Chin. Ann. of Math.
4B (1) 1983

DECOMPOSITION OF BMO FUNCTIONS AND FACTORIZATION OF A_p WEIGHTS IN MARTINGALE SETTING

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Abstract

Let $(\Omega, \mathscr{F}, \mu)$ be a probabilty space with an increasing family $\{\mathscr{F}_t\}_{t\geq 0}$ of sub-σ-fields satisfying the usual conditions. The following results are obtained: for $f\in BMO$, we have $f=g-h$ with $g, h\in BLO$; if in addition, f satisfies

$$E(e^{\alpha(f-f_t)}|\mathscr{F}_t)\leqslant K_\alpha,\ E(e^{-\beta(f-f_t)}|\mathscr{F}_t)\leqslant K_\beta,$$

then for $\varepsilon>0$ arbitrary, g, h can be chosen such that $g, h\in BLO$, and

$$E(e^{(\alpha-\varepsilon)(g-g_t)}|\mathscr{F}_t)\leqslant C_{\alpha,\beta,\varepsilon},\ E(e^{(\beta-\varepsilon)(h-h_t)}|\mathscr{F}_t)\leqslant C_{\alpha,\beta,\varepsilon}$$

and for weights z, we have

$$z\in A_p\cap S\Leftrightarrow z=z_1z_2^{1-p} \text{ with } z_i\in A_1\cap S,\ i=1, 2,$$

where $$S=\{\text{weights } z: Cz_{T-}\leqslant z_T\leqslant Cz_{T-},\ \forall \text{ stopping times } T, \text{ outside a null set}\},$$

§ 1. Introduction

In elaborating a probabilistic proof of Garnett-Jones's theorem on the distance in BMO to L^∞, Varopoulos[1] has defined the notion of γ-graded sequence of stopping times and showed that this notion may be used to prove Jones's theorem on the factorization of A_p weights. But, his argument works only under the hypothesis "H" (i. e. "continuous path hypothesis") owing to the fact that his γ-graded function is defined too restrictedly.

In order to apply the notion of γ-graded sequence of stopping times to martingales with jumps, we have generalized the notion of γ-graded function in Long[2]. Now, we devote the application of this generalization to the subjects, indicated by the title of this paper. We shall consider principally the martingales with discrete times, but the arguments will work effectivelly also for those with continuous times. We shall discuss this case briefly.

Manuscript received September 17, 1981.

Let us begin with several concepts and notations. Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space and $\{\mathscr{F}_n\}_{n\geq 0}$ be an increasing family of sub-σ-fields of $\mathscr{F}$ with $\mathscr{F}_0$ trivial and $\mathscr{F}=\bigvee_n \mathscr{F}_n$. Several spaces or classes of martingales which we shall deal with are cited as follows:

Definitions 1. *BMO and BLO*

A martingale $f=(f_n)_{n\geq 0}$ of L^1 (i. e. an uniformly integrable martingale) is said to be of BMO, if

$$\|f\|_{BMO}=\sup_n \|E(|f-f_{n-1}|\,|\mathscr{F}_n)\|_\infty<\infty. \tag{1}$$

A real martingale $f=(f_n)_{n\geq 0}$ of L^1 is said to be of BLO, if

$$\|f\|_{BLO}=\inf\{C;\ f_n-f\leqslant C,\ |f_n-f_{n-1}|\leqslant C,\ \text{a. e. } \forall n\}<\infty. \tag{2}$$

2. $\log A_{\alpha,\beta}$, $\alpha>0$, $\beta>0$.

A real martingale $f=(f_n)_{n\geq 0}$ of L^1 is said to be of $\log A_{\alpha,\beta}$, $\alpha>0$, $\beta>0$, if

$$\begin{aligned} &E(e^{\alpha(f-f_n)}\,|\mathscr{F}_n)^{\frac{1}{\alpha}}\leqslant K_\alpha<\infty^*, \text{ a. e, } \forall n,\\ &E(e^{-\beta(f-f_n)}\,|\mathscr{F}_n)^{\frac{1}{\beta}}\leqslant K_\beta<\infty, \text{ a. e, } \forall n. \end{aligned} \tag{3}$$

3. BD.

A martingale $f=(f_n)_{n\geq 0}$ is said to be of BD, if

$$\|f\|_{BD}=\sup_n\|f_n-f_{n-1}\|_\infty<\infty. \tag{4}$$

4. γ-graded sequence of stopping times.

A sequence $\{T_i\}_1^\infty$ of stopping times is called a γ-graded sequence, if $\{T_i\}_1^\infty$ is increasing and

$$E(\mathbf{1}(\{T_{i+1}<\infty\})\,|\mathscr{F}_{T_i})\leqslant\gamma, \text{ a. e., } 0<\gamma<1. \tag{5}$$

5. A_p.

Let z be strictly positive and be of L^1, $z=(z_n)_{n\geq 0}$, $z_n=E(z\,|\mathscr{F}_n)^{**}$. Such $z=(z_n)_{n\geq 0}$ is called a weight. A weight $z=(z_n)_{n\geq 0}$ is called a A_p weight, $1\leqslant p\leqslant\infty$, if

$$\sup_n\|z_nE(z^{-\frac{1}{p-1}}\,|\mathscr{F}_n)^{p-1}\|_\infty\leqslant C_p<\infty, \text{ for } 1<p<\infty,\ z_n\leqslant Cz, \text{ a. e, } \forall n, \text{ for } p=1, \tag{6}$$

and there exists q, $1<q<\infty$, such that $z\in A_q$, in symbols $A_\infty=\bigcup_q A_q$, for $p=\infty$.

6. S.

A weight $z=(z_n)_{n\geq 0}$ is said to be of S (or S^+, or S^-), if

$$Cz_{n-1}\leqslant z_n\leqslant Cz_{n-1}, \text{ a. e. } \forall n, \tag{7}$$

(or $z_n\leqslant Cz_{n-1}$, or $z_{n-1}\leqslant Cz_n$, respectively)

We take several most elementary facts concerning these spaces or classes for granted. For example

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* K(or C, …) is denoted as a constant. When the parameters on which the constant depend are needed to be emphasized, we indicate it by subscripts. As usually, the constant denoted by same symbol is not necessarily the same, even in the same expression.

** There is a convention: for $f\in L^1$, f_n stands for $E(f\,|\mathscr{F}_n)$ except when otheawise specified.

Assertions 1. We have

$$\operatorname{Re} L^\infty \subset BLO \subset BMO,$$

$$\frac{1}{3}\|f\|_{BMO} \leqslant \|f\|_{BLO} \leqslant 2\|f\|_\infty.$$

2. We have that if $f \in BMO$ (or BLO, or $\log A_{\alpha,\beta}$), so does $\varphi = f - f_T$, where T is any stopping time. More precisely

$$\|\varphi\|_{BMO} \leqslant \|f\|_{BMO},$$
$$\|\varphi\|_{BLO} \leqslant \|f\|_{BLO}, \tag{8}$$
$$E(e^{\alpha(\varphi-\varphi_n)} | \mathscr{F}_n)^{\frac{1}{\alpha}} \leqslant K_\alpha, \ E(e^{-\beta(\varphi-\varphi_n)} | \mathscr{F}_n)^{\frac{1}{\beta}} \leqslant K_\beta,$$

with K_α, K_β unchanged.

3. We have that $\log A_{\alpha,\beta} \subset \log A_{\alpha',\beta'}$, when $\alpha' \leqslant \alpha$, $\beta' \leqslant \beta$, and that

$$\operatorname{Re} BMO = \bigcup_{\alpha,\beta>0} \log A_{\alpha,\beta} \cap BD.$$

4. We have

$$A_p \subset A_q, \ 1 \leqslant p \leqslant q \leqslant \infty.$$

The main results of this paper are summarized as follows. In § 2, we prove that every real BMO martingale may be decomposed as difference of two BLO martingales, and that every f of $\log A_{\alpha,\beta} \cap BD$, for any $\varepsilon > 0$, may be decomposed as $f = g - h$ with $g \in BLO \cap \log A_{\alpha-\varepsilon,\tau}$, $h \in BLO \cap \log A_{\beta-\varepsilon,\tau}$, $\forall \tau > 0$. In § 3, we prove that every z of $A_p \cap S$ may be fctorized as $z = z_1 z_2^{1-p}$ with $z_i \in A_1 \cap S$, $i = 1, 2$. The significance of this factorization was shown by Jones[3]. In § 4 we show briefly that all results of § § 2, 3, still hold in continuous times case.

§ 2. Decomposition of BMO martingales

Lemma 1. *Real* $f = (f_n)_{n\geqslant 0} \in \log A_{\alpha,\beta}$, *iff*

$$E(e^{\alpha f} | \mathscr{F}_n)^{\frac{1}{\alpha}} E(e^{-\beta f} | \mathscr{F}_n)^{\frac{1}{\beta}} \leqslant K_{\alpha,\beta} < \infty. \tag{3$'$}$$

Corollary 1. $f \in \log A_{\alpha,\beta}$, *iff* $e^{\alpha f} \in A_p$, *with* $\dfrac{\beta}{\alpha} = \dfrac{1}{p-1}$.

Lemma 2. *Let* $\{T_k\}_1^\infty$ *be a* γ*-graded sequence, and* $\{b_k(\omega)\}_1^\infty$ *be a sequence of measurable functions satisfying*

$$0 \leqslant b_k(\omega) \leqslant B, \ b_k(\omega) \text{ measurable with respect to } \mathscr{F}_{T_k}. \tag{9}$$

Then

$$\varphi = \sum_1^\infty b_k(\omega) \mathbf{1}(\{T_k < \infty\}) \in BLO \tag{10}$$

with

$$\|\varphi\|_{BLO} \leqslant \frac{2B}{1-\gamma}, \quad \|\varphi\|_{BMO} \leqslant \frac{2B}{1-\gamma}. \tag{11}$$

Furthermore, for all $\alpha > 0$ *satisfying* $e^{\alpha B} < \dfrac{1}{\gamma}$, *we have*

$$\sup_n \|E(e^{\alpha(\varphi-\varphi_n)} | \mathscr{F}_n)\|_\infty < \infty. \tag{12}$$

Proof Since $\{T_k\}_1^\infty$ is γ-graded, we have

$$|\{T_k<\infty\}|=E(E(\mathbf{1}(\{T_k<\infty\})|\mathscr{F}_{T_{k-1}}))^*$$
$$=E(E(\mathbf{1}(\{T_k<\infty\})\mathbf{1}(\{T_{k-1}<\infty\})|\mathscr{F}_{T_{k-1}}))$$
$$\leqslant\gamma E(\mathbf{1}(\{T_{k-1}<\infty\}))\leqslant\gamma^{k-1}.$$

This shows that $|\{T_k<\infty\}|\to 0$, $T_k\to\infty$, a. e, and $\sum_1^\infty\mathbf{1}(\{T_k<\infty\})\in L^1$. |

Now, for $n\in\mathbf{Z}^+$ fixed, consider the partition $\{X_n^{(m)}\}_{m\geqslant 0}$ of Ω with

$$X_n^{(m)}=\{T_1<n,\ \cdots,\ T_m<n,\ T_{m+1}\geqslant n\},$$
$$X_n^{(0)}=\{T_1\geqslant n\},$$
$$X_n^{(\infty)}=\{T_m<n,\ \forall m=1,\ 2,\ \cdots\}.$$

Note that $|X_n^{(\infty)}|=0$ and $X_n^{(m)}\in\mathscr{F}_{n-1}\cap\mathscr{F}_{T_{m+1}}$. We want to estimate $(\varphi_n-\varphi)\mathbf{1}(X_n^{(m)})$, $m=0,1,\cdots$. Since, when $1\leqslant k\leqslant m$, $b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})$ are measurable with respect to $\mathscr{F}_{n-1}$, and $b_k(\omega)\geqslant 0$, we have

$$\varphi_n\mathbf{1}(X_n^{(m)})=E(\varphi|\mathscr{F}_n)\mathbf{1}(X_n^{(m)})=E(\varphi\mathbf{1}(X_n^{(m)})|\mathscr{F}_n)$$
$$=\sum_{k=1}^m b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})+\sum_{k=m+1}^\infty E(b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_n),$$
$$(\varphi_n-\varphi)\mathbf{1}(X_n^{(m)})=\sum_{k=m+1}^\infty E(b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_n)$$
$$-\sum_{k=m+1}^\infty b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})$$
$$\leqslant\sum_{k=m+1}^\infty E(E(b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_{T_{m+1}})|\mathscr{F}_n)$$
$$\leqslant B\sum_{k=m+1}^\infty\gamma^{k-m-1}=\frac{B}{1-\gamma}.$$

Analogously, we have also

$$E(|\varphi-\varphi_{n-1}||\mathscr{F}_n)\mathbf{1}(X_n^{(m)})=E(|\varphi-\varphi_{n-1}|\mathbf{1}(X_n^{(m)})|\mathscr{F}_n)$$
$$=E\Big(\Big|\sum_{k=m+1}^\infty b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})$$
$$-\sum_{k=m+1}^\infty E(b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_{n-1})\Big|\Big|\mathscr{F}_n\Big)$$
$$\leqslant B\sum_{k=m+1}^\infty E(\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_n)$$
$$+B\sum_{k=m+1}^\infty E(\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})|\mathscr{F}_{n-1})$$
$$\leqslant 2B\sum_{k=m+1}^\infty\|E(\mathbf{1}(\{T_k<\infty\})|\mathscr{F}_{T_{m+1}})\|_\infty\leqslant\frac{2B}{1-\gamma}.$$

This completes the proof of two inequalities in (11).

It remains to prove (12). For $\alpha>0$, we have

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* $|\cdot|$ denote the μ-measure.

$$\sum_{m=0}^{\infty} E(\exp(\alpha(\varphi-\varphi_n))\,|\,\mathscr{F}_n)\mathbf{1}(X_n^{(m)}) = E\Big(\sum_{m=0}^{\infty}\exp(\alpha(\varphi-\varphi_n)\mathbf{1}(X_n^{(m)}))\mathbf{1}(X_n^{(m)})\,|\,\mathscr{F}_n\Big)$$

$$= E\Big(\sum_{m=0}^{\infty}\exp\Big\{\alpha\sum_{k=m+1}^{\infty} b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})$$

$$-\alpha\sum_{k=m+1}^{\infty} E(b_k(\omega)\mathbf{1}(\{T_k<\infty\})\mathbf{1}(X_n^{(m)})\,|\,\mathscr{F}_n)\Big\}\mathbf{1}(X_n^{(m)})\,|\,\mathscr{F}_n\Big)$$

$$\leqslant\sum_{m=0}^{\infty} E\Big(E\Big(\exp\Big\{\alpha B\sum_{k=m+1}^{\infty}\mathbf{1}(\{T_k<\infty\})\Big\}\Big|\mathscr{F}_{T_{m+1}}\Big)\Big|\mathscr{F}_n\Big)\mathbf{1}(X_n^{(m)}). \tag{13}$$

Now, we want to obtain the uniform estimate of

$$\Big\|E\Big(\exp\Big\{\alpha B\sum_{k=m+1}^{\infty}\mathbf{1}(\{T_k<\infty\})\Big\}\Big|\mathscr{F}_{T_{m+1}}\Big)\Big\|_{\infty}$$

for all m. We have

$$\exp\Big\{\alpha B\sum_{k=m+1}^{\infty}\mathbf{1}(\{T_k<\infty\})\Big\} = \sum_{l=0}^{\infty}\frac{(\alpha B)^l}{l!}\Big(\sum_{k=m+1}^{\infty}\mathbf{1}(\{T_k<\infty\})\Big)^l$$

$$=\sum_{l=0}^{\infty}\frac{(\alpha B)^l}{l!}\sum_{k=m+1}^{\infty}(k-m)^l\mathbf{1}(\{T_k<\infty,\ T_{k+1}=\infty\}),$$

$$E\Big(\exp\Big\{\alpha B\sum_{k=m+1}^{\infty}\mathbf{1}(\{T_k<\infty\})\Big\}\Big|\mathscr{F}_{T_{m+1}}\Big)\leqslant\sum_{l=0}^{\infty}\frac{(\alpha B)^l}{l!}\sum_{k=m+1}^{\infty}(k-m)^l\gamma^{k-m-1}$$

$$=\sum_{l=0}^{\infty}\frac{(\alpha B)^l}{l!}\sum_{k=1}^{\infty}k^l\gamma^{k-1} = \sum_{k=1}^{\infty}e^{\alpha Bk}\gamma^{k-1}.$$

Substituting this estimate into (13), we get (12) provided α satisfying $\gamma e^{\alpha B}<1$.

The proof of the lemma is concluded.

Remark. It is in Long[3] that such generalized γ-graded function with $0\leqslant b_k\leqslant B$ replaced by $|b_k|\leqslant B$ was introduced, and its BMO-norm was estimated.

Lemma 3. *Let $\{T_k\}_1^\infty$ be γ_0-graded sequence. Let $\{A_k\}_1^\infty$ be a sequence of sets satisfying $A_k\subset\{T_k<\infty\}$, $A_k\in\mathscr{F}_{T_k}$, and*

$$E(\mathbf{1}(A_{k+1})\,|\,\mathscr{F}_{T_k})\leqslant\gamma_1,\ \text{a. e.} \tag{14}$$

Let $\{b_k\}_1^\infty$ be a sequence of functions satisfying $0\leqslant b_k\leqslant B$, and $b_k(\omega)$ measurable with respect to $\mathscr{F}_{T_k}$. Then, there exists a γ-graded sequence $\Big($with $\gamma=\dfrac{\gamma_1}{1-\gamma_0}$, assuming γ_1 small such that $\gamma<1.\Big)$ $\{S_j\}_1^\infty$ and a sequence $\{C_j\}_1^\infty$ of functions satisfying $0\leqslant C_j(\omega)\leqslant B$, and $C_j(\omega)$ measurable with respect to $\mathscr{F}_{S_j}$, such that

$$\varphi=\sum_1^\infty b_k\mathbf{1}(A_k)=\sum_1^\infty C_j\mathbf{1}(\{S_j<\infty\})=\theta. \tag{15}$$

Proof Define

$$n_0(\omega)\equiv 0,\ n_j(\omega)=\inf\{i>n_{j-1}(\omega);\ \omega\in A_i\},\ j=1,\ 2,\ \cdots.$$

First of all, we prove $\{n_j(\omega)=k\}\in\mathscr{F}_{T_k}$ by induction. We have

$$n_1(\omega)=\inf\{i>0;\ \omega\in A_i\},$$

$$\{n_1=k\}=A_1'\cap\cdots\cap A_{k-1}'\cap A_k\in\mathscr{F}_{T_k}{}^*.$$

Now, suppose that $\{n_{j-1}=l\}\in\mathscr{F}_{T_l}$, $\forall l=1,\ \cdots,\ k-1$. Then

* For a set A, we denote the complementary set of A by A'.

$$\{n_j=k\}=\bigcup_{i=1}^{k-1}\Big((\{n_{j-1}=i\}\cap A_k\cap\Big(\bigcap_{j=i+1}^{k-1}A_j'\Big)\Big)\in\mathscr{F}_{T_k}.$$

Now, $\forall j=1, 2, \cdots$, define

$$S_j(\omega)=\begin{cases}T_{n_j(\omega)}(\omega), & n_j(\omega)<\infty,\\ \infty, & n_j(\omega)=\infty.\end{cases} \tag{16}$$

Then, each S_j is a stopping time. This is due to

$$\{S_j=k\}=\bigcup_{l=1}^{\infty}(\{n_j=l\}\cap\{T_l=k\})\in\mathscr{F}_k,\ \forall k=0, 1, \cdots.$$

And because $\{n_j\}$ and $\{T_k\}$ are both increasing, $\{S_j\}$ is also increasing. In addition, we have also $\{n_j=k\}\in\mathscr{F}_{S_j}$. In fact, because, $\forall n=0, 1, \cdots$

$$\{n_j=k\}\cap\{S_j=n\}=\{n_j=k\}\cap\{T_k=n\}\in\mathscr{F}_n.$$

Now we proceed to prove that $\{S_j\}_1^\infty$ is $\gamma=\dfrac{\gamma_1}{1-\gamma_0}$-graded.

$\forall k\geqslant 1$, we have

$$\begin{aligned}E(\mathbf{1}(\{S_{j+1}<\infty\})\,|\,\mathscr{F}_{S_j})\mathbf{1}(\{n_j=k\})&=E(\mathbf{1}(\{S_{j+1}<\infty\})\mathbf{1}(\{n_j=k\})\,|\,\mathscr{F}_{T_k})\\ &=E\Big(\mathbf{1}\Big(\bigcup_{i=k+1}^{\infty}A_i\Big)\Big|\mathscr{F}_{T_k}\Big)\mathbf{1}(\{n_j=k\})\leqslant\sum_{i=k+1}^{\infty}E(\mathbf{1}(A_i)\,\Big|\mathscr{F}_{T_k}\Big)\mathbf{1}(\{n_j=k\})\\ &\leqslant\sum_{i=k+1}^{\infty}\gamma_1\gamma_0^{i-k-1}=\frac{\gamma_1}{1-\gamma_0},\end{aligned}$$

and for $k=\infty$, we have (noticing $\{n_j=\infty\}\in\mathscr{F}_{S_j}$)

$$E(\mathbf{1}(\{S_{j+1}<\infty\})\,|\,\mathscr{F}_{S_j})\mathbf{1}(\{n_j=\infty\})=E(\mathbf{1}(\{S_{j+1}<\infty\}\cap\{n_j=\infty\})\,|\,\mathscr{F}_{S_j})=0.$$

This proves $$E(\mathbf{1}(\{S_{j+1}<\infty\})\,|\,\mathscr{F}_{S_j})\leqslant\frac{\gamma_1}{1-\gamma_0},\ \forall j\geqslant 1.$$

Now, we define

$$C_j=\begin{cases}b_{n_j}, & n_j<\infty,\\ 0, & n_j=\infty.\end{cases} \tag{17}$$

Then $0\leqslant C_j\leqslant B$, and $C_j(\omega)$ is measurable with respect to $\mathscr{F}_{S_j}$, due to

$$\{C_j\in\Delta\}\cap\{S_j=n\}=\bigcup_k(\{n_j=k\}\cap\{T_k=n\}\cap\{b_k\in\Delta\})\in\mathscr{F}_n(\Delta,\text{ Borels in }\mathbf{C}).$$

It remains to prove $\varphi=\theta$, a. e.

Let $\omega\in\Omega$. Supposing $\omega\notin\bigcup_1^\infty A_i$, then $\varphi(\omega)=0$, and $\theta(\omega)=0$ too, because of $n_j(\omega)=\infty$, $\forall j$. Supposing $\omega\in A_{n_1}\cap\cdots\cap A_{n_j}$, then $\varphi(\omega)=b_{n_1}+\cdots+b_{n_j}$. But due to $n_1(\omega)<\infty$, $S_1(\omega)<\infty$ (since $\omega\in A_{n_1}$), $\cdots$, $n_j(\omega)<\infty$, $S_j(\omega)<\infty$, and $n_{j+1}(\omega)=\infty$, we have also $\theta(\omega)=C_1+\cdots+C_j=b_{n_1}+\cdots+b_{n_j}=\varphi(\omega)$. Notice else $\Big|\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}A_i\Big|=0$. This proves $\varphi=\theta$, a. e.

Thus, the lemma is proved completely.

Remark. When $b_i\equiv 1$, the lemma is due to Varopoulos[1], but the condition there may not be sufficient.

Theorem 1. *Every real martingale f of BMO can be decomposed as $f=g-h+\varphi$, where $g, h\in BLO$, $\varphi\in L^\infty$.*

Proof Since $f\in BMO$, by virtue of John-Nirenberg's theorem, there exists $\alpha>0$ stuch that (see [2] for example)

$$E(e^{\alpha|f-f_n|}|\mathscr{F}_n)\leqslant K_\alpha<\infty,\ \text{a. e.}$$

For $\lambda>0$ to be determined late, define

$$f^{(1)}=f-E(f),\ T_i=\inf\{n;\ |f_n^{(i)}|>\lambda\},\ f^{(i+1)}=f-f_{T_i},\ i=1,\ 2,\ \cdots, \tag{18}$$

where $f_n^{(i)}=E(f^{(i)}|\mathscr{F}_n)$.

Since $f_n^{(i+1)}=f_n-f_{T_i\wedge n}$, then $f_n^{(i+1)}=0$ when $n\leqslant T_i$, hence $T_{i+1}>T_i$ Thus, $\{T_i\}_1^\infty$ is increasing. Furthermore, we have

$$E(e^{\alpha|f^{(i+1)}|}|\mathscr{F}_{T_{i+1}})\geqslant e^{\alpha E(|f^{(i+1)}||\mathscr{F}_{T_{i+1}})}$$
$$\geqslant e^{\alpha|E(f^{(i+1)}|\mathscr{F}_{T_{i+1}})|}=e^{\alpha|f_{T_{i+1}}^{(i+1)}|}\geqslant e^{\alpha\lambda}\mathbf{1}(\{T_{i+1}<\infty\}),$$

and

$$K_\alpha\geqslant E(e^{\alpha|f-f_{T_i}|}|\mathscr{F}_{T_i})=E(E(e^{\alpha|f^{(i+1)}|}|\mathscr{F}_{T_{i+1}})|\mathscr{F}_{T_i}).$$

Thus, we have

$$E(\mathbf{1}(\{T_{i+1}<\infty\})|\mathscr{F}_{T_i})\leqslant K_\alpha e^{-\alpha\lambda}. \tag{19}$$

That is to say, $\{T_i\}_1^\infty$ is a γ-graded sequence with $\gamma=K_\alpha e^{-\alpha\lambda}$ (assuming λ to be enough large).

Since $\{T_i\}_1^\infty$ is γ-graded, $T_i\to\infty$, a. e, and $f^{(i)}\to 0$, a. e., and $\{T_i=\infty\}\subset\{f^{(i+1)}=0\}$. Let $T_0=0$. Then $f_{T_0}=E(f)$. Thus, we get the decomposition of f as follows

$$f-E(f)=\sum_1^\infty(f_{T_i}-f_{T_{i-1}})=\sum_1^\infty f_{T_i}^{(i)}$$
$$=\sum_1^\infty(f_{T_i}^{(i)})^+\mathbf{1}(\{T_i<\infty\})-\sum_1^\infty(f_{T_i}^{(i)})^-\mathbf{1}(\{T_i<\infty\})+\sum_1^\infty f_{T_i}^{(i)}\mathbf{1}(\{T_i=\infty\}),$$

$$f=g-h+\varphi,\ \varphi=\sum_1^\infty f_{T_i}^{(i)}\mathbf{1}(\{T_i=\infty\})+E(f),\ \ g=\sum_1^\infty(f_{T_i}^{(i)})^+\mathbf{1}\{T_i<\infty\}). \tag{20}$$

Noting that $|f_{T_i}^{(i)}\mathbf{1}(\{T_i=\infty\})|\leqslant\lambda$, and that $\{f_{T_i}^{(i)}\mathbf{1}(\{T_i=\infty\})\neq 0\}$'s are mutually disjoint ($\subset\{T_{i-1}<\infty,\ T_i=\infty\}$), we have

$$\|\varphi\|_\infty\leqslant|E(f)|+\lambda.$$

Since $(f_{T_i}^{(i)})^+$(or $(f_{T_i}^{(i)})^-$) is positive, and measurable with respect to $\mathscr{F}_{T_i}$, and bounded uniformly with the uniform bound $\lambda+\|f\|_{BMO}$

$$|f_{T_i}^{(i)}|\leqslant\lambda+\|f^{(i)}\|_{BMO}\leqslant\lambda+\|f\|_{BMO}. \tag{21}$$

Then by virtue of Lemma 2, g, $h\in BLO$.

The proof of the theorem is thus finished.

Remarks 1. The idea of this proof is due to Varopoulos[1]. The notion of BLO and the decomposition of BMO as difference of BLO in classical case are due to Coifman-Rochberg[4].

2. By means of a decomposition of BMO of Garsia[5], we can obtain the decomposition of $f\in BMO$

$$f=g-h+\varphi.$$

with

$$\|g\|_{BLO}+\|h\|_{BLO}+\|\varphi\|_\infty\leqslant C\|f\|_{BMO}.$$

Now, we refine the preceding decomposition.

Theorem 2. *Let $f\in \log A_{\alpha,\beta}\cap BD$. Then for any $\varepsilon>0$, we have the decomposition of f*

$$f=g-h+\varphi$$

with $\varphi\in L^\infty$, and

$$g\in BLO\cap \log A_{\alpha-\varepsilon,\tau},\ h\in BLO\cap\log A_{\beta-\varepsilon,\tau},\ \forall\tau>0.$$

Proof It is easy to see, for min (α, β), say β

$$E(e^{\beta|f-f_n|}\mid\mathscr{F}_n)\leqslant K_\beta<\infty, \text{ a. e.}$$

As done in the proof of Theorem 1, we can get a $\gamma_0=K_\beta e^{-\beta\lambda}$-graded sequence $\{T_i\}_1^\infty$ and a decomposition of f

$$f=g-h+\varphi,$$

$$g=\sum_1^\infty f_{T_i}^{(i)}\mathbf{1}(A_i),\ A_i=\{\omega\in\{T_i<\infty\}\colon f_{T_i}^{(i)}>0\},$$

$$h=-\sum_1^\infty f_{T_i}^{(i)}\mathbf{1}(B_i),\ B_i=\{\omega\in\{T_i<\infty\}\colon f_{T_i}^{(i)}<0\},$$

$$\varphi=\sum_1^\infty f_{T_i}^{(i)}\mathbf{1}(\{T_i=\infty\})+E(f).$$

Note that $A_i\in\mathscr{F}_{T_i}$ and $E(\mathbf{1}(A_i)\mid\mathscr{F}_{T_{i-1}})\leqslant\gamma_1=K_\alpha e^{-\alpha\lambda}$. The latter follows from

$$K_\alpha\geqslant E(e^{\alpha(f-f_{T_i})}\mid\mathscr{F}_{T_i})=E(E(e^{\alpha(f-f_{T_i})}\mid\mathscr{F}_{T_{i+1}})\mid\mathscr{F}_{T_i})$$
$$\geqslant E(e^{\alpha E(f-f_{T_i}\mid\mathscr{F}_{T_{i+1}})}\mid\mathscr{F}_{T_i})\geqslant E(e^{\alpha\lambda}\mathbf{1}(A_{i+1})\mid\mathscr{F}_{T_i}).$$

Note also that $b_i=f_{T_i}^{(i)}$ is measurable with respect to $\mathscr{F}_{T_i}$ and

$$\lambda\leqslant b_i\mathbf{1}(A_i)\leqslant\lambda+\|f\|_{BMO}.$$

Thus, by virtue of Lemma 3, there exist a γ-graded sequence $\{S_j\}_1^\infty$ (with $\gamma=K_\alpha e^{-\alpha\lambda}/(1-K_\beta e^{-\beta\lambda})<1$, provided λ is enough large) and a sequence $\{C_j\}_1^\infty$ of functions, such that $(\{S_j\}_1^\infty, \{C_j\}_1^\infty)$ is a pair satisfying the condition of Lemma 2, and

$$g=\sum_1^\infty f_{T_k}^{(i)}\mathbf{1}(A_i)=\sum_1^\infty C_j\mathbf{1}(\{S_j<\infty\}).$$

We have known $\varphi\in L^\infty$, g, $h\in BLO$. It remains to prove that for $\varepsilon>0$ given arbitrarily, we have that

$$g\in\log A_{\alpha-\varepsilon,\tau},\ h\in\log A_{\beta-\varepsilon,\tau},\ \forall\tau>0,$$

provided λ is chosen enough large. we aim at g first. For $\varepsilon>0$, choose $\delta_1>0$, $\delta_2>0$ such that $(\delta_1+\delta_2)\ \alpha<\varepsilon$, then choose λ such that

$$\lambda+\|f\|_{BMO}\leqslant(1+\delta_1)\lambda,$$
$$K_\alpha e^{-\alpha\lambda}/(1-K_\beta e^{-\beta\lambda})\leqslant e^{-(1-\delta_2)\alpha\lambda}.$$

Thus, we have $\quad e^{(\alpha-\varepsilon)(1+\delta_1)\lambda}e^{-(1-\delta_2)\alpha\lambda}=e^{(-\varepsilon+(\delta_1+\delta_2)\alpha-\varepsilon\alpha)\lambda}<1.$

By means of Lemma 2, we get finally

$$E(e^{(\alpha-\varepsilon)(g-g_n)}\mid\mathscr{F}_n)\leqslant K_{\alpha,\beta,\varepsilon}<\infty. \tag{23}$$

Similarly, but more simply, since $\{T_i\}_1^\infty$ is already $K_\beta e^{-\beta\lambda}$-graded, we have

$$E(e^{(\beta-\varepsilon)(h-h_n)}\mid\mathscr{F}_n)\leqslant K_{\beta,\varepsilon}<\infty. \tag{23$'$}$$

Since g, $h\in BLO$, $\forall\tau>0$, we have $g\in\log A_{\alpha-\varepsilon,\tau}$, $h\in\log A_{\beta-\varepsilon,\tau}$.

The proof of the theorem is concluded.

Remark. We are almost in a position to prove the factorization theorem of A_p weights. We postpone this to § 3.

§ 3. Factorization of A_p weights

Before proving the main theorem of this section, we prove the following lemmas.

Lemma 4. *Real martingale* $f=(f_n)_{n\geq 0}\in BMO$, *iff there exists* $\lambda\neq 0$ *such that* $z=e^{\lambda f}\in A_p\cap S$, $p>1$. *And*, $f=(f_n)\in BLO$, *iff, there exists* $\lambda>0$ *such that*, $z=e^{\lambda f}\in A_1\cap S$.

Proof Let real $f\in BMO$, then $f\in \log A_{\alpha,\beta}\cap BD$. Thus $e^{\alpha f}\in A_p$ with p satisfying $\frac{\beta}{\alpha}=\frac{1}{p-1}$. And since

$$E(e^{\alpha f}\mid \mathscr{F}_n)\leqslant Ke^{\alpha f_n},\text{ a. e.,} \tag{24}$$

we have $E(e^{\alpha f}\mid \mathscr{F}_n)\leqslant Ke^{\alpha f_n}\leqslant Ke^{\alpha f_{n-1}+\alpha\|f\|_{BMO}}\leqslant Ke^{\alpha f_{n-1}}\leqslant KE(e^{\alpha f}\mid \mathscr{F}_{n-1})$.

That is to say $e^{\alpha f}\in S^+$. Similarly, $e^{\alpha f}\in S^-$. If $f\in BLO$, then we have

$$E(e^{\alpha f}\mid \mathscr{F}_n)\leqslant Ke^{\alpha f_n}\leqslant Ke^{\alpha f+\alpha\|f\|_{BLO}}\leqslant Ke^{\alpha f},$$

that is to say $z=e^{\alpha f}\in A_1$.

Conversely, let $z=e^{\lambda f}\in A_p\cap S$. Then we have (24) with α replaced by λ. Thus we have

$$e^{\lambda f_{n-1}}\leqslant E(e^{\lambda f}\mid \mathscr{F}_{n-1})\leqslant KE(e^{\lambda f}\mid \mathscr{F}_n)\leqslant Ke^{\lambda f_n}=e^{\lambda f_n+\lambda C},$$

$$f_{n-1}\leqslant f_n+C.$$

Similarly, we have also

$$f_n\leqslant f_{n-1}+C.$$

That is to say $f\in BD$. Since we have already that $f\in\log A_{\alpha,\beta}$ for certain α, $\beta>0$, so that $f\in BMO$. Furthermore, if $e^{\lambda f}\in A_1\cap S$, we have

$$e^{\lambda f_n}\leqslant E(e^{\lambda f}\mid \mathscr{F}_n)\leqslant Ke^{\lambda f}=e^{\lambda f+\lambda c},$$

$$f_n\leqslant f+c,\text{ a. e, }\forall n.$$

That is to say $f\in BLO$.

The lemma is thus proved.

Lemma 5. (reverse Hölder's inequality). *Let z be a weight of $A_p\cap S^+$, then there exists $\varepsilon>0$ such that*

$$E(z^{1+\varepsilon}\mid \mathscr{F}_n)\leqslant Kz_n^{1+\varepsilon},\text{ a. e, }\forall n. \tag{25}$$

We take this for granted, the proof is referred to Doléans-Dade; Meyer[6].

Lemma 6. *Let z be a weight of $A_p\cap S$, then there is $\varepsilon>0$ such that $z^{1+\varepsilon}\in A_p\cap S$,* $1\leqslant p<\infty$.

Proof To begin with, consider $1<p<\infty$. Denote $U=z^{-\frac{1}{p-1}}$, $U_n=E(U\mid \mathscr{F}_n)$. It is easy to see that $z\in A_p$ iff $U\in A_{p'}$ Furthermore, we have.

$$1=E(z^{\frac{1}{p}}z^{-\frac{1}{p}}\mid \mathscr{F}_n)^p\leqslant E(z\mid \mathscr{F}_n)E(z^{-\frac{p'}{p}}\mid \mathscr{F}_n)^{\frac{p}{p'}}=z_nU_n^{p-1}\leqslant K. \tag{26}$$

Thus, $z\in A_p\cap S$, iff $U\in A_{p'}\cap S$. By means of Lemma 5, we know that there exist $\alpha>0$, $\beta>0$, such that

$$\begin{aligned} z_n &\leqslant E(z^{1+\alpha}|\mathscr{F}_n)^{\frac{1}{1+\alpha}}\leqslant K_1 z_n, \\ U_n &\leqslant E(U^{1+\beta}|\mathscr{F}_n)^{\frac{1}{1+\beta}}\leqslant K_2 U_n. \end{aligned} \tag{27}$$

Obviously, for $\varepsilon=\min(\alpha, \beta)$, (27) with $\alpha=\beta=\varepsilon$ still holds. Thus, from (26), (27) we have

$$\{E(z^{1+\varepsilon}|\mathscr{F}_n)E(U^{1+\varepsilon}|\mathscr{F}_n)^{p-1}\}^{\frac{1}{1+\varepsilon}}\leqslant K_1 z_n(K_2U_n)^{p-1}\leqslant K_1K_2^{p-1}K,$$

$$\therefore\quad E(z^{1+\varepsilon}|\mathscr{F}_n)E((z^{1+\varepsilon})^{-\frac{1}{p-1}}|\mathscr{F}_n)^{p-1}\leqslant (KK_1K_2^{p-1})^{1+\varepsilon}=K.$$

This proves $z^{1+\varepsilon}\in A_p$. As regards $z^{1+\varepsilon}\in S$, it follows from $z\in S$ and (27) with $\alpha=\varepsilon$.

When $p=1$, the proof is much simpler, only a half of (27) is needed.

Theorem 3. *Let $z=e^f$ be a weight. Then $z\in A_p\cap S$ iff $z=z_1z_2^{1-p}$ with $z_i\in A_1\cap S$, $i=1, 2$.*

Proof Suppose that $z=z_1z_2^{1-p}$ with $z_i=e^{f_i}\in A_1\cap S$, $i=1, 2$. We have

$$\begin{aligned} &E(z_1z_2^{1-p}|\mathscr{F}_n)E((z_1z_2^{1-p})^{-\frac{1}{p-1}}|\mathscr{F}_n)^{p-1} \\ &= E(z_1z_{2,n}^{p-1}z_2^{1-p}|\mathscr{F}_n)z_{2,n}^{1-p}E(z_1^{-\frac{1}{p-1}}z_{1,n}^{\frac{1}{p-1}}z_2|\mathscr{F}_n)^{p-1}z_{1,n}^{-1} \\ &\leqslant K_2E(z_1|\mathscr{F}_n)z_{1,n}^{-1}K_1E(z_2|\mathscr{F}_n)^{p-1}z_{2,n}^{1-p}=K_1K_2. \end{aligned}$$

This proves $z\in A_p$. Furthermore, from Lemma 4, we have $f_i\in BLO$, hence $f\in BMO$. This and $E(e^f|\mathscr{F}_n)\leqslant Ke^{f_n}$ together imply $z=e^f\in S$.

Now, suppose $z=e^f\in A_p\cap S$. By virtue of Lemma 6, there is $\varepsilon>0$ such that $z^{1+\varepsilon}\in A_p\cap S$. From Corollary 1 and Lemma 4, we have $f\in\log A_{1+\varepsilon,\frac{1+\varepsilon}{p-1}}\cap BD$. Now an application of Theorem 2 to f gives

$$f=f_1-f_2,$$

with $f_1=g+\varphi$, $f_2=h$ (or $f_1=g$, $f_2=h-\varphi$) satisfying

$$\begin{aligned} &f_1\in BLO\cap\log A_{1+\delta,\tau}, \\ &f_2\in BLO\cap\log A_{\frac{1+\delta}{p-1},\tau}, \end{aligned}\quad 0<\delta<\varepsilon,\ \forall\tau>0.$$

By means of Lemma 4 again, we have

$$e^{(1+\delta)f_1}\in A_1\cap S,\quad e^{\frac{1+\delta}{p-1}f_2}\in A_1\cap S.$$

Since, by virtue of Hölder's inequality, we know that if $\omega\in A_1$ and $0<\delta<1$, so does ω^δ. Then from this

$$z_1=e^{f_1}\in A_1\cap S,\quad z_2=e^{\frac{1}{p-1}f_2}\in A_1\cap S.$$

Thus,

$$z=e^f=e^{f_1}e^{-f_2}=z_1z_2^{1-p}$$

realize the required factorization of z. The theorem is thus proved.

Remarks 1. We don't know if the condition $z\in S$ is superfluous for the truth of the theorem. But, we will show that this condition is reclly necessary for Lemma 4 and Lemma 6. An example of Bonami-Lepingle[7] shows that Lemma 6 is no longer true without $z\in S$. We take an example to show so does this assertion for Lemma 4. On probability space $(\Omega, \mathscr{F}, \mu)$, we take $\mathscr{F}_0=(\phi, \Omega)$, $\mathscr{F}_1=\mathscr{F}_2=\cdots=\mathscr{F}$. Then in this case, $BLO=BMO=L^\infty$, but

$$A_p=\{\text{positive } z;\ z\in L^1,\ z^{-1}\in L^{\frac{1}{p-1}}\},\ 1<p<\infty,$$
$$A_1=\{\text{positive } z;\ z\geqslant a>0\},\ a=\text{const.}$$

Thus, $\log A_p\not\subset BMO$, $\log A_1\not\subset BLO$, i. e. Lemma 4 fails to be true. We know that the proof of Theorem 3 depends heavily on Lemma 4 and Lemma 6. Without Lemma 4, we don't know if $\{f_{T_i}^{(i)}\}$ are still uniformly bounded, and without Lemma 6, we don't know if $e^{f_1}\in A_1$ and, $e^{-\frac{1}{p-1}f_2}\in A_1$ still hold.

2. For weight problem, the condition S occurs often and holds in many cases. For example, under a regular condition considered by many authors such as

$$E(\mathbf{1}(F)\,|\,\mathscr{F}_n)\leqslant dE(\mathbf{1}(F)\,|\,\mathscr{F}_{n-1}),$$

$\forall n$ and $\forall F\in\mathscr{F}$, $d=\text{const.}$, the conditisn S is an immediate consequence of A_p condition (vi'a (26)).

§ 4. Continuous times case

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space, and let $\{\mathscr{F}_t\}_{t\geqslant 0}$ be an increasing family of sub-σ-fields of $\mathscr{F}$ satisfying the usual conditions, i. e. $\mathscr{F}_0$ complete (assuming $\mathscr{F}_0$ be trivial in addition), $\{\mathscr{F}_t\}_{t\geqslant 0}$ right continuous, and $\mathscr{F}=\bigvee_{t\geqslant 0}\mathscr{F}_t$. In this case, Theorems 1, 2 and 3 still hold, and all arguments remain almost unchanged. It is sufficient to show when the care is needed. We begin with the definitions.

A martingale $f=(f_t)_{t\geqslant 0}$ of L^1 is said to be of BMO, if*

$$\|f\|_{BMO}=\sup_T\|E(|f-f_{T-}|\,|\,\mathscr{F}_T)\|_\infty<\infty,$$

$$T \text{ is taken through all stopping times;} \tag{1'}$$

is said to be of BLO, if it is real, and

$$\|f\|_{BLO}=\inf\{C;\ f_t-f\leqslant C,\ |\Delta f_T|=|f_T-f_{T-}|\leqslant C,\ \forall t,\ T,\ \text{outside a null set}\}<\infty \tag{2'}$$

is said to be of $\log A_{\alpha,\beta}$, if (3) or its epuivalence (3)$'$ with n replaced by t holds; it is said to be of BD, if

$$\|f\|_{BD}=\sup_T\|\Delta f_T\|_\infty<\infty. \tag{4'}$$

A weight z is said to be of A_p, if (6) holds with n replaced by t; it is said to be of S (or S^+, or S^-), if (7) holds with n replaced by T, $n-1$ by T^-.

Note that in preceding definitions, all statement concerning times t can be substituted by that concerning stopping times T. Roughly speaking, an assertion (or

* As shown by Meyer (Sém. Prob. Lect. Notes in Math., 511 (1976), p 348) that this definition is equavalent to usual one.

condition) holds for all t's, so it does for all T's (T stopping times), except those concerning left limit. This may be seen sometimes immediately, sometimes by a limit argument such as: for any T, define

$$T_n=\sum_{k=0}^{\infty}\frac{k+1}{2^n}\mathbf{1}(F_{n,k}),\quad \text{with } F_{n,k}=\left\{\frac{k}{2^n}\leqslant T<\frac{k+1}{2^n}\right\},$$

Then $T_n\searrow T$, and T_n takes only discrete values. Thus, an assertion holds for all t's so it does for T_n, and due to the right continuity, so does for T in general.

By means of this observation, we have yet four assertions in § 1 with a slight modification, i. e. $\|\varphi\|_{BMO}\leqslant\|f\|_{BMO}$ is replaced by $\|\varphi\|_{BMO}\leqslant 2\|f\|_{BMO}$ in assertion 2. Furthermore, Lemma 1, Corollary 1, and Lemma 3 still hold obviously. For the proof of Lemma 2, only the part concerning the estimate of $\|\varphi\|_{BMO}$ is slightly complex, but it has been done in Long[2]. For the proof of the Theorem 1, it is needed to appeal to John-Nirenberg Theorem the proof of which has also occured in [2]. The proofs of the Theorem 2, Lemma 5 and 6, and Theorem 3 remain unchanged with a trivial modification. The proof of the Lemma 4 will be finished by the substitution of n by T in the original proof.

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