u_\varepsilon - k) \Big[\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi_1}{\partial x_i} + a^{ij} \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \varphi_1}{\partial x_i} \\
&\quad + (f^i(u_\varepsilon, x, t) - f^i(k, x, t)) \frac{\partial \varphi_1}{\partial x_i} - (f^i_{x_i}(k, x, t) + g(u_\varepsilon, x, t)) \varphi_1 \Big] \Big\} dx\, dt \\
&\quad + \operatorname{sgn} k \int_0^T \int_\Sigma \Big(\varepsilon \frac{\partial u_\varepsilon}{\partial x_i} + a^{ij} \frac{\partial u_\varepsilon}{\partial x_j} \Big) \varphi_1 n_i \, d\sigma\, dt \\
&\quad + \operatorname{sgn} k \int_0^T \int_\Sigma (f^i(0, x, t) - f^i(k, x, t)) \varphi_1 n_i \, d\sigma\, dt \geqslant 0.
\end{aligned} \tag{4.3}$$

Clearly

$$\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt=-\int_0^T\int_\Sigma \operatorname{sgn}k A^{ij}(k,x,t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)\,(A^{ij}(u_\varepsilon,x,t)-A^{ij}(k,x,t))\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}\,dx\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)\left(\int_k^{u_\varepsilon}a^{ij}_{x_j}(\tau,x,t)\,d\tau\right)\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt.$$

Hence

$$\lim_{\varepsilon_n\to0}\iint_{Q_T}\operatorname{sgn}(u_{\varepsilon_n}-k)a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt$$
$$=-\int_0^T\int_\Sigma \operatorname{sgn}kA^{ij}(k,x,t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt-\iint_{Q_T}\operatorname{sgn}(u-k)\,(A^{ij}(u,x,t)$$
$$-A^{ij}(k,x,t))\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}\,dx\,dt-\iint_{Q_T}\operatorname{sgn}(u-k)\left(\int_k^u a^{ij}_{x_j}(\tau,x,t)\,d\tau\right)\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt.$$

Observing that $\operatorname{sgn}(u-k)\,(A^{ij}(u,x,t)-A^{ij}(k,x,t))\in BV(Q_T)$ and

$$\frac{\partial}{\partial x_j}[\operatorname{sgn}(u-k)\,(A^{ij}(u,x,t)-A^{ij}(k,x,t))]$$
$$=\operatorname{sgn}(u-k)\frac{\partial}{\partial x_j}(A^{ij}(u,x,t)-A^{ij}(k,x,t))$$

(see[5]), we derive

$$\lim_{\varepsilon_n\to0}\iint_{Q_T}\operatorname{sgn}(u_{\varepsilon_n}-k)a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt$$
$$=-\int_0^T\int_\Sigma \operatorname{sgn}kA^{ij}(k,x,t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt+\int_0^T\int_\Sigma\operatorname{sgn}(\gamma u-k)\,(A^{ij}(\gamma u,x,t)$$
$$-A^{ij}(k,x,t))\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt+\iint_{Q_T}\operatorname{sgn}(u-k)\hat a^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt. \tag{4.4}$$

In order to compute $\lim\limits_{\varepsilon_n\to0}\int_0^T\int_\Sigma\left(\varepsilon_n\frac{\partial u_{\varepsilon_n}}{\partial x_i}+a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\right)\varphi_1 n_i\,d\sigma\,dt$, we take $\varphi_2\in C^2(\bar Q_T)$ with $\operatorname{supp}\varphi_2\subset\Omega\times(0,T)$, $\varphi_2|_{\Sigma\times(0,T)}=\varphi_1|_{\Sigma\times(0,T)}$. Using (1.3) we have

$$\int_0^T\int_\Sigma\left(\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi_1 n_i\,d\sigma\,dt$$
$$=-\iint_{Q_T}\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt-\iint_{Q_T}\left(\frac{\partial}{\partial x_j}A^{ij}(u_\varepsilon,x,t)-\int_k^{u_\varepsilon}a^{ij}_{x_j}(\tau,x,t)\,d\tau\right)\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt$$
$$-\iint_{Q_T}f^i(u_\varepsilon,x,t)\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt+\iint_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}\,dx\,dt+\iint_{Q_T}g\varphi_2\,dx\,dt-\int_0^T\int_\Sigma f^i(0,x,t)\varphi_1n_i\,d\sigma\,dt.$$

Hence

$$\lim_{\varepsilon_n\to0}\int_0^T\int_\Sigma\left(\varepsilon_n\frac{\partial u_{\varepsilon_n}}{\partial x_i}+a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\right)\varphi_1n_i\,d\sigma\,dt$$
$$=-\iint_{Q_T}\hat a^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2}{\partial u_i}\,dx\,dt-\iint_{Q_T}f^i(u,x,t)\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt+\iint_{Q_T}u\frac{\partial\varphi_2}{\partial t}\,dx\,dt+\iint_{Q_T}g\varphi_2\,dx\,dt$$
$$+\int_0^T\int_\Sigma A^{ij}(\gamma u,x,t)\frac{\partial\varphi_2}{\partial x_i}n_j\,d\sigma\,dt-\int_0^T\int_\Sigma f^i(0,x,t)\varphi_1n_i\,d\sigma\,dt. \tag{4.5}$$

Let $\varepsilon=\varepsilon_n\to 0$ and combine (4.4), (4.5) with (4.3). Using (2.3) and Proposition 1, we deduce the inequality (2.4).

§ 5. Uniqueness and stability of generalized solutions

Similar to Lemma 4, we can prove

Lemma 6. *Let u be a generalized solution of the first boundary value problem for (1.1) with $g^i\in L^\infty(Q_T)$ $(i=1, \cdots, m)$ in (2.1). If $S_1=\Sigma\times[0, T]$, then*

$$\lim_{h\to 0^+}\iint_{Q_T} q\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial \rho_h}{\partial x_i}\,dx\,dt=0 \tag{5.1}$$

where $q\in L^\infty(Q_T)$ and ρ_h is the function in (1.9).

Using this lemma, we can obtain

Proposition 5. *Let u be a generalized solution of the first boundary value problem for (1.1) with $g^i\in L^\infty(Q_T)$ $(i=1, \cdots, m)$ in (2.1). Then*

$$(\operatorname{sgn}(\gamma u-k)+\operatorname{sgn} k)(f^i(\gamma u, x, t)-f^i(k, x, t))n_i\geqslant 0 \tag{5.2}$$

almost everywhere on S_1.

Proof Choose $\varphi_1=\varphi_2=\psi(t)\rho_h(x)\varphi(x)$ in (2.4), with $\varphi(x)\geqslant 0$, $\psi(t)\geqslant 0$, $\psi(t)\in C_0^2(0, T)$, $\varphi(x)\in C^2(\bar{\Omega})$ and $\varphi(x)=0$ in a neighborhood of S_2. Then

$$\begin{aligned}&\iint_{Q_T}\Big\{|u-k|\psi'(t)\varphi(x)\rho_h(x)-\operatorname{sgn}(u-k)\Big[\Big(\hat{a}^{ij}\frac{\partial u}{\partial x_j}+f^i(u, x, t)-f^i(k, x, t)\Big)\\
&\quad\times\Big(\frac{\partial\varphi}{\partial x_i}\rho_h+\varphi\frac{\partial\rho_h}{\partial x_i}\Big)\psi-(f^i_{x_i}(k, x, t)+g)\varphi\psi\rho_h\Big]\Big\}dx\,dt\\
&\quad+\operatorname{sgn} k\iint_{Q_T}\Big[u\psi'\varphi\rho_h-\Big(\hat{a}^{ij}\frac{\partial u}{\partial x_j}+f^i(u, x, t)-f^i(k, x, t)\Big)\Big(\frac{\partial\varphi}{\partial x_i}\rho_h+\varphi\frac{\partial\rho_h}{\partial x_i}\Big)\psi\\
&\quad+(f^i_{x_i}(k, x, t)+g)\varphi\psi\rho_h\Big]dx\,dt\geqslant 0.\end{aligned}$$

Letting $h\to 0$, by virtue of Lemma 6, we have

$$\int_0^T\int_{\Sigma_1}[\operatorname{sgn}(\gamma u-k)+\operatorname{sgn} k][f^i(\gamma u, x, t)-f^i(k, x, t)]\varphi(x)\psi(t)n_i\,d\sigma\,dt\geqslant 0.$$

The inequality (5.2) follows from the arbitrariness of $\psi(t)$ and $\varphi(x)$.

Lemma 7. *Let Ω be a bounded region of m-dimensional, $u(x)\in BV(\Omega)$, $V(x)=(v_1(x), \cdots, v_m(x))\in BV(\Omega)$ and $|V|\leqslant K|u|$ a.e. on Ω with some constant K. If on Σ either $\gamma u=0$ or $\varphi=0$, and $\varphi\in C^2(\bar{\Omega})$, then*

$$\int_\Omega \sigma\operatorname{div}(\varphi, V)\,dx=-\int_{\Gamma_u}\varphi(\operatorname{sgn} u^+-\operatorname{sgn} u^-)(\bar{V}, \nu)\,dH_{m-1},$$

where $\sigma=\frac{1}{2}(\operatorname{sgn} u^++\operatorname{sgn} u^-)$, Γ_u is the set of all jump points of u, $\nu=(\nu_1, \nu_2, \cdots, \nu_m)$ is the normal of Γ_u and $u^+(x_0)$ and $u^-(x_0)$ are the approximate limits of u at $x_0\in\Gamma_u$ with respect to $(\nu, x-x_0)>0$ and $(\nu, x-x_0)<0$, $\bar{V}$ is the symmetric mean value of V.

The proof is similar to the respective proposition in [4]. Since for any $(x, t)\in\Sigma$

at which $\gamma u=0$ we have $\gamma\sigma_s=0$, the boundary integral obtained by integrating by parts is also equal to zero, although in present case φ need not be zero on Σ everywhere.

The following two propositions can be found in [4].

Proposition 6. *Let u be a generalized solution of (1.1). Then for almost all points of Γ_u*

$$(u^+-u^-)\nu_t-(f^i(u^+, x, t)-f^i(u^-, x, t))\nu_{x_i}=0.$$

Proposition 7. *Let u be a generalized solution of (1.1). Then for almost all points of Γ_u.*

$$\begin{aligned}&\operatorname{sgn}(u^+-k)\left[(u^+-k)\nu_t-(f^i(u^+, x, t)-f^i(k, x, t))\nu_{x_i}\right]\\ &\leqslant\operatorname{sgn}(u^--k)\left[(u^--k)\nu_t-(f^i(u^-, x, t)-f^i(k, x, t))\nu_{x_i}\right].\end{aligned} \tag{5.3}$$

Proposition 8. *Let u_1, u_2 be generalized solutions of (1.1) with the respective functions g_1^i, $g_2^i (i=1, \cdots, m)$ in $L^\infty(Q_T)$ and for almost all $t\in[0, T]$, as functions of x, g_1^i, $g_2^i\in \mathrm{BV}(\Omega)$. Then*

$$\begin{aligned}\iint_{Q_T}\Big\{&|u_1-u_2|\frac{\partial\varphi}{\partial t}+\operatorname{sgn}(u_1-u_2)\Big[(g(u_1, x, t)-g(u_2, x, t))\varphi\\ &-(f^i(u_1, x, t)-f^i(u_2, x, t))\frac{\partial\varphi}{\partial x_i}-\Big(\hat{a}^{ij}(u_1, x, t)\frac{\partial u_1}{\partial x_j}\\ &-\hat{a}^{ij}(u_2, x, t)\frac{\partial u_2}{\partial x_j}\Big)\frac{\partial\varphi}{\partial x_i}\Big]\Big\}dx\,dt\geqslant 0,\end{aligned} \tag{5.4}$$

where $\varphi\geqslant 0$, $\varphi\in C^2(Q_T)$ and $\varphi=0$ in a neighborhood of S_1.

Proof Let $z=u_1-u_2$. sgn z in (5.4) may be replaced by $\sigma=\frac{1}{2}(\operatorname{sgn} z^+ +\operatorname{sgn} z^+)$. Using equation (1.1), we can rewrite the left hand side of (5.4) as

$$J(u_1, u_2, \varphi)=J_1(u_1, u_2, \varphi)+J_2(u_1, u_2, \varphi), \tag{5.5}$$

where

$$J_1=\iint_{Q_T}\sigma\left(\frac{\partial\varphi z}{\partial t}-\frac{\partial\varphi\beta^i z}{\partial x_i}\right)dx\,dt,$$

$$J_2=-\iint_{Q_T}\sigma\frac{\partial}{\partial x_i}\left[\varphi(r_1^{ij}g_1^j-r_2^{ij}g_2^j)\right]dx\,dt,$$

$$\beta^i=\int_0^1 f_u^i(\tau u_1+(1-\tau)u_2, x, t)d\tau,$$

$$r_k^{ij}=r^{ij}(u_k, x, t)\quad (k=1, 2).$$

By Lemma 7

$$J_1=-\int_{\Gamma_z}\varphi(\operatorname{sgn} z^+-\operatorname{sgn} z^-)(\bar{z}\nu_t-\overline{\beta^i z}\nu_{x_i})dH_m.$$

By Proposition 6

$$\bar{z}\nu_t-\overline{\beta^i z}\nu_{x_i}=z^+\nu_t-(\beta^i z)^+\nu_{x_i}=z^-\nu_t-(\beta^i z)^-\nu_{x_i}.$$

Thus, taking $k=u_2^+$ and $k=u_1^-$ in (5.3) respectively, we derive

$$\begin{aligned}\operatorname{sgn} z^+(\bar{z}\nu_t-\overline{\beta^i z}\nu_{x_i})&=\operatorname{sgn} z^+(z^+\nu_t-(\beta^i z)^+\nu_{x_i})\\ &\leqslant\operatorname{sgn}(u_2^+-u_1^-)\left[(u_2^+-u_1^-)\nu_t-(f^i(u_2^+, x, t)-f^i(u_1^-, x, t))\nu_{x_i}\right]\\ &\leqslant\operatorname{sgn} z^-(z^-\nu_t-(\beta^i z)^-\nu_{x_i})=\operatorname{sgn} z^-(\bar{z}\nu_t-\overline{\beta^i z}\nu_{x_i}).\end{aligned}$$

Hence $J_1(u_1, u_2, \varphi)\geqslant 0$.

Obviously

$$J_2(u_1, u_2, \varphi) = -\lim_{\varepsilon\to 0}\int_0^T dt\int_\Omega\left(\frac{\bar z}{(z^2+\varepsilon)^{1/2}}\right)\frac{\partial}{\partial x_i}[\varphi(r_1^{ij}g_1^j-r_2^{ij}g_2^j)]\,dx\,dt.$$

Since we have $\gamma z=0$ on S_2 or $\varphi=0$ on S_1, integrating by parts yields

$$-\int_\Omega\left(\frac{\bar z}{(z^2+\varepsilon)^{1/2}}\right)\frac{\partial}{\partial x_i}[\varphi(r_1^{ij}g_1^j-r_2^{ij}g_2^j)]\,dx$$
$$=\int_\Omega\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j-r_2^{ij}g_2^j})\,\frac{\partial z}{\partial x_i}\,dx.$$

The last integral can be rewritten as

$$\int_\Omega\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\,\frac{\partial z}{\partial x_i}\,dx$$
$$=\int_{\Omega\setminus(\Gamma_z(t)\cup\Gamma_g(t)}\varphi\frac{\varepsilon}{(z^2+\varepsilon)^{3/2}}(\bar r_1^{ij}g_1^j-\bar r_2^{ij}g_2^j)\frac{\partial z}{\partial x_i}\,dx$$
$$+\int_{\Gamma_z(t)\cup\Gamma_g(t)}\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\,dx$$
$$=\int_{\Omega\setminus(\Gamma_z(t)\cup\Gamma_g(t))}\varphi\,\frac{\varepsilon}{(z^2+\varepsilon)^{3/2}}\,|g_1-g_2|^2dx$$
$$-\int_{\Omega\setminus(\Gamma_z(t)\cup\Gamma_g(t))}\varphi\,\frac{\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\bar r_1^{ij}-\bar r_2^{ij})\left(g_1^j\frac{\partial u_2}{\partial x_i}-g_2^j\frac{\partial u_1}{\partial x_i}\right)dx$$
$$+\int_{\Gamma_z(t)\cup\Gamma_g(t)}\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\frac{\partial z}{\partial x_i}\,dx,\tag{5.6}$$

where $\Gamma_z(t)$ denotes the set of all jump points of z for fixed t, $\Gamma_g(t)$ denotes the union of the sets of all jump points of g_1^j, g_2^j $(j=1, \cdots, m)$.

The first term on the right hand side of (5.6) is nonnegative. Since on the complement of $\Gamma_z(t)$

$$|\bar r_1^{ij}(u_1, x, t)-\bar r_2(u_2, x, t)|\leqslant K|z|$$

and $\dfrac{\varepsilon z}{(z^2+\varepsilon)^{3/2}}$ is bounded uniformly for ε, we can assert by Lebesgue's theorem that the second term on the right hand side of (5.6) tends to zero as $\varepsilon\to 0$. Observe that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_z(t)\cup\Gamma_g(t)}\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\,\frac{\partial z}{\partial x_i}\,dx$$
$$=\lim_{\varepsilon\to 0}\int_{\Gamma_z(t)}\varphi\frac{\hat\varepsilon}{(z^2+\varepsilon)^{3/2}}\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\,(z^+-z^-)\nu_{x_i}dH_{m-1}$$
$$=\int_{\Gamma_z(t)}\varphi(\operatorname{sgn}z^+-\operatorname{sgn}z^-)\,(\overline{r_1^{ij}g_1^j}-\overline{r_2^{ij}g_2^j})\,\nu_{x_i}dH_{m-1}.\tag{5.7}$$

It is clear that the integrand of the last integral vanishes whenever $z^+\cdot z^->0$. If $z^+\cdot z^-\leqslant 0$, then

$$r^{ij}(\tau, x, t)\nu_{x_j}=0,\quad \min(u_1^+, u_1^-, u_2^+, u_2^-)\leqslant\tau\leqslant\max(u_1^+, u_2^-, u_2^+, u_1^-)$$

(see [4]). From this it follows that the integrand of the last integral vanishes too. Therefore $J_2(u_1, u_2, \varphi)\geqslant 0$. The proof is completed.

Theorem 3. *Suppose $S_1\cap\bar S_2=\phi$, u_1, u_2 are generalized solutions of equation* (1.1) *with homogeneous boundary value and*

$$\gamma u_1(x, 0)=u_0'(x), \quad \gamma u_2(x, 0)=u_0^2(x).$$

If the respective functions g_1^i, g_2^i $(i=1, \cdots, m)$ in (2.1) are bounded and for almost all $t\in[0, T]$, as functions of x, g_1^i, $g_2^i\in \mathrm{BV}(\Omega)$, then for almost all $t\in[0, T]$,

$$\int_\Omega |u_1(x, t)-u_2(x, t)|dx\leqslant N\int_\Omega |u_0^1(x)-u_2^0(x)|dx$$

with constant N independent of u_1, u_2.

As a consequence, we have

Theorem 4. *Under the conditions of Theorem 3, the first boundary value problem for (1.1) has at most one generalized solution.*

Proof of Theorem 3 By Proposition 8

$$\iint_{Q_T}\Big\{|u_1-u_2|\frac{\partial\varphi}{\partial t}+\mathrm{sgn}\,(u_1-u_2)\Big[(g(u_1, x, t)-g(u_2, x, t))\varphi-(f^i(u_1, x, t)$$
$$-f^i(u_2, x, t))\frac{\partial\varphi}{\partial x_i}-\Big(\widehat{a^{ij}(u_1}, x, t)\frac{\partial u_1}{\partial x_j}-\widehat{a^{ij}(u_2}, x, t)\frac{\partial u_2}{\partial x_j}\Big)\frac{\partial\varphi}{\partial x_i}\Big]\Big\}dx\,dt\geqslant 0,$$

where $\varphi\geqslant 0$, $\varphi\in C^2(\bar{Q}_T)$, supp $\varphi\subset\bar{\Omega}\times(0, T)$ and $\varphi=0$ near S_1.

Since $\Sigma_1\cap\Sigma_2=\phi$, we can construct function $\rho_h'(x)\in C^2(\bar{\Omega})$ such that

$$\begin{cases}\rho_h'(x)=1 \quad \text{near} \quad \Sigma_1\\ \rho_h'(x)=0 \quad \text{near} \quad \Sigma_2\\ \lim\limits_{h\to 0^+}\rho_h'(x)=0 \text{ for any } x\in\Omega\\ 0\leqslant\rho_h'(x)\leqslant 1, \quad |\mathrm{grad}\,\rho_h'|\leqslant C/h,\end{cases}$$

where the constant c is independent of h.

Take $\varphi=(1-\rho_h'(x))\psi(t)$, $\psi(t)\in C_0^2(0, T)$, $\psi(t)\geqslant 0$. Then

$$\iint_{Q_T}\Big\{|u_1-u_2|(1-\rho_h')\psi'+\mathrm{sgn}\,(u_1-u_2)\Big[(g(u_1, x, t)-g(u_2, x, t))(1-\rho_h')\psi$$
$$+(f^i(u_1, x, t)-f^i(u_2, x, t))\frac{\partial\rho_h'}{\partial x_i}\psi$$
$$+\Big(\widehat{a^{ij}(u_1}, x, t)\frac{\partial u_1}{\partial x_j}-\widehat{a^{ij}(u_2}, x, t)\frac{\partial u_2}{\partial x_j}\Big)\frac{\partial\rho_h'}{\partial x_i}\psi\Big]\Big\}dx\,dt\geqslant 0.$$

Let $h\to 0$. By Lemma 7 and Lemma 3, we derive

$$\iint_{Q_T}|u_1-u_2|\psi'\,dx\,dt+\iint_{Q_T}\mathrm{sgn}\,(u_1-u_2)(g(u_1, x, t)-g(u_2, x, t))\psi(t)\,dx\,dt$$
$$\geqslant\int_0^T\int_\Sigma \mathrm{sgn}\,(\gamma u_1-\gamma u_2)(f^i(\gamma u_1, x, t)-f^i(\gamma u_2, x, t))n_i\,d\sigma\,dt.$$

Now we define

$$k(x, t)=\begin{cases}\gamma u_1(x, t), & \text{if} \quad \gamma u_1\in I(0, \gamma u_2),\\ 0, & \text{if} \quad 0\in I(\gamma u_1, \gamma u_2),\\ \gamma u_2(x, t), & \text{if} \quad \gamma u_2\in I(0, \gamma u_1).\end{cases}$$

It is easy to show that

$$\operatorname{sgn}(\gamma u_1-\gamma u_2)(f^i(\gamma u_1, x, t)-f^i(\gamma u_2, x, t))n_i$$
$$=\operatorname{sgn}(\gamma u_1-k)(f^i(\gamma u_1, x, t)-f^i(k, x, t))n_i$$
$$+\operatorname{sgn}(\gamma u_2-k)(f^i(\gamma u_2, x, t)-f^i(k, x, t))n_i.$$

This and Proposition 5 give

$$\operatorname{sgn}(\gamma u_1-\gamma u_2)(f^i(\gamma u_1, x, t)-f^i(\gamma u_2, x, t))n_i\geqslant 0 \quad \text{a. e. on } \Sigma_1.$$

Hence

$$\iint_{Q_T}|u_1-u_2|\psi'\,dx\,dt+\iint_{Q_T}\operatorname{sgn}(u_1-u_2)(g(u_1, x, t)-g(u_2, x, t))\psi(t)\,dx\,dt\geqslant 0. \tag{5.8}$$

Let $0<s<\tau<T$. Taking

$$\psi(t)=\int_{t-\tau}^{t-s}\delta_h(\sigma)\,d\sigma,$$

where $\delta_h(\sigma)$ is the function in § 1, we obtain

$$\int_0^T[\delta_h(t-s)-\delta_h(t-\tau)]\,|u_1-u_2|_{L^1(\Omega)}dt+\int_0^T C_0|u_1-u_2|_{L^1(\Omega)}\left(\int_{t-\tau}^{t-s}\delta_h(\sigma)\,d\sigma\right)dt\geqslant 0,$$

where C_0 is a constant. From this we deduce by letting $h\to 0$

$$|u_1(\cdot, \tau)-u_2(\cdot, \tau)|_{L^1(\Omega)}\leqslant|u_1(\cdot, s)-u_2(\cdot, t)|_{L^1(\Omega)}$$
$$+C_0\int_s^\tau|u_1(\cdot, t)-u_2(\cdot, t)|_{L^1(\Omega)}dt.$$

Hence, by Gronwall's lemma we obtain

$$|u_1(\cdot, \tau)-u_2(\cdot, \tau)|_{L^1(\Omega)}\leqslant|u_1(\cdot, s)-u_2(\cdot, s)|_{L^1(\Omega)}e^{C_0(\tau-s)}$$

and the desired result follows by letting $s\to 0$ and using initial value conditions.

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