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A GENERIC PROPERTY FOR DIFFEOMORPHISMS OF THE 2-SPHERE

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Abstract

S. Smale posed the following problem as one of "Problems of Present Day Mathematics":

Is having an attracting periodic orbit (i.e. having a periodic sink or source) a generic property for diffeomorphisms of the 2-sphere S^2?

In this paper, we will prove that having a periodic sink or source is a generic property for $f \in \mathrm{Diff}^1(S^2)$, and therefore give an affirmative answer to the above Smale's problem.

Let Diff^r (M) be the set of all C^r-diffeomorphism from a C^r manifold M to itself, endowed with the Whitney C^r-topology. Diff^λ (M) becomes then a Baire space (see [1] chap. 0, [2] chap. II). A property (of $f \in \mathrm{Diff}^r$ (M)) $P(f)$ is said to be C^r generic, if the set $\{f \in \mathrm{Diff}^r(\mathrm{M}) \mid P(f)\}$ contains a Baire residual subset of Diff^r (M) (see [3]). A point $p \in M$ is called an n-periodic point of $f \in \mathrm{Diff}^r$ (M), if n is the least positive integer m such that $f^m(p) = p$. An n-periodic point p of f is said to be hyperbolic, if every eigenvalue λ of $Df^n(p)$ satisfies $|\lambda| \neq 1$. In particular an n-periodic point p of f is called a periodic source (sink), if every eigenvalue λ of $Df^n(p)$ satisfies $|\lambda| > 1$ ($|\lambda| < 1$).

The following problem has attracted people's attention:

Is having a periodic source or sink a generic property for diffeomorphisms of the 2-sphere?

Actually, since 1962, some similarly formulated problems have been mentioned by S. Smale and others at symposia (see [4, p. 494], [5, p. 350], [6, p. 61]). For instance, in 1974, at the symposium entitled "Mathematical Developments Arising from Hilbert Problems", S. Smale posed the following problem as one of "Problems of Present Day Mathematics":

Is having an attracting periodic orbit a generic property for diffeomorphisms of the two-sphere S^2? ([6, p. 61], a positive or negative attracting periodic orbit is an orbit through a periodic sink or source.)

In 1974, Plykin (Плыкин) [7, p. 259] proved that every Axiom A diffeomorphism of 2-sphere has a periodic source or sink.

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Now, by combining the use of the recent results of Liao Shan Tao (S. D. Liao)[8] and the result of Plykin mentioned above, we can prove the following theorem.

Theorem. *Having a periodic source or sink is a generic property for* $f\in Diff^1(S^2)$.

Actually we shall prove that the set

$$\mathscr{S}=\left\{f\in \mathrm{Diff}^1(S^2)\ \middle|\ f \text{ has at least a periodic source or sink}\right\}$$

is an open and dense subset of $\mathrm{Diff}^1(S^2)$. We remind that every homeomorphism of S^2 has at least a periodic point by usual arguments on topological degree and Lefschetz number.

Proof of Theorem Because of the local stability property of a hyperbolic periodic point (see [1] chap. 2), it is obvious that $\mathscr{S}$ is an open subset of $\mathrm{Diff}^1(S^2)$. So the only thing to be proved is the density of $\mathscr{S}$ in $\mathrm{Diff}^1(S^2)$.

Write

$$\mathscr{G}=\{g\in \mathrm{Diff}^1(S^2)\ |\text{all the periodic points of } g \text{ are hyperbolic}\}$$

and $\mathscr{G}^0=\mathrm{int}\,\mathscr{G}$ (the interior of $\mathscr{G}$ in the whitney C^1 topology of $\mathrm{Diff}^1(S^2)$.)
By the Kupka-Smale Theorem (see [1] chap. 2), $\mathscr{G}$ is a residual subset in $\mathrm{Diff}^1(S^2)$. So it is dense in $\mathrm{Diff}^1(S^2)$. By Theorem II in [8, p. 9], $g\in\mathscr{G}^0$ is Ω-stable, and then by Theorem I in [8, p. 9] it satisfies Axiom A. It follows then from the result in [7, p. 259] that every $g\in\mathscr{G}^0$ has a periodic source or sink. So $\mathscr{G}^0\subset\mathscr{S}$.

Now, consider any $g\in\mathscr{G}\setminus\mathscr{G}^0$ Clearly, any C^1 neighborhood of g contains some $h\in\mathrm{Diff}^1(S^2)\setminus\mathscr{G}$. Then h has an n-periodic point p such that $Dh^n(p)$ has an eigenvalue λ_1 satisfying $|\lambda_1|=1$. Let λ_2 be the other eigenvalue of $Dh^n(p)$. There are two possible cases, namely $|\lambda_2|>1$ or $|\lambda_2|\leqslant 1$.

Write $\quad p_j=h^j(p)\quad (0\leqslant j\leqslant n,\ p_0=p_n=p)$.

First choose open neighborhoods of p_j, U_j, V_j, W_j, satisfying the following conditions:

i) $W_j(1\leqslant j\leqslant n)$ are disjoint, $W_0\subset W_n$;

ii) $\overline{U}_j\subset V_j\subset\overline{V}_j\subset W_j\ (0\leqslant j\leqslant n)$;

iii) $h(W_j)\subset U_{j+1}\ (0\leqslant j\leqslant n-1)$;

iv) there are local coordinate maps φ_j such that $\varphi_j(p_j)=0$,

$$\varphi_j(U_j)=D(1),\ \varphi_j(V_j)=D(2),\ \varphi_j(W_j)=D(3)\quad (0\leqslant j\leqslant n),$$

where $D(r)$ denotes the open disc of radius r in R^2 with center at origin.

Next, choose C^∞"bump" function δ: $R^2\to R^1$ to satisfy the following conditions:

$$\delta(u)=\begin{cases}1, & u\in D(1),\\ 0, & u\notin D(2),\end{cases}$$

$$0\leqslant\delta(u)\leqslant 1,\ u\in R^2.$$

Define η_ε: $R^2\to R^1$ and k: $S^2\to S^2$ as follows:

$$\eta_\varepsilon(u)=1+\varepsilon\delta(u),\ u\in R^2, \text{ in case } |\lambda_2|>1,$$

$$\eta_\varepsilon(u)=1-\varepsilon\delta(u),\ u\in R^2, \text{ in case } |\lambda_2|\leqslant 1,$$

$$k(x)=\begin{cases}\varphi_{j+1}^{-1}(\eta_\varepsilon(\varphi_j(x))\varphi_{j+1}(h(x))), & x\in W_j,\ 0\leqslant j\leqslant n-1,\\ h(x), & x\in S^2\setminus\bigcup_{j=0}^{n-1}W_j.\end{cases}$$

Obviously such k belongs to $C^1(S^2, S^2)$. In W_j $(0\leqslant j\leqslant n-1)$, the local representations of k are

$$\tilde{k}_j(u)=\varphi_{j+1}\circ k\circ\varphi_j^{-1}(u)=\eta_\varepsilon(u)\tilde{h}_j(u),\quad u\in D(3),$$

where

$$\tilde{h}_j(u)=\varphi_{j+1}\circ h\circ\varphi_j^{-1}(u),\ u\in D(3).$$

With ε sufficiently small, k can be arbitrarily C^1 close to h in each $\overline{V}_j$ $(0\leqslant j\leqslant n-1)$, while $k\equiv h$ outside the set $\bigcup_{j=0}^{n-1}V_j$. So k can be C^1 close to h on whole S^2, and $k\in \mathrm{Diff}^1(\mathrm{S}^2)$.

obviously p is an n-periocic point of k and

$$\begin{aligned}Dk^n(p)&=Dk(p_{n-1})\circ Dk(p_{n-2})\circ\cdots\circ Dk(p)\\&=((1\pm\varepsilon)Dh(p_{n-1}))\circ((1\pm\varepsilon)Dh(p_{n-2}))\circ\cdots\circ((1\pm\varepsilon)Dh(p))\\&=(1\pm\varepsilon)^nDh^n(p).\end{aligned}$$

The eigenvalues of $Dk^n(p)$ is $\mu_i=(1\pm\varepsilon)^n\lambda_i$ $(i=1, 2)$. In case $|\lambda_2|>1$, $\mu_i=(1+\varepsilon)^n\lambda_i$ satisfy $|\mu_i|>1$ $(i=1, 2)$, and thus p is a periodic source of k. In case $|\lambda_2|\leqslant 1$, $\mu_i=(1-\varepsilon)^n\lambda_i$ satisfy $|\mu_i|<1$ $(i=1, 2)$, and thus p is a periodic sink of k.

To sum up, for arbitrary $f\in \mathrm{Diff}^1(\mathrm{S}^2)$ there is $g\in\mathscr{G}$ sufficiently C^1 close to f. If $g\in\mathscr{G}^0$, we already have $g\in\mathscr{S}$. If $g\in\mathscr{G}\setminus\mathscr{G}^0$, then there is $h\in \mathrm{Diff}^1(\mathrm{S}^2)\setminus\mathscr{G}$ sufficiently C^1 close to g, and hence there is $k\in\mathscr{S}$ constructed above which is sufficiently C^1 close to h. This proves the density of $\mathscr{S}$ in $\mathrm{Diff}^1(\mathrm{S}^2)$. The proof of our theorem is complete.

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