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ON THE STRUCTURE OF PRIMITIVE RINGS

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Abstract

In this paper the author introduces two concepts, i. e. the concept of so-colled ν-socles of primitive rings and the concept of a pair of dual modules. Then the author establishes a general structure theorem for primitive rings with ν-socles, which implies the well-known structure theorem for primitive rings with usual non-zero socles.

It is well known that the investigation of structure of primitive rings is usually restricted by their non-zero socles. There is almost nothing to do with the structure of primitive rings without non-zero socles. Even if we study the structure of primitive rings with non-zero socles, we are always concerned for their properties of finite-fold transitivity. But in general, primitive rings are infinite-fold transitive. Thus for the purpose of studying deeply the structure of primitive rings it is useful to introduce more general concept of so-called ν-socles[3]. Using the concept of ν-socles and $\aleph_\nu$-fold transitivity we shall in this paper characterize some basic properties of ν-socles. Then in § 2 we extend the notion of a pair of dual vector spaces to the one of a pair of dual modules. Besides, it permits us to associate with every primitive ring having ν-socle a pair of dual modules and then we establish a general structure theorem for primitive rings with ν-socles, which implies the well-known structure theorem for primitive rings with usual non-zero socles.

1. Before preceding our theory we shall discuss a few preliminaries. Throughout this paper the term "vector space" without modifies will always mean left vector space over a division ring and primitive ring R always mean dense subring of the complete ring Ω of all linear transformations of a vector space. A primitive ring R is called $\aleph_\nu$-fold transitive if and only if for any subset $\{x_i\}_{i\in I}$ of linearly independent elements x_i and any subset $\{y_i\}_{i\in I}$ of vector space $\mathfrak{M}$ there exists an element $r\in R$ such that $x_ir=y_i$ for $i\in I$, where the cardinal number of I, denoted by $|I|$, is smaller than $\aleph_\nu$. Specially, we say that R is finitefold transitive if $\aleph_\nu=\aleph_0$. A primitive ring R is called having the lergest $\aleph_\nu$-transitivity if R is $\aleph_\nu$-fold transitive and not $\aleph_{\nu+1}$-

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fold transitive. Two primitive rings are called the same fold transitivity if their largest transitivities are the same. Let Ω be the complete ring of linear transformations of vector space $\mathfrak{M}$, R a dense subring of Ω. We always denote $T_\nu=\{\omega\in\Omega \mid \rho(\omega)<\aleph_\nu\}$, where $\rho(\omega)$ denotes the rank of ω. And we always mean $N(\sigma)=\{m\in\mathfrak{M} \mid m\sigma=0\}$ for any $\sigma\in\Omega$, and call $N(\sigma)$ the annihilator of σ in $\mathfrak{M}$.

Lemma 1.1. *Let $\mathfrak{M}$ be a left vector space over division ring F, Ω the ring of linear transformations, $T_\nu=\{\omega\in\Omega \mid \rho(\omega)<\aleph_\nu\}$. Let R be a subring of Ω which is $\aleph_\nu$-fold transitive, and $\mathfrak{S}_\nu=T_\nu\cap R$. Suppose that $\mathfrak{S}_\nu\neq\mathfrak{S}_\mu$ for any ordinal number $\mu<\nu$. Then $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive.*

Proof If $\mathfrak{S}_\mu=0$ for any ordinal number $\mu<\nu$, then ν is not a limit ordinal number, because if ν is a limit ordinal number, then $T_\nu=\bigcup_{\mu<\nu}T_\mu$. From this it follows that $\mathfrak{S}_\nu=T_\nu\cap R=\bigcup_{\mu<\nu}(R\cap T_\mu)=0$. Hence ν is not a limit number. It is easy to see that there exists an elemnt $\sigma\in\mathfrak{S}_\nu$ with $\rho(\sigma)=\aleph_{\nu-1}$. Now we prove that $R\sigma R$ is $\aleph_\nu$-fold transitive. In fact, we have $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(\sigma)$, where $N(\sigma)$ is the annihilator of σ in $\mathfrak{M}$, $|I|=\aleph_{\nu-1}$. Hence $\mathfrak{M}\sigma=\sum_{i\in I}\oplus Fu_i\sigma$. Denote $\{\bar{u}_j\}_{j\in J}$ as a set of F-linearly independent elements, $\{b_j\}_{j\in J}$ an arbitrary set of elements of $\mathfrak{M}$ and $|J|<\aleph_\nu$. Then there exists an element $r\in R$ such that $\bar{u}_jr=u_j$f or $j\in J\subseteq I$, since R is $\aleph_\nu$-fold transitive. On the other hand, there exists an element $s\in R$ such that $u_i\sigma s=b_i$ for $i\in J$, since the set $\{u_i\sigma\}_{i\in I}$ is linearly independent. Therefore $\bar{u}_ir\sigma s=b_i$ for $i\in J$. But $r\sigma s\in R\sigma R\subseteq\mathfrak{S}_\nu$. This implies that $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive.

Now we may assume that there exists an $\mu<\nu$ such that $\mathfrak{S}_\mu\neq0$. By hypothesis for our lemma we can choose an element $\sigma\in\mathfrak{S}_\nu$ such that $\rho(\sigma)\geqslant\aleph_\mu$. From the above proof we know that $R\sigma R$ is $\aleph_\mu$-fold transitive, hence $\mathfrak{S}_\nu$ is $\aleph_\mu$-fold transitive. On the other hand, by hypothesis we have $\mathfrak{S}_\alpha\neq\mathfrak{S}_\nu$ where $\mu<\alpha<\nu$. Therefore $\mathfrak{S}_\nu$ is $\aleph_\alpha$-fold transitive. This completes our proof.

Lemma 1.2. *suppose $\mathfrak{S}_\nu=T_\nu\cap R$* is *$\aleph_\nu$-fold transitive, then $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive for any $\mu<\nu$.*

Paoof Let $\{x_i\}_{i\in I}$ denotes a set of linearly independent elements of $\mathfrak{M}$, $|I|<\aleph_\mu$, $\{y_i\}_{i\in I}$ an arbitrary set of elements of $\mathfrak{M}$. Then from the $\aleph_\nu$-fold transitivity of $\mathfrak{S}_\nu$ it follows that there exists an element $\sigma\in\mathfrak{S}_\nu$ such that $x_i\sigma=x_i$ for $i\in I$. Hence $\mathfrak{M}=\sum_{i\in I}\oplus Fx_i\oplus\sum_{j\in J}\oplus Fu_j\oplus N(\sigma)$, where $N(\sigma)=\{x\in\mathfrak{M} \mid x\sigma=0\}$, $|J|<\aleph_\nu$. This implies that $\mathfrak{M}\sigma=\sum_i\oplus Fx_i\sigma\oplus\sum_j\oplus Fu_j\sigma$. Hence there exists an element $\sigma'\in\mathfrak{S}_\nu$ such that $x_i\sigma\sigma'=x_i\sigma$ for $i\in I$ and $u_j\sigma'\sigma=0$ for $j\in J$. Clearly $N(\sigma)\sigma\sigma'=0$. Because $\{x_i\sigma\sigma'\}_{i\in I}$ is linearly independent, there exists an element $\sigma''\in\mathfrak{S}_\nu$ such that $x_i\sigma\sigma'\sigma''=y_i$ for $i\in I$. Let $\tau=\sigma\sigma'\sigma''$, then $x_i\tau=y_i$ for $i\in I$ and $u_j\tau=0$ for $j\in J$, $N(\sigma)\tau=0$. But $\tau=\sigma\sigma'\sigma''\in R\cap T_\mu=\mathfrak{S}_\mu$. This proved that $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive.

Theorem 1.1. *Let R be a primitive ring which is $\aleph_\nu$-fold transitive, then R have zero socle if and only if $\mathfrak{S}_\nu=0$.*

Proof The sufficiency of the condition is clear. Now we are going to prove the necessity of the condition. If $\mathfrak{S}_\mu=0$ for all $\mu<\nu$, then by Lemma 1.1 either $\mathfrak{S}_\nu=0$ or $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive. If the latter case occurs, then $\mathfrak{S}_0$ is $\aleph_0$-fold transitive by Lemma 1.2, hence R would have non-zero socle. This contradicts the assumption of our lemma. Hence $\mathfrak{S}_\nu=0$. Now we may assume that there exists an ordinal number $\mu<\nu$ such that $\mathfrak{S}_\mu\neq0$. Let α be the least ordinal number of all number $\tau\leqslant\nu$ with $\mathfrak{S}_\tau=\mathfrak{S}_\nu$. Then we have $\mathfrak{S}_\rho\neq\mathfrak{S}_\alpha$ for $\rho<\alpha$. By the property of $\aleph_\nu$-fold transitivity of R and Lemma 1.1 we can easily see that $\mathfrak{S}_\alpha$ is $\aleph_\alpha$-fold transitive. Hence $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive by Lemma 1.2 for $\mu<\alpha$. This implies that R has non-zero socle. Thus we have again a contradiction.

Definition 1.1. *Let Ω be the complete ring of linear transformations of $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i$, R be a subring of Ω. $\mathfrak{S}_\nu=T_\nu\cap R$. We call $\mathfrak{S}_\nu$ ν-socle of R if and only if it satisfies the following conditions:* (i)$\mathfrak{S}_\nu\Omega\subseteq\mathfrak{S}_\nu$ (ii)$\mathfrak{S}_\nu$ *is $\aleph_\nu$-fold transitive* (iii)*if $\sigma\in T_\nu$ and $\sigma\mathfrak{S}_\nu\subseteq\mathfrak{S}_\nu$ then $\sigma\in\mathfrak{S}_\nu$.*

Theorem 1.2. *Let R be a primitive ring with $\aleph_\nu$-fold transitivity, $\mathfrak{S}_\nu\neq0$. Then there exists an ordinal number $\mu\leqslant\nu$ such that $\mathfrak{S}_\mu$ is μ-socle and $\mathfrak{S}_\rho$ is also ρ-socle for any $\rho<\mu$.*

Proof First we show that if $\mathfrak{S}_\nu$ is ν-socle, then $\mathfrak{S}_\rho$ is ρ-socle for any $\rho<\nu$. For this purpose we need only to check the conditions of Definition 1.1.

(i) Let $\sigma\in\mathfrak{S}_\rho$, then $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(\sigma)$, where $|I|<\aleph_\rho$ and $N(\sigma)$ as before. Thus we have $\mathfrak{M}\sigma=\sum_I\oplus Fu_i\sigma$. Let ω be an element of the ring Ω of linear transformations of $\mathfrak{M}$. Because R is $\aleph_\nu$-fold transitive, there exists an element $r\in R$ such that $u_i\sigma r=u_i\sigma\omega$ for $i\in I$. From this it is easy to see that $\sigma\omega=\sigma r$. This proves $\sigma\Omega=\sigma R\subset\mathfrak{S}_\rho$. (ii) Since $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive, it follows that $\mathfrak{S}_\rho$ is $\aleph_\rho$-fold transitive by Lemma 1.2. (iii) If $\sigma\in T_\rho$ and $\sigma\mathfrak{S}_\rho\subseteq\mathfrak{S}_\rho$, then $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(\sigma)$, and $\mathfrak{M}\sigma=\sum_{i\in I}\oplus Fu_i\sigma$, where $|I|<\aleph_\rho$. Let $u_i1=u_i$ for $i\in I$, $N(\sigma)1=0$, then $u_i1\sigma=u_i\sigma$ for $i\in I$, $N(\sigma)1\sigma=N(\sigma)\sigma=0$, hence $\sigma=1\sigma$. Now we want to prove that $1\in\mathfrak{S}_\rho$. In fact, since $\{u_i\sigma\}_{i\in I}$ is the set of F-linearly independent elements, there exisits an element $\tau\in\mathfrak{S}_\rho$ such that $u_i\sigma\tau=u_i=u_i1$ for $i\in I$, and $N(\sigma)\sigma\tau=N(\sigma)1=0$. Hence $\sigma\tau=1\in\sigma\mathfrak{S}_\rho\subseteq\mathfrak{S}_\rho$ by the assumption. From above relation $\sigma=1\sigma$ we get $\sigma\in\mathfrak{S}_\rho\Omega\subseteq\mathfrak{S}_\rho$.

Now we want to show that if $\mathfrak{S}_\nu\neq0$, then there exists $\mu\leqslant\nu$ such that $\mathfrak{S}_\mu$ is μ-socle. Certainly, we assume that $\mathfrak{S}_\nu$ is not ν-socle. Then from the proof of (i) we know that $\mathfrak{S}_\nu\Omega\subseteq\mathfrak{S}_\nu$ is always true only if R is $\aleph_\nu$-fold transitive. From the proof of (iii) it follows that if $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive then $\mathfrak{S}_\nu$ satisfies the condition (iii) of Definition 1.1. Therefore, when $\mathfrak{S}_\nu$ is not ν-socle, $\mathfrak{S}_\nu$ is not $\aleph_\nu$-fold transitive too. By the proof of Theorem 1.1, there exisits an ordinal number $\mu<\nu$ such that $\mathfrak{S}_\mu\neq 0$ and from this it follows that there exists an $\alpha<\nu$ such that $\mathfrak{S}_\alpha$ is $\aleph_\alpha$-fold transitive. From the above we can conclude that $\mathfrak{S}_\alpha$ is α-socle. This completes the proof of our theorem.

Now from the proof of Theorem 1.2 we can further formulate the following theorem.

Theorem 1.3. *Let R be $\aleph_\nu$-fold transitive primitive ring, then $\mathfrak{S}_\nu=T_\nu\cap R$ is ν-socle if and only if $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive.*

Lemma 1.3. *Let R be a primitive ring, $\mathfrak{S}_\nu=T_\nu\cap R$ $\aleph_\nu$-fold transitive, then $\mathfrak{S}_\nu$ is a principle ideal if and only if ν is not a limit ordinal number.*

Proof If ν is not a limit ordinal number, then $\nu-1$ exists. By the property of $\aleph_\nu$-fold transitivity there exists an element $\sigma\in\mathfrak{S}_\nu$ such that $\rho(\sigma)=\aleph_{\nu-1}$. By the proof of Lemma 1.1 we know that $R\sigma R$ is $\aleph_\nu$-fold transitive. It needs only to proof $R\sigma R=\mathfrak{S}_\nu$. In fact, we need to prove that every $\aleph_\nu$-fold transitive ideal L contains $\mathfrak{S}_\nu$. For this purpose we let $\sigma\in\mathfrak{S}_\nu$, then $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(\sigma)$, and $\mathfrak{M}\sigma=\sum_{i\in I}\oplus Fu_i\sigma$, where $|I|<\aleph_\nu$. Write l: $u_il=u_i$ for $i\in I$, $N(\sigma)l=0$, then there exists an $\tau\in L$ such that $u_i\sigma\tau=u_i=u_il$ for $i\in I$, $N(\sigma)\sigma\tau=N(\sigma)l=0$, hence $l=\sigma\tau\in RL\subseteq L$. On the other hand we have $l\sigma=\sigma\in LR\subseteq L$. Thus $\mathfrak{S}_\nu\subseteq L$. This proves $\mathfrak{S}_\nu=R\sigma R$.

Conversely, let ν is a limit ordinal number and $\mathfrak{S}_\nu=R\sigma R$, $\sigma\in\mathfrak{S}_\nu$. Suppose that $\rho(\sigma)=\aleph_\mu$ and $\mu<\nu$. Then $\mathfrak{S}_\nu\subset T_\mu$. This contradicts that $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive. Thus $\mathfrak{S}_\nu$ cannot be a principle ideal.

Theorem 1.4. *Let R be a primitive ring with ν-socle $\mathfrak{S}_\nu=T_\nu\cap R$. Then R contains an ideal chain $\mathfrak{S}_\nu\supsetneqq\mathfrak{S}_{\nu+1}\supsetneqq\cdots\supsetneqq\mathfrak{S}_\mu\supsetneqq\cdots\supsetneqq\mathfrak{S}_0$, where $\mu<\nu$, and every ideal $\mathfrak{S}_\mu$ is μ-socle of R, every $\aleph_\nu$-fold ideal of R contains $\mathfrak{S}_\mu$ as well. If μ is not a limit ordinal number, then $\mathfrak{S}_\mu$ is principle. Let L be an non-zero ideal of R with $L\subset T_\nu$, then L must be one of the $\mathfrak{S}_\mu$ of the above chain.*

Proof By Lemma 1.2 and Theorem 1.3, $\mathfrak{S}_\mu$ is μ-socle. Hence $\mathfrak{S}_\mu\neq\mathfrak{S}_\alpha$ if and only if $\alpha\neq\mu$, where α, $\mu<\nu$. By the above lemma, if μ is not limit ordinal number, then $\mathfrak{S}_\mu$ is a principle ideal and any $\aleph_\mu$-fold transitive ideal contains $\mathfrak{S}_\mu$. Hence we need only to prove the last assertion of the theorem. Since $L\neq 0$, $L\subset T_\nu$, there exists an μ such that $L\subset T_\mu$ and $L\not\subset T_\lambda$ where $\lambda<\mu<\nu$. If μ is not a limit ordinal number, then $L\not\subset T_{\mu-1}$. Hence there exists an element $\sigma\in L$ with $\rho(\sigma)=\aleph_{\mu-1}$. From the property of $\aleph_\nu$-fold transitivity it follows that $R\sigma R$ is $\aleph_\mu$-fold transitive. Thus $\aleph_\mu\subseteq R\sigma R\subseteq L\subseteq\mathfrak{S}_\mu$. If μ is a limit ordinal number, then $L\not\subset T_\lambda$ for every non-limit ordinal number $\lambda<\mu$. Hence there exists an element $\sigma\in L$ with $\rho(\sigma)=\aleph_\alpha$ such that $\aleph_\alpha\geqslant\aleph_\lambda$. A similar argument as above can show that $R\sigma R$ is $\aleph_{\alpha+1}$-fold transitive. Hence $R\sigma R=\mathfrak{S}_{\alpha+1}\supset\mathfrak{S}_\lambda$. Thus $L\supset\mathfrak{S}_\lambda$. But λ is arbitrary, hence $L\supseteq\mathfrak{S}_\mu$. From $L\subseteq T_\mu\cap R$ it follows that $L=\mathfrak{S}_\mu$.

From this theorem we get the following well known result.

Corollary *Let Ω be the ring of linear transformations of $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$, then every ideal of Ω must be an T_ν.*

Proposition 1.1 *Let R be primitive ring, then every $\aleph_\nu$-fold transitive left ideal of R contains $\mathfrak{S}_\nu$.*

Proof We write $\sigma\in\mathfrak{S}_\nu$. Then $\mathfrak{M}=\sum_I\oplus Fu_i\oplus N(\sigma)$, $N(\sigma)=\{m\in\mathfrak{M}\,|\,m\sigma=0\}$, and $|I|<\aleph_\nu$. Since L is $\aleph_\nu$-fold transitive, there exists an element $\tau\in L$ such that $u_i\sigma\tau=u_i\sigma$ for $i\in I$. Clearly $N(\sigma)\sigma\tau=0=N(\sigma)\sigma$. Thus $\sigma=\sigma\tau\in L$.

Theorem 1.5. *Let R be a primitive ring with zero socle. Then every non-zero ideal of R have the same transitivity.*

Proof Let R be $\aleph_\nu$-fold transitive but not $\aleph_{\nu+1}$-fold transitive. Since the socle of R is zero, then $\mathfrak{S}_\nu=0$ by Theorem 1.1. Hence every non-zero element σ of R has rank $\geqslant\aleph_\nu$. It is clear that we may assume that there exists an ordinal number μ such that $\mathfrak{S}_\mu=T_\mu\cap R\neq0$ and $\mathfrak{S}_\lambda=0$ for all $\lambda<\mu$. Hence μ is not a limit ordinal number. Therefore, every element of $\mathfrak{S}_\mu$ has rank $\aleph_{\mu-1}$, where $\mu-1\geqslant\nu$. Now we prove that $\mathfrak{S}_\mu$ is $\aleph_\nu$-fold transitive. In fact, we can prove that the ideal $R\sigma R$ generated by any non-zero element σ of $\mathfrak{S}_\mu$ is $\aleph_\nu$-fold transitive. This is, because $\mathfrak{M}=\sum_I\oplus Fu_i\oplus N(\sigma)$, $\mathfrak{M}\sigma=\sum_I\oplus Fu_i\sigma$, $|I|=\aleph_{\mu-1}$. Let $\{\bar{u}_i\}_{i\in J}$ be a set of linearly independent elements, $\{b_i\}_J$ a set of elements of $\mathfrak{M}$. Since $|J|<\aleph_\nu$, it follows from the $\aleph_\nu$-fold transitivity of R that there exists $r\in R$ such that $\bar{u}_ir=u_i$. Hence $\bar{u}_ir\sigma=u_i\sigma$ for $i\in J\subset I$. We have also an element $s\in R$ such that $\bar{u}_ir\sigma s=u_i\sigma s=b_i$ for $i\in J$. Clearly $r\sigma s\in R\sigma R\subset\mathfrak{S}_\mu$. Thus $\mathfrak{S}_\mu$ is $\aleph_\nu$-fold transitive.

On the other hand, let L be an ideal of R and $\sigma\in L$, then σ belongs to some $\mathfrak{S}_\tau=T_\tau\cap R\neq0$ where $\tau\geqslant\mu$. We can also show as before that $R\sigma R$ is $\aleph_\nu$-fold transitive. Hence L is $\aleph_\nu$-fold transitive. This completes the proof of our theorem.

Theorem 1.6. *Let R be a primitiv ring, then R has zero socle if and only if the rank of any non-zero element of R is greater than the largest transitivity of R.*

Proof The necessary part follows immediately from the proof of Theorem 1.5. Now we want to show the sufficient part. If R has non zero socle, then R has element with rank 1. This contradicts the assumption.

Theorem 1.7. *Let R be a primitive ring, then R has zero socle if and only if R contains no right ideal of Ω, where Ω is the closure of R in the finite topology.*

Proof If R contains a right ideal L of Ω and $\sigma\in L,\sigma\neq0$, then $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(\sigma)$, $\mathfrak{M}\sigma=\sum_{i\in I}\oplus Fu_i\sigma$, hence there exists an element $\omega\in\Omega$ such that $u_i\sigma\omega=u_i$, $u_j\sigma\omega=0$ for $i\neq j$, i, $j\in I$. Let E_i be an element of Ω such that $u_iE_i=u_i$, $u_jE_i=0$ for $i\neq j$, i, $j\in I$. $N(\sigma)E_i=0$. Hence $E_i=\sigma\omega\in L$, where $i\in I$. This follows that R has an non-zero socle. Conversely, if R has an non-zero socle, then according to the proof of Theorem 1.2, R

contains an non-zero right ideal of Ω.

2. In this section we first introduce the concept of a pair of modules over ring with identity, which extends the concept of a pair of dual vector space over division ring. After this we study further the structure of primitive rings.

Let $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$ be a vector space over division ring F, $\mathfrak{N}=\sum_{j\in I}\oplus Fu_j$ be a subspace of $\mathfrak{M}$. Clearly, for any complementary vector space $\overline{\mathfrak{N}}$ of $\mathfrak{N}$, i. e. $\mathfrak{M}=\mathfrak{N}\oplus\overline{\mathfrak{N}}$, there exists an idempotent element l such that $nl=n$ for $n\in\mathfrak{N}$ and $\overline{\mathfrak{N}}l=0$. In this situation we say that l corresponds to $\overline{\mathfrak{N}}$ and denote $l=l(\overline{\mathfrak{N}})$, then it is easy to see that for any different complementary $\overline{\mathfrak{N}}_1$ from $\overline{\mathfrak{N}}$, the correponding idempotent elements $l(\overline{\mathfrak{N}}_1)$ and $l(\mathfrak{N})$ are different. Now we choose an arbitrary such idempotent element l. Let $\mathscr{A}^*$ be the set of linear transformations from $\mathfrak{M}$ into $\mathfrak{N}$ and Ω the ring of all linear transformations of $\mathfrak{M}$, then $\mathscr{A}^*=\Omega l$. In fact, if $a^*\in\mathscr{A}^*$, then it is clear $a^*=a^*l\in\Omega l$. Conversely, Ωl is a set of linear transformations of $\mathfrak{M}$ into $\mathfrak{N}$. Hence $\Omega l\subseteq\Omega^*$. Therefore $\mathscr{A}^*=\Omega l$. Suppose that $\overline{\mathfrak{N}}_1$ is an another complementary space of $\mathfrak{N}$, and l_1 is the corresponding idempotent element, we can show that $\Omega l=\Omega l_1$. Since for $n\in\mathfrak{N}$ it follows $nl=nl_1=nll_1$ and for $\bar{n}\in\overline{\mathfrak{N}}$ it follows $\bar{n}l=\bar{n}ll_1=0$, hence $l=ll_1$, $\Omega l\subseteq\Omega l_1$. Similarly, we have $\Omega l_1\subseteq\Omega l$. This means that $\mathscr{A}^*=\Omega l$ is independent on the choice of complementary spaces of $\mathfrak{N}$, it is uniquely determinated by $\mathfrak{N}$. Of course, $\mathscr{A}^*=\Omega l$ determinats the subspace $\mathfrak{N}=\mathfrak{M}\Omega l$. We have proved that the subspaces $\mathfrak{N}$ and the left ideals $\mathscr{A}^*=\Omega l$ of Ω as above are one to one correspondent.

Now we consider the set $\mathscr{A}$ of linear transformations from $\mathfrak{N}$ to $\mathfrak{M}$. We want to show that $\mathscr{A}=l\Omega$. In fact, for any element $a\in\mathscr{A}$ there exists an element $\omega\in l\Omega$ such that $na=n\omega$ for all $n\in\mathfrak{N}$. Hence $\mathscr{A}\subseteq l\Omega$. Conversely, $l\Omega$ is clearly a set of linear transformations from $\mathfrak{N}$ to $\mathfrak{M}$, hence $\mathscr{A}=l\Omega$. Therefore we have a pair of modules $\mathscr{A}=l\Omega$ and $\mathscr{A}^*=\Omega l$.

Let $\mathscr{K}=l\Omega l$, then l is the identity of $\mathscr{K}$. $\mathscr{A}=l\Omega$ is a left $\mathscr{K}$-module and $\mathscr{A}^*=\Omega l$ is a right $\mathscr{K}$-module.

We still denote Ω as the complete ring of $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$, and l is an idempotent element of Ω. Let $\mathscr{K}=l\Omega l$, $\mathscr{A}=l\Omega$, $\mathscr{A}^*=\Omega l$, then $\mathfrak{M}=\mathfrak{N}\oplus N(l)$, where $\mathfrak{N}$ is a subspace and $nl=n$ for all $n\in\mathfrak{N}$, $N(l)=\{x\in\mathfrak{M}\,|\,xl=0\}$. Clearly, $\mathscr{A}^*=\Omega l$ and $\mathscr{A}=l\Omega$ are the complete rings of linear transformations of $\mathfrak{M}$ to $\mathfrak{N}$and of $\mathfrak{N}$ to $\mathfrak{M}$ respectively. This means that the pair of dual modules $\mathscr{A}=l\Omega$ and $\mathscr{A}^*=\Omega l$ over $\mathscr{K}=l\Omega l$ are uniquely correspondent to the subspace $\mathfrak{N}$ of $\mathfrak{M}$.

Definition 2.1. *As stated above, we call the subspace $\mathfrak{N}$ the underlying space of the pair of dual modules $\mathscr{A}=l\Omega$ and $\mathscr{A}^*=\Omega l$ over $\mathscr{K}$. Meanwhile, we call the $\mathscr{A}$ and $\mathscr{A}^*$ are the underlying modules over $\mathscr{K}$ of $\mathfrak{N}$.*

Consider the pair of dual modules $\mathscr{A}=l\Omega$, $\mathscr{A}^*=\Omega l$ over $\mathscr{K}=l\Omega l$. As usual we define the bilinear form as follows: $(a, a^*)=aa^*$ for $a\in\mathscr{A}$, $a^*\in\mathscr{A}^*$. Clearly, $(\mathscr{A}, \mathscr{A}^*)=\mathscr{K}$. We want to show that the bilinear form $(\mathscr{A}, \mathscr{A}^*)$ is non-singular. In fact, if $a^*\in\mathscr{A}^*$ and $\mathscr{A}a^*=0$, then we have $\Omega l\Omega a^*=0$, hence $a^*=0$. Similarly, if $a\in\mathscr{A}$ and $a\mathscr{A}^*=0$, then $a=0$.

Definition 2.2. *Let $\mathscr{A}=l\Omega$, $\mathscr{A}^*=\Omega l$ be a pair of dual modules over $\mathscr{K}=l\Omega l$, $\mathscr{A}'$ a submodule of $\mathscr{A}^*$. Suppose that $a\mathscr{A}'=0$, then $a=0$ for $a\in\mathscr{A}$. Then $(\mathscr{A}, \mathscr{A}')$ is called a pair of dual modules over $\mathscr{K}$.*

Definition 2.3. *Let $\mathfrak{N}$ be the underlying space of the pair of dual modules $\mathscr{A}=l\Omega$, $\mathscr{A}^*=\Omega l$ over $\mathscr{K}=l\Omega l$. We call the pair of dual modules $(\mathscr{A}, \mathscr{A}')$ over $\mathscr{K}$ the $\aleph_\mu$-typical dual modules over $\mathscr{K}$ if $\mathscr{A}'$ is $\aleph_\mu$-fold transitivity of $\mathfrak{M}$ to $\mathfrak{N}$, i.e. for any set of F-linearly independent elements $\{x_i\}_{i\in I}$ of $\mathfrak{M}$ and any set of elements $\{y_i\}_{i\in I}$ of $\mathfrak{N}$ with $|I|<\aleph_\mu$ there exists an element $a'\in\mathscr{A}'$ such that $x_ia'=y_i$ for $i\in I$.*

Lemma 2.1. *Let $\mathscr{A}=l\Omega$, $\mathscr{K}=l\Omega l$. Then the set of $\mathscr{K}$-endomorphisms of left $\mathscr{K}$-module $\mathscr{A}$ is Ω.*

Proof Denote the set of $\mathscr{K}$-endomorphisms of $\mathscr{A}$ by $\tilde{\Omega}$. If $\sigma\in\Omega$ and $\mathscr{A}\sigma=0$, then clearly $\sigma=0$, hence $\Omega\subseteq\tilde{\Omega}$. Now we want to prove that $\Omega=\tilde{\Omega}$. In fact, it is clear that $\mathscr{A}=l\Omega=l\tilde{\Omega}$. For $l^2=l$ we have $\mathfrak{M}=\mathfrak{N}\oplus N(l)$, $\mathfrak{N}=\sum_{i\in I}\oplus Fu_i$, $|I|=\rho(l)$, the rank of l. Then there exists a set $\{E_i\}_{i\in I}$ of idempotent elements with ranks 1 such that $u_iE_i=u_i$, $u_jE_i=0$ for $i\neq j$, $i, j\in I$ and $N(l)E_i=0$. Clearly, $E_il=lE_i=E_i$ for $i\in I$. Write $A_i=E_i\mathscr{A}$, then $A_i=E_il\mathscr{A}=E_i\Omega=E_i\tilde{\Omega}$ for $i\in I$. It is clear $K_i=E_i\Omega E_i\subseteq l\Omega l=\mathscr{K}$, $A_i\subset\mathscr{A}$, hence every element of $\tilde{\Omega}$ can be induced in space A_i a K_i-linear transformation. Now we want to show that if $\tilde{\sigma}\in\tilde{\Omega}$ and $A_i\tilde{\sigma}=0$, then $\mathscr{A}\tilde{\sigma}=0$. For this purpose we prove first, if $\tilde{\sigma}\in\tilde{\Omega}$ and $A_i\tilde{\sigma}=0$ for some A_i, $i\in I$, then $A_j\tilde{\sigma}=0$ for all $A_j=E_j\Omega$, $j\in I$. In fact, if it were false, i. e. there would exist $a_j\tilde{\sigma}\neq 0$ for some element a_j, then as above it follows $\sigma\in A_j\subset\Omega$, if we set $\sigma=a_j\tilde{\sigma}$. By [2] we know that $A_j=E_j\Omega$ as vector space over $K_j=E_j\Omega E_j$ is (ψ, I)-isomorphic to $A_i=E_i\Omega$ as vector space over $K_i=E_i\Omega E_i$. We denote this (ψ, I)-isomorphism by S, then from $\sigma\in\Omega$, $A_i\tilde{\sigma}=0$ it follows that $(E_j\sigma)\ S=(E_jS)\sigma\subseteq A_i\sigma=(A_ia_j)\tilde{\sigma}\subseteq A_i\tilde{\sigma}=0$, hence $E_j\sigma=0$ and $a_j\tilde{\sigma}=0$. This implies the contradiction with $a_j\tilde{\sigma}\neq 0$. On the other hand $\mathfrak{M}=\sum_{i\in I}\oplus Fu_i\oplus N(l)$, $\Omega\subset\tilde{\Omega}$, $l\Omega\tilde{\sigma}\subseteq l\Omega$. It follows from $E_i\Omega\tilde{\sigma}=0$ that $u_i(l\Omega\tilde{\sigma})=u_i((E_il)\Omega\tilde{\sigma})=0$ for $i\in I$ and $N(l)(l\Omega\tilde{\sigma})=0$, hence $\mathscr{A}\tilde{\sigma}=l\Omega\tilde{\sigma}=0$. This proves the above assertion. Now $\tilde{\sigma}$ is a $\mathscr{K}$-endomorphism of $\mathscr{A}$, hence from $\mathscr{A}\tilde{\sigma}=0$ it follows $\tilde{\sigma}=0$. This proves that every element of $\tilde{\Omega}$ must be a zero endomorphism of $\mathscr{A}$ if its induced linear transformation in A_i is a zero one. Again, if $\tilde{\sigma}\in\tilde{\Omega}$, then $\tilde{\sigma}$ is an induced K_i-linear transformation of A_i. Hence there exists an element σ of Ω such that σ is equal to $\tilde{\sigma}$ in A_i, i.e. $A_i(\sigma-\tilde{\sigma})=0$. But $\Omega\subset\tilde{\Omega}$, hence $\mathscr{A}(\sigma-\tilde{\sigma})=0$ by the above assertion. Then it follows $\sigma=\tilde{\sigma}\in\Omega$. Therefore $\Omega=\tilde{\Omega}$.

Definition 2.4. *Let $\mathscr{A}=l\Omega$ be a left module over $\mathscr{K}=l\Omega l$, f is said to be a $\mathscr{K}$-linear function from $\mathscr{A}$ to $\mathscr{K}$ if and only if f is a $\mathscr{K}$-homomorphism from left module $\mathscr{A}$ over $\mathscr{K}$ to left module $\mathscr{K}$ over $\mathscr{K}$. Denote the set of such linear functions by $\mathscr{A}^*$, then $\mathscr{A}^*$ is said to be conjugate module of $\mathscr{A}$. Clearly $\mathscr{A}^*$ is a right module over $\mathscr{K}$.*

Theorem 2.1. *The conjugate module of $\mathscr{A}=l\Omega$ is $\mathscr{A}^*=\Omega l$.*

Proof It is clear that $\Omega l\subseteq\mathscr{A}^*$. It needs to prove $\mathscr{A}^*\subseteq\Omega l$. Let $f\in\mathscr{A}^*$, then f is also a $\mathscr{K}$-endomorphism of $\mathscr{K}$-module $\mathscr{A}$. Hence $f\in\Omega$ by Lemma 2.1. But for any element a of $\mathscr{A}$ we have $af=h=afl$ for $h\in\mathscr{K}$. Therefore $f=fl\in\Omega l$.

Definition 2.5. *Let Ω be the complete ring of linear transformations of vector space $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$ over F. An element l of Ω is called idempotent relative to basis $\{u_i\}_{i\in\Gamma}$ if and only if there exists a subset I of Γ such that $u_il=u_i$ for $i\in I$ and $u_jl=0$ for $j\in\Gamma\setminus I$.*

Theorem 2.2. *Let R be a dense ring of the complete ring of linear transformations, then R is $\aleph_\nu$-fold transitive if and only if $lR=l\Omega$ for all idempotent relative to a basis $\{u_i\}_{i\in\Gamma}$ with rank of $l<\aleph_\nu$.*

Proof The necessary condition is clear from the proof of Theorem 1.2, (i). Now we prove the sufficient condition. Let $\{x_i\}_{i\in I}$ be a set of linearly independent elements of the vector space $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$ and $\{y_i\}_{i\in I}$ an arbitrary set of $\mathfrak{M}$, $|I|<\aleph_\nu$. Then there exists a subset $\{u_i\}_{i\in I^*}$ of $\{u_i\}_{i\in\Gamma}$ such that $x_i\in\sum_{i\in I^*}\oplus Fu_i$ for $i\in I$. We have $\mathfrak{M}=\sum_{i\in I^*}\oplus Fu_i\oplus\sum_{\Gamma\setminus I^*}\oplus Fu_j$, where $|I^*|<\aleph_\nu$. Hence there exists an element $\omega\in\Omega$ such that $x_i\omega=y_i$ for $i\in I$, therefore there exists an idempotent element l^* such that $u_il^*=u_i$ for $i\in I^*$, and $u_jl^*=0$ for $j\in\Gamma\setminus I^*$. Since $l^*R=l^*\Omega$, there exists an element $r\in R$ such that $x_ir=y_i$ for $i\in I$. This means that R is $\aleph_\nu$-fold transitivity.

Corollary. *Let R be a dense subring of the complete ring Ω of linear transformations, $\mathfrak{S}_\nu=T_\nu\cap R$. Then $\mathfrak{S}_\nu$ is ν-socle if and only if $l\mathfrak{S}_\nu=l\Omega$ for all idempotent elements l relative to a basis $\{u_i\}_{i\in\Gamma}$ with rank of $l<\aleph_\nu$.*

Denote $(\mathscr{A}, \mathscr{A}')$ as a pair of dual modules over $\mathscr{K}$ and $\mathscr{L}(\mathscr{A}, \mathscr{A}')=\{\omega\in\Omega\,|\,\omega\mathscr{A}'\subseteq\mathscr{A}'\}$, $G_\nu(\mathscr{A}, \mathscr{A}')=\{l\,|\,l\in\mathscr{L}(\mathscr{A}, \mathscr{A}')$, and the rank of $l<\aleph_\nu\}=T_\nu\mathscr{L}(\mathscr{A}, \mathscr{A}')$. Then we have the following theorem:

Theorem 2.3. (*Structure theorem with non-limit ordinal number*) *Let ν be an non-limit ordinal number, then the following conditions are equivalent:*

(I) *R is a primitive ring with ν-socle $\mathfrak{S}_\nu$,*

(II) *R is a $\aleph_\nu$-fold transitive ring of linear transformations of a vector space $\mathfrak{M}$ over division ring F containing an non-zero element with rank$<\aleph_{\nu-1}$,*

(III) *there exists a pair of dual modules $(\mathscr{A}, \mathscr{A}')$ over $\mathscr{K}$ such that R is a subring of $\mathscr{L}(\mathscr{A}, \mathscr{A}')$ containing the $G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A}$.*

Proof (I)$\to$(III). By assumption $\mathfrak{S}_\nu=T_\nu\cap R$ is ν-socle. Hence $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive. Then there exists an idempotent element $l \in \mathfrak{S}_\nu$ with rank of $l = \aleph_{\nu-1}$ such that $Rl\Omega = \mathfrak{S}_\nu$ (see the proof of Theorem 1.2(III)). Write $\mathscr{A} = lR$, $\mathscr{A}' = Rl$. By Theorem 2.2 we have $\mathscr{A} = lR = l\Omega$. We want to prove that $G_\nu(\mathscr{A}, \mathscr{A}') = \mathfrak{S}_\nu$. In fact, if $\sigma \in G_\nu(\mathscr{A}, \mathscr{A}')$, then $\sigma\mathscr{A}' \subset \mathscr{A}'$, hence $\sigma\mathfrak{S}_\nu \sqsubseteq \mathfrak{S}_\nu$. From the property of ν-socle it follows that $\sigma \in \mathfrak{S}_\nu$. Hence $G_\nu(\mathscr{A}, \mathscr{A}') \sqsubseteq \mathfrak{S}_\nu$. Conversely, it is clear $\mathfrak{S}_\nu \sqsubseteq G_\nu(\mathscr{A}, \mathscr{A}')$ by the definition of $G_\nu(\mathscr{A}, \mathscr{A}')$. We want to prove that $G_\nu(\mathscr{A}, \mathscr{A}') = G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A}$. Since $\mathscr{A} = lR = l\Omega$, it then follows immediately. Finally we want to prove that $(\mathscr{A}, \mathscr{A}')$ is a pair of $\aleph_\nu$-typical dual modules. Let $\mathfrak{N}$ be the underlying vector space of the pair of modules $\mathscr{A} = l\Omega$ and $\mathscr{A}^* = \Omega l$ over $\mathscr{K} = l\Omega l$, $\mathfrak{N} = \sum_{i \in I} \oplus Fu_i$, $\mathfrak{M} = \sum_{i \in I} \oplus Fu_i \oplus N(l)$, $N(l) = \{x \in \mathfrak{M} \mid xl = 0\}$, and the rank of $l = |I| = \aleph_{\nu-1}$. Then by Definition 2.3 we need only to prove that $\mathscr{A}' = Rl$ is $\aleph_\nu$-fold transitive from $\mathfrak{M}$ to $\mathfrak{N}$. Let $\{x_i\}_{i \in I'}$ be a set of linearly independent elements of $\mathfrak{M}$ with $|I'| < \aleph_\nu$, $\{y_i\}_{i \in I'}$ be a set of arbitrary elements of $\mathfrak{N}$, then there exists an element $r \in R$ such that $x_i r = y_i$ for $i \in I'$, since R is $\aleph_\nu$-fold transitive. From $y_i l = y_i$ it follows immediately that $r = rl \in Rl = \mathscr{A}'$.

(III)→(I). Let $(\mathscr{A}, \mathscr{A}')$ be a pair of $\aleph_\nu$-typical dual modules over $\mathscr{K}$, and $\mathscr{A} = l\Omega$ $\mathscr{A}^* = \Omega l$, $\mathscr{K} = l\Omega l$. Let $\mathfrak{N}$ be the underlying subspace of the pair of $\mathscr{A}$ and $\mathscr{A}'$ over $\mathscr{K}$, $\mathfrak{N} = \sum_{i \in I} \oplus Fu_i$, $|I| =$ the rank of $l = \aleph_{\nu-1}$, $u_i l = u_i$ for $i \in I$. Since $\mathscr{A}'$ is a $\mathscr{K}$-subspace of $\mathscr{A}^*$, $\mathscr{A}'\mathscr{A}' \sqsubseteq \mathscr{A}'$. Let $\{x_i\}_{i \in I'}$ be a set of linearly independent elements of $\mathfrak{M}$, $|I'| < \aleph_\nu$ and let $\{y_i\}_{i \in I'}$ be linearly independent elements of $\mathfrak{N}$, $\{z_i\}_{i \in I'}$ any set of $\mathfrak{M}$. Then from the $\aleph_\nu$-fold transitivity of $\mathscr{A}'$ it follows that there exists an element $a' \in \mathscr{A}'$ such that $x_i a' = y_i$ for $i \in I'$ and an element $\omega \in \Omega$ such that $y_i \omega = z_i$ for $i \in I'$. Hence $x_i a' \omega = z_i$ for $i \in I'$. Since $a'\omega \in \mathscr{A}'\mathscr{A}'$, $\mathscr{A}'\mathscr{A}$ is $\aleph_\nu$-fold transitive. On the other hand, from $\mathscr{A}'\mathscr{A}' \subset \mathscr{A}'$ it follows $\mathscr{A}' \subset \mathscr{L}(\mathscr{A}, \mathscr{A}')$. Since $\mathscr{A}' \subset T_\nu$, then $\mathscr{A}' \subset G_\nu(\mathscr{A}, \mathscr{A}')$. Therefore $G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A}$ is $\aleph_\nu$-fold transitive. Since $\mathfrak{S}_\nu = T_\nu \cap R \sqsupseteq G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A}$, $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive. By Theorem 1.3 $\mathfrak{S}_\nu$ is ν-socle.

(I)→(III) is clear. Now we need only to prove (II)→(I). If R is $\aleph_\nu$-fold transitive and contains an element σ with rank $= \aleph_{\nu-1}$, then, by the foregoing proof, $R\sigma\Omega$ is $\aleph_\nu$-fold transitive and contains in R. Hence $R\sigma\Omega = \mathfrak{S}_\nu$. This completes the proof.

Corollary *Let ν be an non-limit ordinal number. Suppose that there exists a pair of $\aleph_\nu$-typical dual modules $(\mathscr{A}, \mathscr{A}')$ such that $\mathscr{L}(\mathscr{A}, \mathscr{A}') \supset R \supset G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A}$, then $G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A} = G_\nu(\mathscr{A}, \mathscr{A}') = \mathscr{A}'\mathscr{A} = \mathfrak{S}_\nu$ and $\mathscr{A}' = \mathfrak{S}_\nu\mathscr{K}$, $\mathfrak{S}_\nu = \mathscr{A}'\Omega$.*

Proof In the proof of Theorem 2.3, we see that $\mathscr{A}'\mathscr{A}$ is $\aleph_\nu$-fold transitive and $\mathfrak{S}_\nu \sqsupseteq G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A} \sqsupseteq \mathscr{A}'\mathscr{A}$. On the other hand, since $\mathscr{A}'\mathscr{A}$ is a left ideal of R, by Proposition 1.1 $\mathscr{A}'\mathscr{A} \sqsupseteq \mathfrak{S}_\nu$. This proves that $G_\nu(\mathscr{A}, \mathscr{A}')\mathscr{A} = \mathscr{A}'\mathscr{A} = \mathfrak{S}_\nu$. It is clear that $\mathfrak{S}_\nu = T_\nu \cap R \sqsubseteq T_\nu \cap \mathscr{L}(\mathscr{A}, \mathscr{A}') = G_\nu(\mathscr{A}, \mathscr{A}')$. If $\sigma \in G_\nu(\mathscr{A}, \mathscr{A}')$, then $\sigma\mathscr{A}' \sqsubseteq \mathscr{A}'$, hence $\sigma\mathfrak{S}_\nu \sqsubseteq \mathfrak{S}_\nu$. Using the property of ν-socle we have $\sigma \in \mathfrak{S}_\nu$. Hence $G_\nu(\mathscr{A}, \mathscr{A}') = \mathfrak{S}_\nu$.

Finally we see that $\mathfrak{S}_\nu\mathscr{K}=\mathscr{A}'\mathscr{A}\mathscr{K}=\mathscr{A}'\mathscr{K}=\mathscr{A}'$, $\mathscr{A}'\Omega=\mathfrak{S}_\nu\mathscr{K}\Omega=\mathscr{A}'\mathscr{A}\Omega=\mathscr{A}'\mathscr{A}=\mathfrak{S}_\nu$. This completes our proof.

Remark. If we set $\nu=0$, i. e. $\mathfrak{S}_0$ the socle of R, then it follows immediately the well-known structure theorem of primitive ring with non-zero socle.

Theorem 2.4. (*Structure theorem with limit ordinal numbers*). *Let ν be a limit ordinal number, then the following conditions are equivalent:*

(i) *R is a primitive ring with ν-socle,*

(ii) *R is $\aleph_\nu$-fold transitive and R contains an element with rank$=\aleph_\mu$ for any non-limit ordinal number $\mu<\nu$,*

(iii) *there exists a pair of $\aleph_\mu$-typical dual modules $(\mathscr{A}_\mu, \mathscr{A}'_\mu)$ such that $\mathscr{L}(\mathscr{A}_\mu, \mathscr{A}'_\mu)\supset R\supset G_\mu(\mathscr{A}_\mu, \mathscr{A}'_\mu)\mathscr{A}_\mu$ for every non-limit ordinal number $\mu<\nu$.*

Proof (i)→(iii). Suppose that $\mathfrak{S}_\nu=T_\nu\cap R$ is ν-socle, then for any $\mu<\nu$, $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive by Lemma 1.2. Hence if $\mu_1<\mu_2<\nu$, then we have $\mathfrak{S}_{\mu_1}\subsetneqq\mathfrak{S}_{\mu_2}$ and $\mathfrak{S}_\nu=\bigcup_{\mu<\nu}\mathfrak{S}_\mu$, where μ may be assumed as non-limit ordinal number. From Theorem 2.3 it follows that for every non-limit ordinal number $\mu<\nu$ there exists a pair of $\aleph_\mu$-typical dual modules $(\mathscr{A}_\mu, \mathscr{A}'_\mu)$ over $\mathscr{K}_\mu=l_\mu\Omega l_\mu$ such that $\mathscr{L}(\mathscr{A}_\mu, \mathscr{A}'_\mu)\supset R\supset G_\mu(\mathscr{A}_\mu, \mathscr{A}'_\mu)\mathscr{A}_\mu$. This completes the proof of (i)→(iii). Now we prove (iii)→(i). In fact, for every non-limit ordinal number $\mu<\nu$ there exists a pair of $\aleph_\mu$-typical dual modules $(\mathscr{A}_\mu, \mathscr{A}'_\mu)$ by the assumption such that $\mathscr{L}(\mathscr{A}_\mu, \mathscr{A}'_\mu)\supset R\supset G_\mu(\mathscr{A}_\mu, \mathscr{A}'_\mu)\mathscr{A}_\mu$. From the above corollary, it follows that $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive. Hence $\mathfrak{S}_\nu=\bigcup_{\mu<\nu}\mathfrak{S}_\mu$ is $\aleph_\nu$-fold transitive. Therefore $\mathfrak{S}_\nu$ is ν-socle by Theorem 1.3.

Finally we want to prove that (i) and (ii) are equivalent. If (i) is true, then R is $\aleph_\nu$-fold transitive. Accoding to Lemma 1.2, $\mathfrak{S}_\mu$ is $\aleph_\mu$-fold transitive for every non-limit ordinal number $\mu<\nu$. Hence there exists an element $l_\mu\in\mathfrak{S}_\mu$ with rank$=\aleph_{\mu-1}$. This implies that (ii) is true. Conversely, if (ii) is true, then $\mathfrak{S}_\mu\neq 0$ for every μ and $\mathfrak{S}_{\mu_1}\subsetneqq\mathfrak{S}_{\mu_2}\subsetneqq\mathfrak{S}_\nu$ for $\mu_1<\mu_2<\nu$. By Lemma 1.1, $\mathfrak{S}_\nu$ is $\aleph_\nu$-fold transitive. Applying Theorem 1.3 we see that $\mathfrak{S}_\nu$ is ν-socle. Hence (i) is true. This completes the proof.

Remark 1. Theorem 2.3 implies the well-known structure theorem, if we set $\nu=0$, i. e. $\aleph_\nu=\aleph_0$.

Remark 2. Denote $\mathfrak{M}=\sum_{i\in\Gamma}\oplus Fu_i$, and Ω the complete ring of F-linear transformations of $\mathfrak{M}$. Let l: $u_il=u_i$ for $i\in I$ and $u_jl=0$ for $j\in\Gamma\backslash I$. Then there exists a set $\{E_i\}_{i\in\Gamma}$ such that $u_iE_j=\delta_{ij}u_i$ for $i, j\in\Gamma$. Write $\mathscr{A}=l\Omega$, $\mathscr{A}^*=\Omega l$, and $\mathscr{A}'$ a submodule of $\mathscr{A}^*$ over $\mathscr{K}=l\Omega l$, let $A_i=E_i\Omega$, $A'_i=\mathscr{A}'E_i$, $K_i=E_i\Omega E_j$ for $i\in I$. Denote $\mathscr{L}(A_i, A'_i)=\{\omega\in\Omega\,|\,\omega A'_i\subseteq A'_i\}$. Then it is clear $\mathscr{L}(\mathscr{A}, \mathscr{A}')\subset\mathscr{L}(A'_i, A'_i)$, $\mathfrak{F}(A_i, A'_i)=\{\omega\in\mathscr{L}(A_i, A'_i)$, and the rank of $\omega<\infty\}\subset\mathfrak{S}_\nu$. In fact, A'_i is a subspace of $A^*_i=\Omega E_i$ over K_i, i. e. $A'_iK_i\subseteq A'_i$, $A'_i\subset\mathscr{A}^*E_i=\Omega E_i=A^*_i$. On the other hand, if $\sigma\mathscr{A}'\subset\mathscr{A}'$, then $\sigma\mathscr{A}'E_i\subseteq\mathscr{A}'E_i$, i. e. $\sigma A'_i\subseteq A'_i$. Hence $\mathscr{L}(\mathscr{A}, \mathscr{A}')\subset\mathscr{L}(A_i, A'_i)$. We have $\mathscr{L}(A_i, A_i') \supset \mathscr{L}(\mathscr{A}, \mathscr{A}') \supset R \supset \mathfrak{S}_\nu \supset \mathfrak{F}(A_i, A_i')$. This refines the well-known chain $\mathscr{L}(A_i, A_i') \supset R \supset \mathfrak{F}(A_i, A_i')$ of usual structure theorem.

Remark 3. Suppose that the condition (iii) of Theorem 2.4 is true, then $\bigcap_{\mu<\nu} \mathscr{L}(\mathscr{A}_\mu, \mathscr{A}'_\mu) \supset R \supset \mathfrak{S}_\nu = \bigcup_{\mu<\nu} G_\mu(\mathscr{A}_\mu, \mathscr{A}'_\mu)\mathscr{A}_\mu = \bigcup_{\mu<\nu} G_\mu(\mathscr{A}_\mu, \mathscr{A}'_\mu) = \bigcup_{\mu<\nu} \mathscr{A}'_\mu \mathscr{A}_\mu$.

Definition 2.6. *A primitive ring R with ν-socle $\mathfrak{S}_\nu$ is said to be maximal if R cannot be imbedded properly in another primitive ring R' with the same ν-socle.*

Theorem 2.5. *Let ν be a non-limit ordinal number. Then a primitive ring R with ν-socle $\mathfrak{S}_\nu$ is maximal if and only if R is isomorphic to a ring $\mathscr{L}(\mathscr{A}, \mathscr{A}')$ where $(\mathscr{A}, \mathscr{A}')$ is a pair of $\aleph_\nu$-typical dual modules.*

Proof Of course, we may assume that R is a subring of the complete ring Ω of linear transformations of a vector space. We prove first the necessity of the condition. Let l be an idempotent element of $\mathfrak{S}_\nu$ with rank $\aleph_{\nu-1}$, and denote $\mathscr{A}=lR$, $\mathscr{A}'=Rl$, $\mathscr{K}=lRl$, then we have $\mathfrak{S}_\nu \subset R \subset \mathscr{L}(\mathscr{A}, \mathscr{A}')$. Since R is maximal by assumption, hence $R=\mathscr{L}(\mathscr{A}, \mathscr{A}')$. Now we prove the sufficiency of the condition. If $R=\mathscr{L}(\mathscr{A}, \mathscr{A}') \supset \mathfrak{S}_\nu$, then by Corollary of Theorem 2.3 $\mathfrak{S}_\nu \mathscr{K} = \mathscr{A}'$. If $L \supset R = \mathscr{L}(\mathscr{A}, \mathscr{A}') \supset \mathfrak{S}_\nu$ and $\mathfrak{S}_\nu$ is ν-socle of L, then $L\mathfrak{S}_\nu \subset \mathfrak{S}_\nu$. Hence $L\mathscr{A}' \subseteq \mathscr{A}'$. This proves that $L \subseteq \mathscr{L}(\mathscr{A}, \mathscr{A}')$. Hence $\mathscr{L}(\mathscr{A}, \mathscr{A}')$ is maximal.

Remark. Theorem 2.5 generalizes the well-known theorem, if $\mathfrak{S}_0 = \mathfrak{S}_\nu$ is usual socle of primitive ring (see p. 88 Theorem 1[1]).

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