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# GLOBAL SMOOTH SOLUTION OF CAUCHY PROBLEMS FOR A CLASS OF QUASILINEAR HYPERBOLIC SYSTEMS

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**Abstract**

In this article we prove that certain kinds of inhomogeneous terms can smoothen the solution for first order quasilinear hyperbolic systems globally in time provided that the initial data are small.

## § 1. Introduction

For the Cauchy problem of first order quasilinear hyperbolic systems it is well-known that even if the initial data are very smooth, in general, a smooth solution exists only locally in time and singularities may appear in a finite time. It was discussed in [1] how the presence of various damping and dissipation mechanisms may influence the smoothness of the solution. In the same direction, we consider here the influence of a special kind of dissipative inhomogeneous term and establish the existence and uniqueness of global smooth solutions.

Consider the following Cauchy problem for a first order system of inhomogeneous hyperbolic balance laws:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial f(u)}{\partial x}+g(u)=0, & -\infty<x<\infty,\ 0\leqslant t<\infty, \quad (1)\\ u(x,0)=\phi(x), & -\infty<x<\infty, \quad (2)\end{cases}$$

where $u=(u_1, \cdots, u_n)^T$ is the unknown vector function, $f(u)$ and $g(u)$ are given smooth vector functions of $u$ with

$$g(0)=0 \tag{3}$$

and $\phi(x)$ is smooth.

System (1) is hyperbolic on the domain under consideration if

1) The $n\times n$ matrix $\nabla f(u)$ has $n$ real eigenvalues:

$$\lambda_1(u),\ \lambda_2(u),\ \cdots,\ \lambda_n(u); \tag{4}$$

2) $\nabla f(u)$ is diagonalizable, i. e.

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$$\det \zeta(u) \neq 0, \tag{5}$$

where

$$\zeta(u) = \begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \vdots \\ \zeta_n(u) \end{pmatrix} \tag{6}$$

and $\zeta_l(u) = (\zeta_{l1}(u), \zeta_{l2}(u), \cdots, \zeta_{ln}(u))$ is a left eigenvector corresponding to $\lambda_l$:

$$\zeta_l(u) \nabla f(u) = \lambda_l(u) \zeta_l(u). \tag{7}$$

By (3), $g(u)$ can be rewritten as

$$g(u) = B(u)u, \tag{8}$$

where $B(u) = (B_{ij}(u))$ is a $n \times n$ matrix with

$$B(0) = \nabla g(0). \tag{9}$$

Let

$$\lambda(u) = \mathrm{diag}\{\lambda_1(u), \lambda_2(u), \cdots, \lambda_n(u)\}. \tag{10}$$

The main results in this paper are the following:

**Theorem 1.** *Assume that* (3) *holds and the matrix*

$$\mathscr{A} \equiv (a_{ij}) = \zeta(0) \nabla g(0) \zeta^{-1}(0) \tag{11}$$

*is row diagonal dominant:*

$$a_{ll} > \sum_{j \neq l} |a_{lj}| \quad (l = 1, \cdots, n), \tag{12}$$

*where, in* (11), $\zeta^{-1}$ *denotes the inverse matrix of* $\zeta$. Furthermore, let $\zeta(u)$, $\lambda(u)$ *and* $B(u)$ *be smooth, then the Cauchy problem* (1), (2) *admits a unique global smooth solution* $u = u(x, t)$ *on* $t \geq 0$ *which decays exponentially as* $t \to \infty$, *provided that the* $C^1$ *norm of the initial data* $\phi(x)$ *is sufficiently small.*

**Theorem 2.** *The conclusion of Theorem* 1 *holds if* (12) *is replaced by the following weaker condition that there exists a diagonal matrix* $\gamma$ *with* $\gamma_l \neq 0$ $(l = 1, \cdots, n)$ *such that the matrix* $\gamma \mathscr{A} \gamma^{-1}$ *is row diagonal dominant.*

The results in this paper generalize and make more precise the results in [4] for the one-dimensional case.

We should note that our assumption (12) or even the corresponding weaker condition in Theorem 2 do not cover, for example, the case of the one-dimensional damped wave equation

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma(u)_x + v = 0. \end{cases} \tag{13}$$

For a global existence theorem for this special system see [5].

The proofs of Theorem 1 and 2 will be given in the next section.

## § 2. Proofs of Theorems

Multiplying (1) by $\zeta$ from the left and using (7), we obtain the characteristic form

$$\zeta(u)\frac{\partial u}{\partial t}+\lambda(u)\zeta(u)\frac{\partial u}{\partial x}+A(u)\zeta(u)u=0, \tag{14}$$

where
$$A(u)\equiv(A_{ij}(u))=\zeta(u)B(u)\zeta^{-1}(u). \tag{15}$$

We note, in view of (9), that $A(0)=\mathscr{A}$, i. e., $A_{ij}(0)=a_{ij}(i, j=1, \cdots, n)$. It is now clear that the assertion of Theorem 1 follows easily from the following lemma.

**Lemma 1.** *Assume that the matrix* $A(0)=\mathscr{A}$ *satisfies* (12) *and that* $\zeta(u)$, $\lambda(u)$ *and* $A(u)$ *are smooth. Then, if the* $C^1$ *norm of the initial data* $\phi(x)$ *is sufficiently small, there exists a unique global smooth solution* $u=u(x, t)$ *on* $t\geqslant 0$ *for* the Cauchy problem (14), (2) *and this solution decays exponentially as* $t\to\infty$.

*Proof* Without loss of generality, we can suppose in what follows that the $C^1$ norm of $\zeta(u)$, $\lambda(u)$ and $B(u)$ is bounded and

$$|\det\zeta|u|\geqslant D_0>0 \quad (D_0 \text{ is constant}). \tag{16}$$

In order to obtain the existence of the global smooth solution it suffices to prove that if the initial data $\phi(x)$ have a small $C^1$ norm then the $C^1$ norm of the smooth solution $u=u(x, t)$ defined on the domain

$$R(T)=\{(t, x)\,|\,0\leqslant t\leqslant T, -\infty<x<\infty\} \tag{17}$$

does not depend on $T$.

Rewrite (14) in the form

$$\sum_j\zeta_{lj}(u)\Big(\frac{\partial u_j}{\partial t}+\lambda_l(u)\frac{\partial u_j}{\partial x}\Big)+\sum_j(A_{lj}(u)-A_{lj}(0))\sum_k\zeta_{jk}(u)u_k$$
$$+\sum_j a_{lj}\sum_k\zeta_{jk}(u)u_k=0 \quad (l=1, \cdots, n). \tag{18}$$

Let
$$\xi=f_l(\tau;\ t, x) \tag{19}$$

be the $l$-th characteristic curve passing through the point $(t, x)$, which satisfies

$$\begin{cases}\dfrac{d\xi}{d\tau}=\lambda_l(u(\tau, f_l(\tau;\ t, x))),\\ f_l(t; t, x)=x.\end{cases} \tag{20}$$

Setting
$$U_l=\sum_k\zeta_{lk}(u)u_k \tag{21}$$

and
$$V_l=\exp(a_{ll}t)\cdot U_l, \tag{22}$$

it is easy to see that

$$\begin{cases}\dfrac{\partial V_l}{\partial t}+\lambda_l(u)\dfrac{\partial V_l}{\partial x}=\exp(a_{ll}t)\Big\{\sum_j\Big[\dfrac{\partial\zeta_{lj}(u)}{\partial t}+\lambda_l(u)\dfrac{\partial\zeta_{lj}(u)}{\partial x}\Big]u_j\\ \qquad -\sum_j[A_{lj}(u)-A_{lj}(0)]\sum_k\zeta_{jk}(u)u_k\Big\}-\exp(a_{ll}t)\sum_{j\neq l}a_{lj}U_j\,(l=1, \cdots, n),\\ V_l=U_l^0(x)\equiv\sum_k\zeta_{lk}(\phi)\phi_k,\ t=0,\ -\infty<x<\infty.\end{cases} \tag{23}$$

Integrating the $l$-th equation of (23) along the $l$-th characteristic curve, we get

$$V_l(t, x)=U_l^0(f_l(0; t, x))+\int_0^t\Big\{\exp(a_{ll}\tau)\cdot\Big[\sum_j\Big(\frac{\partial\zeta_{lj}(u)}{\partial t}+\lambda_l(u)\frac{\partial\zeta_{lj}(u)}{\partial x}\Big)u_j$$
$$-\sum_j(A_{lj}(u)-A_{lj}(0))\sum_k\zeta_{jk}(u)u_k\Big]-\exp(a_{ll}\tau)\cdot\sum_{j\neq l}a_{lj}U_j\Big\}\,d\tau, \tag{24}$$

then

$$U_l(t, x) = \exp(-a_{ll}t)\cdot U_l^0(f(0;\ t,\ x)) + \int_0^t \Big\{\exp(a_{ll}(\tau-t))\cdot\Big[\sum_j\Big(\frac{\partial\zeta_{lj}(u)}{\partial t} + \lambda_l(u)\frac{\partial\zeta_{lj}(u)}{\partial x}\Big)u_j - \sum_j(A_{lj}(u)-A_{lj}(0))\sum_k\zeta_{jk}(u)u_k\Big] - \exp(a_{ll}(\tau-t))\cdot\sum_{j\neq l}a_{lj}U_j\Big\}\,d\tau, \tag{25}$$

in which the integration is carried along the characteristic $(\tau,\ f_l(\tau;t,x))$.

We have

$$\frac{\partial\zeta_{lj}(u)}{\partial t} + \lambda_l(u)\,\frac{\partial\zeta_{lj}(u)}{\partial x} = \sum_k\frac{\partial\zeta_{lj}(u)}{\partial u_k}\Big(\frac{\partial u_k}{\partial t} + \lambda_l(u)\,\frac{\partial u_k}{\partial x}\Big). \tag{26}$$

System (18) together with (21) imply that

$$\frac{\partial u_k}{\partial t} = \sum_{p,m}\zeta^{kp}(u)\zeta_{pm}(u)\lambda_p(u)\frac{\partial u_m}{\partial x} - \sum_{p,m}\zeta^{kp}(u)A_{pm}(u)U_m, \tag{27}$$

where $\zeta^{kp}$ $(k,\ p=1,\ \cdots,\ n)$ denote the elements of $\zeta^{-1}$. Thus, introducing

$$w_i = \frac{\partial u_i}{\partial x},\quad W_l = \sum_j\zeta_{lj}(u)w_j, \tag{28}$$

it follows from (26) and (27) that

$$\begin{cases}\dfrac{\partial\zeta_{lj}(u)}{\partial t} + \lambda_l(u)\,\dfrac{\partial\zeta_{lj}(u)}{\partial x} = \sum\limits_s P_{ljs}(u)W_s + \sum\limits_{p,s}Q_{ljp}(u)A_{ps}(u)U_s,\\ \dfrac{\partial u_k}{\partial t} = \sum\limits_s R_{ks}(u)W_s + \sum\limits_{p,s}S_{kp}(u)A_{ps}(u)U_s,\end{cases} \tag{29}$$

in which

$$\begin{cases}P_{ljs}(u) = \sum\limits_{k,p,m}\dfrac{\partial\zeta_{lj}(u)}{\partial u_k}\,\zeta^{kp}(u)\zeta_{pm}(u)\lambda_p(u)\zeta^{ms}(u) + \sum\limits_k\dfrac{\partial\zeta_{lj}(u)}{\partial u_k}\lambda_l(u)\zeta^{ks}(u),\\ Q_{ljp}(u) = -\sum\limits_k\dfrac{\partial\zeta_{lj}(u)}{\partial u_k}\,\zeta^{kp}(u),\\ R_{ks}(u) = \sum\limits_{p,m}\zeta^{kp}(u)\zeta_{pm}(u)\lambda_p(u)\zeta^{ms}(u),\\ S_{kp}(u) = -\zeta^{kp}(u).\end{cases} \tag{30}$$

Then, from (25) we have

$$U_l(t, x) = \exp(-a_{ll}t)\cdot U_l^0(f_l(0;\ t,\ x)) + \int_0^t\{\exp(a_{ll}(\tau-t))\cdot[\sum_j(\sum_s P_{ljs}(u)W_s + \sum_{p,s}Q_{ljp}(u)A_{ps}(u)U_s)\cdot\sum_r\zeta^{jr}(u)U_r - \sum_j(A_{lj}(u)-A_{lj}(0))U_j] - \exp(a_{ll}(\tau-t))\cdot\sum_{j\neq l}a_{lj}U_j\}d\tau. \tag{31}$$

Moreover, if we set

$$H_l = \exp(a_{ll}t)\cdot W_l \tag{32}$$

and differentiate (18) with respect to $x$, it follows in a similar way that

$$\begin{cases} \frac{\partial H_l}{\partial t}+\lambda_l(u)\frac{\partial H_l}{\partial x}=\exp(a_{ll}t)\cdot\sum_j\left(\frac{\partial\zeta_{lj}(u)}{\partial t}+\lambda_l(u)\frac{\partial\zeta_{lj}(u)}{\partial x}\right)w_j \\ \quad -\exp(a_{ll}t)\left(\sum_{j,k}\frac{\partial\zeta_{lj}}{\partial u_k}\cdot\frac{\partial u_j}{\partial t}w_k+\sum_{j,k}\frac{\partial(\zeta_{lj}\lambda_l)}{\partial u_k}w_jw_k+\sum_{j,k,p}\frac{\partial(A_{lj}\zeta_{jk})}{\partial u_p}w_pu_k\right. \\ \quad \left.+\sum_j(A_{lj}(u)-A_{lj}(0))W_j\right)-\exp(a_{ll}t)\sum_{j\neq l}a_{lj}W_j, \\ H=w_l^0(x)\equiv\sum_k\zeta_{lk}(\phi)\phi_k'(x),\ t=0,\ -\infty<x<\infty. \end{cases} \tag{33}$$

Integrating the $l$-th equation along the $l$-th characteristic curve and using (29), we obtain

$$\begin{aligned} W_l(t,x)&=\exp(-a_{ll}t)\cdot W_l^0(f_l(0;\,t,x)) \\ &+\int_0^t\Big\{\exp a_{ll}(\tau-t)\Big[\sum_j\Big(\sum_s P_{ljs}(u)W_s+\sum_{p,s}Q_{ljp}(u)\,A_{ps}(u)U_s\Big)\cdot\sum_r\zeta^{jr}W_r \\ &-\sum_{j,k}\frac{\partial\zeta_{lj}}{\partial u_k}\Big(\sum_s R_{ks}(u)W_s+\sum_{p,s}S_{kp}(u)A_{ps}(u)U_s\Big)\cdot\sum_r\zeta^{kr}W_r \\ &-\sum_{j,k,s,r}\frac{\partial(\zeta_{lj}\lambda_l)}{\partial u_k}\zeta^{js}\zeta^{kr}W_sW_r-\sum_{j,k,p,r,s}\frac{\partial(A_{lj}\zeta_{jk})}{\partial u_p}\zeta^{kr}\zeta^{ks}W_rU_s \\ &-\sum_j(A_{lj}(u)-A_{lj}(0))W_j\Big]-\exp(a_{ll}(\tau-t))\sum_{j\neq l}a_{lj}W_j\Big\}\,d\tau. \end{aligned} \tag{34}$$

Let

$$U_l(t)=\sup_{\substack{0\le\tau\le t\\ -\infty<x<\infty}}|U_l(\tau,x)|,\quad U(t)=\sup_l U_l(t), \tag{35}$$

$$W_l(t)=\sup_{\substack{0\le\tau\le t\\ -\infty<x<\infty}}|W_l(\tau,x)|,\quad W(t)=\sup_l W_l(t), \tag{36}$$

and

$$a=\min_l\{a_{ll}\}. \tag{37}$$

We can easily obtain from (31) that

$$\begin{aligned} U_l(t)\le e^{-at}C_0+\int_0^t\{&D_1\exp(a_{ll}(\tau-t))\cdot[W(\tau)U(\tau)+U^2(\tau)] \\ &+\exp(a_{ll}(\tau-t))\cdot\sum_{j\neq l}|a_{lj}|U_j(\tau)\}d\tau, \end{aligned} \tag{38}$$

where

$$C_0=\sup_{\substack{l\\ -\infty<x<\infty}}|U_l^0(x)|=\sup_{\substack{l\\ -\infty<x<\infty}}\Big|\sum_k\zeta_{lk}(\phi)\phi_k\Big| \tag{39}$$

and $D_i$ ($i=1,2,\cdots$) will denote throughout various fixed constants.

Since
$$\int_0^t\exp(a_{ll}(\tau-t))\,d\tau=\frac{1}{a_{ll}}(1-\exp(-a_{ll}t)),$$

it follows on account of (12)

$$\begin{aligned} \int_0^t\exp(a_{ll}(\tau-t))\cdot\sum_{j\neq l}|a_{lj}|U_j(\tau)d\tau&\le\int_0^t\exp(a_{ll}(\tau-t))\sum_{j\neq l}|a_{lj}|\,d\tau\cdot U(t) \\ &\le\frac{\sum_{j\neq l}|a_{lj}|}{a_{ll}}U(t)\le dU(t), \end{aligned} \tag{40}$$

where

$$d=\max_l\frac{\sum_{j\neq l}|a_{lj}|}{a_{ll}}<1. \tag{41}$$

Then (38) gives

$$U(t) \leqslant \frac{1}{1-d} \exp(-at) \cdot C_0 + \int_0^t D_2 \exp(a(\tau - t)) \cdot (W(\tau) U(\tau) + U^2(\tau)) d\tau. \quad (42)$$

In a similar way (34) gives

$$W_l(t) \leqslant \exp(-at) \cdot C_1 + \int_0^t \{D_3 \exp(a(\tau - t)) \cdot (W^2(\tau) + W(\tau) U(\tau)) + \exp(a_{ll}(\tau - t)) \sum_{j \neq l} |a_{lj}| W_j(\tau) d\tau, \quad (43)$$

where

$$C_1 = \sup_{\substack{l \\ -\infty < x < \infty}} |W_l^0(x)| = \sup_{\substack{l \\ -\infty < x < \infty}} |\sum_k \zeta_{lk}(\phi) \phi'_k|. \quad (44)$$

Therefore

$$W(t) \leqslant \frac{1}{1-d} \exp(-at) \cdot C_1 + \int_0^t D_4 \exp(a(\tau - t)) \cdot (W^2(\tau) + W(\tau) U(\tau)) d\tau. \quad (45)$$

Let

$$X(t) = U(t) + W(t). \quad (46)$$

Combining (42) and (45), it is easily seen that

$$X(t) \leqslant \frac{1}{1-d} \exp(-at) (C_0 + C_1) + \int_0^t D_5 \exp(a(\tau - t)) \cdot X^2(\tau) d\tau. \quad (47)$$

Letting

$$Y(t) = X(t) \exp(at), \quad (48)$$

(47) can be written as

$$Y(t) \leqslant \frac{1}{1-d} (C_0 + C_1) + \int_0^t D_5 \exp(-at) \cdot Y^2(\tau) d\tau. \quad (49)$$

Consider

$$\begin{cases} \dfrac{dZ(t)}{dt} = D_5 \exp(-as) \cdot Z^2(t), \\ Z = \dfrac{1}{1-d} (C_0 + C_1), \ t = 0. \end{cases} \quad (50)$$

We then have the estimation

$$Y(t) \leqslant Z(t). \quad (51)$$

But

$$Z(t) = \frac{1}{\dfrac{1-d}{C_0 + C_1} - \dfrac{D_5}{a} (1 - \exp(-at))}. \quad (52)$$

Therefore, if $C_0 + C_1$ is so small that

$$a > 2D_5 \cdot \frac{C_0 + C_1}{1-d}, \quad (53)$$

then

$$Z(t) \leqslant \frac{2(C_0 + C_1)}{1-d}. \quad (54)$$

hence

$$X(t) \leqslant \frac{2(C_0 + C_1)}{1-d} \exp(-at). \quad (55)$$

Thus, by the definition of $X(t)$, the lemma and thenceforth Theorem 1, is proved.

**Remark 1.** Assumption (12) in Theorem 1 is similar to the assumption in [2] for establishing the corresponding global existence of discontinuous solutions for the Cauchy problem (1), (2).

By means of the method in [3], multiplying the $l$-th equation of (18) by a real number $\gamma_l \neq 0$, the matrix $\mathscr{A}$ can be replaced by $\gamma\mathscr{A}\gamma^{-1}$, where

$$\gamma = \mathrm{diag}\,\{\gamma_1, \gamma_2, \cdots, \gamma_n\}. \tag{56}$$

Thus Theorem 2 follows from Theorem 1.

**Remark 2.** In a similar way as in Lemma 1, we can also prove the following

**Lemma 2.** *Suppose that the matrix* $A(0) = \mathscr{A}$ *satisfies* (12) *and that* $\zeta(u)$, $\lambda(u)$ *are smooth. Suppose further that any one of the following additional hypotheses holds*

(i) $$A(u) = A(0) \tag{57}$$

or (ii) $$\zeta(u) = I. \tag{58}$$

*Then, if* $a$ *defined by* (37) *is sufficiently large, there exists a unique global smooth solution* $u = u(x, t)$ *on* $t \geqslant 0$ *for the Cauchy problem* (14), (2) *and this solution decays exponentially as* $t \to \infty$.

From Lemma 2. we can get the corresponding result for the Cauchy problem (1), (2).

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