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# RATES OF A. S. CONVERGENCE OF THE ESTIMATION OF ERROR VARIANCE IN LINEAR MODELS

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**Abstract**

Under appropriate conditions we obtain the best rates of a. s. convergence of estimates $\hat{\sigma}_n^2$ of error variance $\sigma^2$ and establish the law of iterated logarithm about $\hat{\sigma}_n^2$.

## § 1. Introduction and Main Results

Consider the usual linear model

$$y_i = x_i'\beta + e_i,\ i=1,\ \cdots,\ n,\ \cdots, \tag{1}$$

where $\{x_i\}$ is a known sequence of $p$-vectors, $\beta$ is an unknown $p$-vector of regression coefficients, $\{e_i\}$ is an independent random error sequence, satisfying the following conditions

$$\int_0^\infty x^{2\lambda-1} q(x)\,dx < \infty \text{ for some } \lambda>0, \text{ where } q(x) = \sup_i P(|e_i|>x), \tag{2}$$

$$Ee_i = 0,\ i=1,\ 2,\ \cdots, \text{ when } \lambda>\frac{1}{2}. \tag{3}$$

$$0<Ee_i^2=\sigma^2<\infty,\ i=1,\ 2,\ \cdots, \text{ when } \lambda\geqq 1. \tag{4}$$

When $\lambda\geqq 1$, on the bassis of the first $n$ observations of sequence (1), one may calculate the estimate $\hat{\sigma}_n^2$ of $\sigma^2$, based on the residual sum of squares, as follows

$$\hat{\sigma}_n^2 = \frac{1}{n-r_n}\left\{\sum_{k=1}^n e_k^2 - \sum_{j=1}^{r_n}\left(\sum_{k=1}^n c_{njk}e_k\right)^2\right\}, \tag{5}$$

where $r_n = rk(x_1|\cdots|x_n)$, $\{c_{njk},\ j=1,\ \cdots,\ r_n;\ \ k=1,\ \cdots,\ n\}$ is a group of real numbtrs determined by $x_1,\ \cdots,\ x_n$, satisfying

$$\sum_{k=1}^n c_{nik}c_{njk} = \delta_{ij}, \tag{6}$$

where $\delta_{ij}$ is Kronecker sign. Even in case $\lambda<1$, we may still define $\hat{\sigma}_n^2$ by (5). For rates of convergence of $\hat{\sigma}_n^2$, we prove the following theorem:

**Theorem 1.** *Suppose $e_1,\ e_2,\ \cdots$ are mutually independently distributed, we have*

(i) *If* (2) *is satisfied with* $0<\lambda\leqq\frac{1}{2}$, or (2)—(3) are satisfied with $\frac{1}{2}<\lambda<1$, then

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$$\frac{1}{n^{\frac{1}{\lambda}-1}}\hat{\sigma}_n^2 \xrightarrow{\text{a.s.}} 0 \quad as\ n\to\infty. \tag{7}$$

(ii) If (2)—(4) *are satisfied with* $1\leqslant\lambda<2$, then

$$n^{1-\frac{1}{\lambda}}(\hat{\sigma}_n^2-\sigma^2)\xrightarrow{\text{a.s.}} 0 \quad as\ n\to\infty. \tag{8}$$

*when* $1<\lambda<2$, *this theorem gives the rates of convergence of* $\hat{\sigma}_n^2\xrightarrow{\text{a.s.}}\sigma^2$, *that is*

$$\hat{\sigma}_n^2-\sigma^2=o(n^{-\left(1-\frac{1}{\lambda}\right)})\ \text{a. s.} \tag{8'}$$

*When* $\lambda\geqq 2$, (8′) *is not true, in this case the rates of convergence are described by the following law of iterated logarithm. For* $\lambda=2$, *we have*

**Theorem 2.** *Suppose* $e_1$, $e_2$, $\cdots$ *are mutually independently distributed, and* (2)—(4) *are satisfied with* $\lambda=2$, *and*

$$\liminf_{n\to\infty} B_n/n>0, \tag{9}$$

*where* $B_n=\sum_{k=1}^{n}\mathrm{Var}(e_k^2)$, *then*

$$\left.\begin{aligned}\limsup_{n\to\infty}\frac{n(\hat{\sigma}_n^2-\sigma^2)}{\sqrt{2B_n\log\log B_n}}&=1\ \text{a. s.}\\ \liminf_{n\to\infty}\frac{n(\hat{\sigma}_n^2-\sigma^2)}{\sqrt{2B_n\log\log B_n}}&=-1\ \text{a. s.}\end{aligned}\right\} \tag{10}$$

When (2) is not satisfied, we have

**Theorem 3.** *Suppose* $e_1$, $e_2$, $\cdots$ *are mutually independently distributed,* $Ee_i=0$, $Ee_i^2=\sigma^2$ $(i=1, 2, \cdots)$, *write* $Z_i=|e_i^2-\sigma^2|$, *then under the condition* (9) *and*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}E(Z_k^2|\log Z_k|^{1+\delta})<\infty\ for\ some\ \delta>0, \tag{11}$$

(10) *holds true.*

We remark that the condition (2) is equivalent to the following condition:

There exists a r. v. $\tilde{\epsilon}$ such that, $E|\tilde{\epsilon}|^{2\lambda}<\infty$ and $P(|e_i|>x)\leqq P(|\tilde{\epsilon}|>x)$ for any $x\geqq 0$ and any $i$. (2′)

In this case we often say that $|\tilde{\epsilon}|$ is stochastically large than $|\tilde{\epsilon}|$ for all $i$. If $\{e_i\}$ is an i.i.d. sequence, and $E|e_1|^{2\lambda}<\infty$, then (2) holds trivially.

It is well known that, if $X_1$, $X_2$, $\cdots$i.i.d., $E|X_1|^\lambda<\infty$, $\bar{X}_n=\frac{1}{n}\sum_1^n X_i$, then the Marcinkiewicz strong law of large numbers is true, that is

$$\lim_{n\to\infty}\bar{X}_n/n^{\frac{1}{\lambda}-1}=0\ \text{a. s., when } 0<\lambda<1, \tag{12}$$

$$\lim_{n\to\infty}n^{1-\frac{1}{\lambda}}(\bar{X}_n-EX_1)=0\ \text{a. s., when } 1\leqq\lambda<2. \tag{13}$$

Conversely, if

$$n^{1-\frac{1}{\lambda}}(\bar{X}_n-b_n)\xrightarrow{\text{a.s.}} 0 \quad \text{as } n\to\infty \tag{14}$$

for some $0<\lambda<2$ and centering constants $\{b_n\}$, then $E|X_1|^\lambda<\infty$. This shows that, under the conditions of Theorem 1, the orders given by (7) and (8) are the best possible.

## § 2. Some Lemmas

**Lemma 1.** *Suppose* $y\leqq 1$, *then*

$$e^y\leqq 1+2|y|^{2q},\ \textit{when}\ 0<q\leqq\frac{1}{2}, \tag{15}$$

$$e^y\leqq 1+y+|y|^{2q},\ \textit{when}\ \frac{1}{2}<q\leqq 1. \tag{16}$$

The proof of the Lemma is easy. For example, when $|y|\leqq 1$ and $0<q\leqq\frac{1}{2}$, we have

$$e^y=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots\leqq 1+|y|\left(1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots\right)=1+2|y|\leqq 1+2|y|^{2q}. \tag{17}$$

Other cases can be dealt with similarly.

**Lemma 2.** *Suppose* $e_1, e_2, \cdots$ *are mutually independently distributed,* $E|e_k|^{2q}\leqq M<\infty$ $(k=1, 2, \cdots)$ *for some* $0<q\leqq 1$, and $Ee_k=0$ *for each* $k$ *when* $q>\frac{1}{2}$. *Also, for some* $\beta>0$

$$\sum_{k=1}^{\infty}P(|e_k|>k^\beta\varepsilon)<\infty\ \textit{for every}\ \varepsilon>0. \tag{18}$$

*Let* $\{a_{nk},\ k,\ n=1, 2, \cdots\}$ *be a double sequence of real number, satisfying*

$$|a_{nk}|\leqq Dk^{-\beta}\ \textit{for all}\ k,\ n, \tag{19}$$

*and for some* $\alpha>0$

$$a_n\triangleq\sum_k|a_{nk}|^{2q}\leqq Dn^{-\alpha}\ \textit{for all}\ n, \tag{20}$$

*where* $D>0$ *is a constant. Then* $T_n=\sum_{k=1}^{\infty}a_{nk}e_k$ *are a. s. finite, and*

$$\lim_{n\to\infty}T_n=0\quad \text{a. s.} \tag{21}$$

*Proof* It is well known that, if $X_1, X_2, \cdots$ are mutually independently distributed, and $\sum_k E|X_k|^\delta<\infty$ for some $0<\delta\leqq 2$, then $\sum_{k=1}^{\infty}X_k$ or $\sum_{k=1}^{\infty}(X_k-EX_k)$ converges a. s. according to $0<\delta\leqq 1$ or $1<\delta\leqq 2$ respectively. For any fixed $n$, under the conditions of the Lemma, $\sum_k|a_{nk}|^{2q}E|e_k|^{2q}<\infty$ with $0<2q\leqq 2$, and $Ee_k=0$ when $1<2q\leqq 2$, therefore $T_n$ are all a. s. finite.

Without loss of generality, we assume that all $a_{nk}\geqq 0$. Choose $N=\left[\frac{2}{\alpha}+1\right]$, where $[x]$ denotes the maximum integral number not exceeding $x$. By (18), there exists $\varepsilon_k\downarrow 0$ such that

$$\sum_{k=1}^{\infty}P\left(|e_k|>\frac{\varepsilon_k}{ND}k^\beta\right)<\infty. \tag{22}$$

By Borel-Cantelli's lemma

$$P\left(|e_k|>\frac{\varepsilon_k}{ND}k^\beta, \text{ i. o.}\right)=0, \tag{23}$$

therefore for any $t>0$

$$\sum_{k=1}^{\infty}|e_k|^t I\left(|e_k|>\frac{\varepsilon_k}{ND}k^\beta\right)<\infty \quad \text{a. s.,} \tag{24}$$

where $I(A)$ denotes the indicator of the event $A$.

Let

$$\left.\begin{aligned} &e'_k=e_kI\left(e_k>\frac{\varepsilon_k}{ND}k^\beta\right), \quad T'_n=\sum_k a_{nk}e'_k,\\ &e''_{nk}=e_kI(a_{nk}e_k\leq n^{-\alpha/4q}), \quad T''_n=\sum_k a_{nk}e''_{nk},\\ &e'''_{nk}=e_k-e'_k-e''_{nk}, \quad T'''_n=\sum_k a_{nk}e'''_{nk}. \end{aligned}\right\} \tag{25}$$

For any fixed $n$, when $T_n=\sum_k a_{nk}e_k$ is finite, $e_k=e''_{nk}$ for $k$ large enough, hence $|T_n|<\infty$ a. s. implies $|T''_n|<\infty$ a. s., and $|T'_n|<\infty$ a. s. in view of the inequality (26) to be deduced in the following, we have $|T'''_n|<\infty$ a. s. also.

When $0<q\leq\frac{1}{2}$, let $q'=2/3$, otherwise let $q'=q$, then

$$a'_n\triangleq\sum_k|a_{nk}|^{2q'}\leq D'n^{-\alpha} \text{ for all } n,$$

where $D'>0$ is a constant. By Hölder's inequality

$$|T'_n|\leq|a'_n|^{\frac{1}{2q'}}\left\{\sum_k|e_k|^{\frac{2q'}{2q'-1}}I\left(|e_k|>\frac{\varepsilon_k}{ND}k^\beta\right)\right\}^{\frac{2q'-1}{2q'}}\to 0 \quad \text{a. s.}$$
$$\text{as } n\to\infty. \tag{26}$$

Let $y_{nk}=n^{\frac{\alpha}{4q}}a_{nk}e''_{nk}$, then $y_{nk}\leq 1$, and $Ey_{nk}\leq 0$ when $\frac{1}{2}<q\leq 1$. By Lemma 1

$$\exp(y_{nk})\leq\begin{cases}1+2|y_{nk}|^{2q}, \text{ when } 0<q\leq\frac{1}{2},\\ 1+y_{nk}+|y_{nk}|^{2q}, \text{ when } \frac{1}{2}<q\leq 1.\end{cases}$$

Thus for $0<q\leq 1$

$$E\exp(y_{nk})\leq 1+2E|y_{nk}|^{2q}\leq\exp(2E|Y_{nk}|^{2q}). \tag{27}$$

Therefore by Fatou's lemma, noticing (20) and $E|e''_{nk}|^{2q}\leq M$, we have

$$\begin{aligned} E\{\exp(n^{\frac{\alpha}{4q}}T''_n)\}&=E\left\{\liminf_{K\to\infty}\exp\left(\sum_{k=1}^K y_{nk}\right)\right\}\\ &\leq\liminf_{K\to\infty}E\left\{\exp\left(\sum_{k=1}^K y_{nk}\right)\right\}=\liminf_{K\to\infty}\prod_{k=1}^K E\{\exp(y_{nk})\}\\ &\leq\liminf_{K\to\infty}\prod_{k=1}^K\exp(2E|y_{nk}|^{2q})\leq\liminf_{K\to\infty}\exp\left(2n^{\alpha/2}\sum_{k=1}^K|a_{nk}|^{2q}E|e''_{nk}|^{2q}\right)\\ &\leq\exp(2n^{\alpha/2}\cdot Dn^{-\alpha}\cdot M)\leq c, \end{aligned} \tag{28}$$

where $c>0$ is a constant. For any $\varepsilon>0$

$$P(T''_n \geqq \varepsilon) \leqq \exp(-n^{\frac{\alpha}{4q}}\varepsilon) E\{\exp(n^{\frac{\alpha}{4q}} T''_n)\} \leqq c \exp(-n^{\frac{\alpha}{4q}}\varepsilon), \tag{29}$$

hence

$$\sum_{n=1}^{\infty} P(T''_n \geqq \varepsilon) < \infty, \tag{30}$$

by Borel-Cantelli's lemma

$$P(\limsup_{n\to\infty} T''_n > \varepsilon) \leqq P(T''_n \geqq \varepsilon, \text{ i. o.}) = 0, \tag{31}$$

since $\varepsilon$ is arbitrary, we have

$$P(\limsup_{n\to\infty} T''_n > 0) = 0,$$

i. e.

$$\limsup_{n\to\infty} T''_n \leqq 0 \text{ a. s.} \tag{32}$$

For any given $\varepsilon > 0$, choose a fixed integral number $K$ such that $\varepsilon_k < \varepsilon/2$ for all $k > K$. By (20), there exists $n_0$ such that, when $n \geqq n_0$

$$\sum_{k=1}^{K} a_{nk} e'''_{nk} \leqq (Dn^{-\alpha})^{\frac{1}{2q}} \sum_{k=1}^{K} \frac{\varepsilon_k}{ND} k^{\beta} < \frac{\varepsilon}{2}. \tag{33}$$

For $n \geqq n_0$, let

$$D_n = \{k \mid k > K,\ a_{nk} e_k \geqq n^{-\frac{\alpha}{4q}}\}, \tag{34}$$

then $D_n$ is a finite set with probability one since $|T_n| < \infty$ a. s., denoting $d_n$ the number of elements in $D_n$, we have

$$\sum_{k>K} a_{nk} e'''_{nk} \leqq \sum_{k\in D_n} Dk^{-\beta} \cdot \frac{\varepsilon_k}{ND} k^{\beta} \leqq d_n \frac{\varepsilon}{2N}, \tag{35}$$

thus $\sum_{k<K} a_{nk} e'''_{nk} \geqq \varepsilon/2$ implies $d_n \geqq N$. So we have

$$\begin{aligned} P(T'''_n \geqq \varepsilon) &\leqq P\{\sum_{k<K} a_{nk} e'''_{nk} \geqq \frac{\varepsilon}{2}\} \\ &\leqq P\ \{\text{There exist at least } N \text{ of subscripts } k > K \\ &\quad \text{such that } |e_k| \geqq (a_{nk} n^{\frac{\alpha}{4q}})^{-1}\} \\ &\leqq \{\sum_k P\,[|e_k| \geqq (a_{nk} n^{\frac{\alpha}{4q}})^{-1}]\}^N \leqq (\sum_k |a_{nk}|^{2q} n^{\alpha/2} E|e_k|^{2q})^N \\ &\leqq (DMn^{-\alpha+\frac{\alpha}{2}})^N = (DM)^N n^{-\frac{\alpha}{2}N}, \end{aligned} \tag{36}$$

by the choice of $N$, $\frac{\alpha}{2} N > 1$, therefore

$$\sum_{n=1}^{\infty} P(T'''_n \geqq \varepsilon) < \infty \text{ for any given } \varepsilon > 0. \tag{37}$$

The same reasoning as before gives

$$\limsup_{n\to\infty} T'''_n \leqq 0 \quad \text{a. s.} \tag{38}$$

From (26), (32) and (38), it follows that

$$\limsup_{n\to\infty} T_n \leqq 0 \quad \text{a. s.} \tag{39}$$

From (39), replacing $e_k$ by $-e_k$, one gets

$$\liminf_{n\to\infty} T_n \geqq 0 \quad \text{a. s.} \tag{40}$$

Now (21) follows from (39) and (40). The Lemma is proved.

**Lemma 3.** *Suppose $e_1, e_2, \cdots$ are mutually independently distributed $E|e_k|^{2\lambda}\leqq M<\infty$ for some $0<\lambda\leqq 1$, $Ee_k=0$ when $\frac{1}{2}<\lambda\leqq 1$ $(k=1, 2, \cdots)$, and*

$$\sum_{k=1}^{\infty} P(|e_k|<k^{\frac{1}{2\lambda}}\varepsilon)<\infty \text{ for every } \varepsilon>0. \tag{41}$$

*Then we have*

$$T_n \triangleq n^{-\frac{1}{2\lambda}} \sum_{k=1}^{n} c_{nk} e_k \xrightarrow{\text{a.s.}} 0 \text{ as } n\to\infty, \tag{42}$$

*where $\{c_{nk},\ k=1, \cdots, n;\ n=1, 2, \cdots\}$ is any triangle sequence of real numbers, satisfying the condition*

$$\sum_{k=1}^{n} c_{nk}^2 \leqq 1. \tag{43}$$

*Proof* Let

$$a_{nk}=\begin{cases} n^{-\frac{1}{2\lambda}} c_{nk}, & \text{when } k\leqq n,\\ 0, & \text{when } k>n,\end{cases}$$

then

$$\left.\begin{aligned} &|a_{nk}|\leqq n^{-\frac{1}{2\lambda}}\leqq k^{-\frac{1}{2\lambda}}, \text{ when } k\leqq n,\\ &a_n\triangleq\sum_k |a_{nk}|^{2\lambda}=\sum_{k=1}^{n}|c_{nk}^2|^{\lambda}/n\leqq\left(\frac{1}{n}\sum_{k=1}^{n} c_{nk}^2\right)^{\lambda}\leqq n^{-\lambda} \text{ for all } n,\end{aligned}\right\} \tag{44}$$

so that, by Lemma 2 with $q=\lambda$, $\beta=\frac{1}{2\lambda}$ and $\alpha=\lambda$, (42) holds.

Similarly, we have

**Lemma 4.** *Suppose $e_1, e_2, \cdots$ are mutually independentlly distributed, $Ee_k=0$, $Ee_k^2 \leqq M<\infty$ $(k=1, 2, \cdots)$, and (41) holds with $\lambda\geqq 1$, then (43) implies (42).*

## § 3. Proof of the Theorems

*Proof of Theorem* 1. First of all, we prove that the condition (2) is equivalent to (2′). To do this, write $\tilde{q}(x)=P(|\tilde{\epsilon}|>x)$, $q_i(x)=P(|e_i|>x)$, $\tilde{F}(x)=1-\tilde{q}(x)$, $F_i(x)=1-q_i(x)$. Assume that (2′) holds, then for $\forall A>0$, $\lambda>0$

$$\int_0^A x^{2\lambda}d\tilde{F}(x)=-\int_0^A x^{2\lambda}d\tilde{q}(x)=-A^{2\lambda}\tilde{q}(A)+2\lambda\int_0^A x^{2\lambda-1}\tilde{q}(x)\,dx, \tag{45}$$

$$A^{2\lambda}\tilde{q}(A)\leqq\int_{x\geqq A} x^{2\lambda}d\tilde{F}(x)\to 0 \text{ as } A\to\infty, \tag{46}$$

therefore

$$E|\tilde{\epsilon}|^{2\lambda}=\int_0^\infty x^{2\lambda}\,d\tilde{F}(x)=2\lambda\int_0^\infty x^{2\lambda-1}\tilde{q}(x)\,dx. \tag{47}$$

By $q(x)=\sup_i q_i(x)\leqq\tilde{q}(x)$, we have

$$\int_0^\infty x^{2\lambda-1}q(x)\,dx\leqq\int_0^\infty x^{2\lambda-1}\tilde{q}(x)\,dx<\infty, \tag{48}$$

i. e., condition (2) is satisfied. Conversely, suppose that (2) holds, then $q(x)$ is a decreasing function with $q(0_-)=1$ and $q(\infty)=0$. In fact, if $q(\infty)=c>0$, then

$$\int_0^\infty x^{2\lambda-1}q(x)\,dx \geqq c\int_0^\infty x^{2\lambda-1}\,dx=\infty \text{ for } \lambda>0, \tag{49}$$

this contradicts (2). Choose any $x_n \downarrow x$, then $q(x_n)\leqq q(x)$; on the other hand, for $\forall \varepsilon >0$, there exists $i_0$ such that $q_{i_0}(x)\geqq q(x)-\varepsilon/2$, and $q_{i_0}(x_n)\geqq q_{i_0}(x)-\varepsilon/2$ for all sufficiently large $n$ by the right continuity of $q_{i_0}(x)$, thus $q(x_n)\geqq q_{i_0}(x_n)\geqq q_{i_0}(x)-\varepsilon/2 \geqq q(x)-\varepsilon$ for sufficiently large $n$, from this, right continuity of $q(x)$ is obtained. Therefore, $F(x)=1-q(x)$ furnishes the distribution function of some random variable $\epsilon$, and by (45)

$$\int_0^\infty x^{2\lambda}dF(x)\leqq 2\lambda\int_0^\infty x^{2\lambda-1}q(x)\,dx<\infty, \tag{50}$$

i. e., $E\epsilon^{2\lambda}<\infty$, thus the condition (2′) is satisfied.

It is well known that, if $\{X_n\}$ is a sequence of independent r. v.'s, and there exists a r. v. $X$ such that $E|X|^p<\infty$ $(0<p<2)$, and $P(|X_n|>x)\leqq P(|X|>x)$ for $\forall\, x\geqq 0$ and $\forall$ n, then

$$\lim_{n\to\infty} n^{-\frac{1}{p}}\sum_{k=1}^n (X_k-a_k)=0 \quad \text{a. s.},$$

where $a_k=0$ or $EX_k$, according to $0<p<1$ or $1\leqq p<2$ respectively (see [1], p. 242). Therefore

$$\lim_{n\to\infty} n^{-\frac{1}{\lambda}}\sum_{k=1}^n (e_k^2-a)=0 \quad \text{a. s.}, \tag{51}$$

where $a=0$ or $\sigma^2$, according to $0<\lambda<1$ or $1\leqq\lambda<2$ respectively.

By (45) and (47), noticing $q_k(x)\leqq\tilde{q}(x)$, we have

$$\int_0^A x^{2\lambda}dF_k(x)\leqq 2\lambda\int_0^\infty x^{2\lambda-1}q_k(x)\,dx\leqq 2\lambda\int_0^\infty x^{2\lambda-1}\tilde{q}(x)\,dx=E|\tilde{\epsilon}|^{2\lambda},$$

i. e.

$$E|e_k|^{2\lambda}\leqq E|\tilde{\epsilon}|^{2\lambda}\triangleq M<\infty \text{ for all } k. \tag{52}$$

On the other hand, for any $\varepsilon>0$,

$$\begin{aligned}\sum_{k=1}^\infty P(|e_k|>k^{\frac{1}{2\lambda}}\varepsilon) &\leqslant \sum_{k=1}^\infty P(|\tilde{\epsilon}|>k^{\frac{1}{2\lambda}}\varepsilon)=\sum_{k=1}^\infty P(|\tilde{\epsilon}|^{2\lambda}>k\varepsilon^{2\lambda})\\ &=\sum_{k=1}^\infty\sum_{j=k}^\infty P(j\varepsilon^{2\lambda}<|\tilde{\epsilon}|^{2\lambda}\leqq(j+1)\varepsilon^{2\lambda})\\ &=\sum_{j=1}^\infty jP(j\varepsilon^{2\lambda}<|\tilde{\epsilon}|^{2\lambda}\leqq(j+1)\varepsilon^{2\lambda})\leqq\frac{1}{\varepsilon^{2\lambda}}E|\tilde{\epsilon}|^{2\lambda}<\infty.\end{aligned} \tag{53}$$

Applying Lemma 3 (when $0<\lambda\leqq 1$) or Lemma 4 (when $1<\lambda<\infty$), noticing $r_n\leqq p$, we have

$$n^{-\frac{1}{\lambda}}\sum_{j=1}^{r_n}\Big(\sum_{k=1}^n c_{njk}e_k\Big)^2 \xrightarrow{\text{a.s.}} 0 \quad \text{as } n\to\infty. \tag{54}$$

By (5), (51) and (54), Theorem 1 is proved.

*Proof of Theorem* **2**. To begin with, we quote a result in [2] (refer to [3], p. 317): Let $\{X_n\}$ be a sequence of independent r. v.'s with $EX_n=0$, there exists a r. v. $X$ such that $EX^2<\infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}P(|X_k|>x)\leq P(|X|>x) \text{ for all sufficiently large } x \text{ and } n, \tag{55}$$

write $B_n=\sum_{k=1}^{n}V_{ar}(X_k)$, then when

$$\liminf_{n\to\infty} B_n/n>0, \tag{56}$$

we have

$$\limsup_{n\to\infty}\frac{\sum_{k=1}^{n}X_k}{\sqrt{2B_n\log\log B_n}}=1 \quad \text{a. s.} \tag{57}$$

Suppose that the conditions of Theorem 2 are satisfied, then for $x>\sigma^2$ and all $k$

$$P(|e_k^2-\sigma^2|>x)=P(e_k^2>\sigma^2+x)\leq P(\tilde{\epsilon}^2>\sigma^2+x)\leq P(|\tilde{\epsilon}^2-\sigma^2|>x). \tag{58}$$

Thus $\{e_k^2-\sigma^2\}$ and $\tilde{\epsilon}^2-\sigma^2$ can serve as $\{X_k\}$ and X respectively in (55). Writing $h_n=\sqrt{2B_n\log\log B_n}$, we have

$$\left.\begin{aligned}&\limsup_{n\to\infty}\frac{1}{h_n}\sum_{k=1}^{n}(e_k^2-\sigma^2)=1 \quad \text{a. s.}\\ &\liminf_{n\to\infty}\frac{1}{h_n}\sum_{k=1}^{n}(e_k^2-\sigma^2)=-1 \quad \text{a. s.}\end{aligned}\right\} \tag{59}$$

By (9), there exists a constant $A>0$ such that

$$\frac{B_n}{n}>\frac{1}{2A^2} \text{ for all sufficiently large } n, \tag{60}$$

so that, by (54) with $\lambda=2$

$$\lim_{n\to\infty}\frac{1}{h_n}\sum_{j=1}^{r_n}\left(\sum_{k=1}^{n}c_{njk}e_k\right)^2\leq\lim_{n\to\infty}An^{-\frac{1}{2}}\sum_{j=1}^{r_n}\left(\sum_{k=1}^{n}c_{njk}e_k\right)^2=0 \quad \text{a. s.} \tag{61}$$

On the other hand, we have

$$\frac{n(\hat{\sigma}_n^2-\sigma^2)}{\sqrt{2B_n\log\log B_n}}=\frac{n}{(n-r_n)h_n}\sum_{k=1}^{n}(e_k^2-\sigma^2)+\frac{nr_n\sigma^2}{(n-r_n)h_n}+\frac{n}{(n-r_n)h_n}\sum_{j=1}^{r_n}\left(\sum_{k=1}^{n}c_{njk}e_k\right)^2, \tag{62}$$

and

$$\lim_{n\to\infty}\frac{nr_n\sigma^2}{(n-r_n)h_n}\leq\lim_{n\to\infty}An^{-\frac{1}{2}}p\sigma^2=0, \tag{63}$$

therefore (10) holds, and Theorem 2 is proved.

*Proof of Theorem* 3. Without loss of generality, assume that $\sigma^2=1$. By a well known theorem (see[3], p. 306), under conditions (9) and (11) we have

$$\left.\begin{aligned}&\limsup_{n\to\infty}\frac{1}{h_n}\sum_{k=1}^{n}(e_k^2-1)=1 \quad \text{a. s.}\\ &\liminf_{n\to\infty}\frac{1}{h_n}\sum_{k=1}^{n}(e_k^2-1)=-1 \quad \text{a. s.}\end{aligned}\right\} \tag{64}$$

To simplify the writing, let $s_n=\sum_{k=1}^{n}E(Z_k^2|\log Z_k|^{1+\delta})$, and use $c$ to denote a positive constant which may take diffrent value in each appearence. By (11), we have

$$s_n \leqq cn, \tag{65}$$

so that, when $n>16$ we have

$$\begin{aligned}\sum_{k=16}^{n} P(|e_k|>k^{\frac{1}{4}}) &\leqq \sum_{k=16}^{n} \frac{E(Z_k^2|\log Z_k|^{1+\delta})}{(\sqrt{k}-1)^2[\log(\sqrt{k}-1)]^{1+\delta}} \leqq c\sum_{k=16}^{n} \frac{s_k - s_{k-1}}{k(\log k)^{1+\delta}} \\ &\leqq c\left\{\sum_{k=16}^{n} s_k\left[\frac{1}{k(\log k)^{1+\delta}} - \frac{1}{(k+1)(\log(k+1))^{1+\delta}}\right] + \frac{s_n}{(n+1)(\log(n+1))^{1+\delta}}\right\} \\ &\leqq c\left\{\sum_{k=16}^{n} k\left[\frac{1}{k(\log k)^{1+\delta}} - \frac{1}{(k+1)(\log(k+1))^{1+\delta}}\right] + \frac{n}{(n+1)(\log(n+1))^{1+\delta}}\right\} \\ &\leqq c\sum_{k=16}^{\infty} \frac{1}{k(\log k)^{1+\delta}} < \infty, \end{aligned} \tag{66}$$

hence

$$\sum_{k=1}^{\infty} P(|e_k|>k^{\frac{1}{4}}) < \infty, \tag{67}$$

argueing in the same way, for any $\varepsilon>0$, we have

$$\sum_{k=1}^{\infty} P(|e_k|>k^{\frac{1}{4}}\varepsilon) < \infty. \tag{68}$$

Therefore, By Lemma 4, (61) holds, and (10) is proved.

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