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ON THE ZERO-ONE LAWS FOR THE ERGODIC QUASI-INVARIANT MEASURES

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Abstract

Many zero-one laws for the quasi-additive functional and quasi-additive operator are proved. The results may be especially applied to the abstract Wiener measure space. As an application, we give a generalization of a remarkable result of the σ-additivity of a Gaussian cylinder probability due to L. Gross.

Using the results and the ideas in [1], we discussed the zero-one law for the ergodic quasi-invariant measures in [2], and it was shown that a sequence of quasi-linear functional either converges a. e. or diverges a. e. The results may be especially applied to the abstract Wiener measure space. As an application, we gave a generalization of Ladan-Shepp's Theorem for Gaussian measure.

In [2], we showed that the ergodis quasi-invariant measures is one which preserves some important properties of Gaussian measure. In this paper, using Xia Dao-Xing inequality, we discuss further the zero-one laws for the ergodic quasi-invariant measures and give the more general results.

Let G be a linear topological space, $\mathscr{B}$ be the Borel σ-field in G, and $\Omega=(G, \mathscr{B}, \mu)$ a regular probability measure space, which is quasi-invariant and ergodic with respect to measurable transfomation group $\mathfrak{G}$. There exists a suitable (see [1]) topology $\mathscr{T}$ on $\mathfrak{G}$ such that $(\mathfrak{G}, T)$ is a connected topological group of the second category with the first axiom of countability satisfied. In this paper, we always assume that $\Omega=(G, \mathscr{B}, \mu)$ satisfies the conditions described above, unless it is noted specially.

Let $f(g)$ be a real measurable function on $(G, \mathscr{B})$ with the following property: for each $h\in\mathfrak{G}$, there exists a real number $\tilde{f}(h)$ and a subset $E_h\in\mathscr{B}$, $\mu(E_h)=0$ such that

$$f(hg)=f(g)+\tilde{f}(h)$$

for any $g\overline{\in} E_h$. Then f is called a quasi-additive functional on $(G, \mathscr{B}, \mu)$ with respect to $\mathfrak{G}$ (see [1]), and we say that $\tilde{f}$ is induced by f. It is easy to know that $\tilde{f}(I)=0$, and $\tilde{f}(h_1^{-1})=-\tilde{f}(h_1), \tilde{f}(h_1\cdot h_2)=\tilde{f}(h_1)+\tilde{f}(h_2)$ for any $h_1, h_2\in\mathfrak{G}$. In particular, if $\mathfrak{G}$ is a maximal translation quasi-invariant quasi-continuous linear subspace of G, $\mathfrak{G}$ may be regarded as a translation quasi-invariant transfomation group. According to [1], the

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s–topology on $\mathfrak{G}$ satisfies the conditions described above. Clearly, the quasi-linear functional is a quasi-additive functional.

According to [1], we introduce the notion of a pair of quasi-convex functions. Let $\Omega=(G, \mathscr{B}, \mu)$ be a measure space which is quasi-invariant with respect to $\mathfrak{G}$, $p(g)$ be a measurable function on G and $\tilde{p}(h)$ be a function on group $\mathfrak{G}$, with the following properecties:

(i) $0\leqslant p(g)\leqslant\infty$,

(ii) for each $h\in\mathfrak{G}$, there exists a μ–null set E_h such that

$$\tilde{p}(h)\leqslant p(g)+p(hg)$$

for any $g\bar{\in}E_h$,

(iii) $\tilde{p}(h)$ is a convex function on the group $\mathfrak{G}$, i. e. $\tilde{p}(h)\geqslant 0$, $\tilde{p}(I)=0$, and

$$\tilde{p}(h_1h_2^{-1})\leqslant\tilde{p}(h_1)+\tilde{p}(h_2)$$

for any $h_1, h_2\in\mathfrak{G}$.

Then $p(g)$ and $\tilde{p}(h)$ is called a pair of quasi-convex functions on Ω with respect to $\mathfrak{G}$.

Here we quote Xia Dao-xing inequality (see [1]) as our lemma.

Lemma 1. *Let $\Omega=(G, \mathscr{B}, \mu)$ be a regular probability measure space, which is quasi-invariant with respect to measurable transfomation group $\mathfrak{G}$; $\mathscr{T}$ a suitable topology on $\mathfrak{G}$ such that $(\mathfrak{G}, \mathscr{T})$ is a topological group of the second category with the first axiom of countability satisfied, then for every set $A\in\mathscr{B}$ with $0<\mu(A)<\infty$ there is a neighborhood V_A of unit element I in $(\mathfrak{G}, \mathscr{T})$ and a positive constant C such that*

$$\sup_{h\in V_A}\tilde{\rho}(h)\leqslant C\int_A \rho(g)\,d\mu(g)$$

for every pair of quasi-convex functions on Ω w. r. t. $\mathfrak{G}$.

By Lemma 1, we may give the following theorem, the proof of which is similar to the Theorem 3 in [2].

Theorem 1. *Let $\Omega=(G, \mathscr{B}, \mu)$ be a regular probability measure space, ergodic and quasi-invariant with respect to the measurable transfomation group $\mathfrak{G}$ on Ω; and $\mathscr{T}$ a suitable topology* (see [1]) *on $\mathfrak{G}$ such that $(\mathfrak{G}, \mathscr{T})$ is a connected topological group of the second category with the first axiom of countability satisfied. If $f_i(g)\, i=1, 2, \cdots$ is a sequence of quasi-additive functional on Ω* w. r. t. $\mathfrak{G}$, *then we have the following zero-one law*

$$\mu\left\{g\,\middle|\,\sum_{i=1}^{\infty}f_i(g)\ \text{converges}\right\}=0 \text{ or } 1.$$

In the following theorem we give a zero-one law for the convergence in measure of quasi-additive functionals. It may be regarded as a generalization of the Theorem 3.2.10 in [1]. Let $A\subset G$, $A\in\mathscr{B}$, we restrict Ω to A, and induce a measure space $\Omega_A=(A, \mathscr{B}\cap A, \mu)$. A sequence $\{f_i(g)\}$ of the measurable functions on Ω is convergent in measure on A, if we restrict the sequence to A and it is convergent in measure on Ω_A.

Theorem 2. *Under the assumptions of Theorem* 1 *and assumption that* $\{f_i(g)\} i=1, 2, \cdots$ *is a sequence of quasi-additive functional,* $\sum_{i=1}^{\infty} f_i(g)$ *either converges in measure on* Ω *to a quasi-additive functional or doesn't converge in measure on any set* $A \in \mathscr{B}$ *with* $\mu(A)>0$.

Proof If $\sum_{i=1}^{n} f_i(g)$ converges in measure on some set $A \in \mathscr{B}$ with $\mu(A)>0$, there exists a subsequence $\{k_n\}$ such that $\sum_{i=1}^{k_n} f_i(g)$ converges almost everywhere on A. By Theorem 1, $\sum_{i=1}^{k_n} f_i(g)$ converges a. e. to some measurable function $f(g)$ on Ω, i. e. there exists a μ-null set E such that for any $g \in G \backslash E$ we have $f(g)=\lim_{n\to\infty}\sum_{i=1}^{k_n} f_i(g)$.

It is easy to know that the a. e. limit $f(g)$ of a sequence of quasi-additive functionals is a quasi-additive functional. In fact, for each $h \in \mathfrak{G}$ there exists a set $E_h^n \in \mathscr{B}$, $\mu(E_h^n)=0$ such that

$$f_{k_n}(hg)=f_{k_n}(g)+\tilde{f}_{k_n}(h)$$

for any $g \overline{\in} E_h^n$. Set $E_h=\bigcup_{n=1}^{\infty}(E_h^n \cup h^{-1}E_h^n) \cup E \cup h^{-1}E$, where E is mentioned above, then $\mu(E_h)=0$. Passing to the limit with n, we see that for $g \overline{\in} E_h$ $\lim_{n\to\infty}\tilde{f}_{k_n}(h)$ exists; denoting the corresponding limit by $\tilde{f}(h)$, we have

$$f(hg)=f(g)+\tilde{f}(h), \quad g \overline{\in} E_h.$$

Thus, $f(g)$ is a quasi-additive functional on Ω.

It follows that for any subsequence of $S_n(g)=\sum_{i=1}^{n} f_i(g)$ we may find again a subsequence converging a. e. to some quasi-additive functional $f_0(g)$. In order to show that $\sum_{i=1}^{n} f_i(g)$ converges in measure to quasi-additive functional $f(g)$ on whole G, it is sufficient to prove that for every subsequence of $\{S_n(g)\}$, we have $f_0(g)=f(g)$ almost everywhere on Ω. In fact, by the assumption, $\sum_{i=1}^{n} f_i(g)$ converges in measure on A with $\mu(A)>0$, hence, we have $f_0(g)=f(g)$ a. e. on A. Let $F(g)=f_0(g)-f(g)$, then $F(g)$ is also a quasi-additive functional and for $g \in A$, $F(g)=0$ a. e. By Lemma 1 we see that there exists a neighborhood V_A of I in $(\mathfrak{G}, \mathscr{T})$ and a positive constant C such that

$$\sup_{h \in V_A} |\tilde{F}(h)| \leqslant C\int_A |F(g)|\, d\mu(g)=0,$$

hence, for $h \in V_A$, $\tilde{F}(h)=0$, it follows that $\tilde{F}(h)=0$ on the whole $\mathfrak{G}$. Let $E=\{g \mid F(g)=0\}$, then $E \supset A$. Since μ is ergodic with respect to $\mathfrak{G}$, in order to prove $\mu(E)=1$, it will suffice to show that E is a quasi-invariant set under $\mathfrak{G}$ in $\mathscr{B}$. In fact, for each $h\in \mathfrak{G}$ there exists a μ-null set F. If $g\in E\backslash F$, we have

$$F(hg)=F(g)+\tilde{F}(h)=0.$$

Thus, $hg\in E$ for every $g\in E\backslash F$, i. e. $E\backslash h^{-1}E\subset F$ and then $\mu(E\backslash h^{-1}E)=0$. In the same way, we can get $\mu(E\backslash hE)=0$, and by the assumption that μ is quasi-invariant *w. r. t.* $\mathfrak{G}$, $\mu(h^{-1}E\backslash E)=0$, i. e. $\mu(E\Delta h^{-1}E)=0$, and E is a quasi-invariant set. Thus, the theorem is completely proved.

Corollary 1. *If $\{f_i(g),\ i=1,\ 2,\ \cdots\}$ is a sequence of quasi-additive functionals, then $\{f_i(g)\}$ either converges in measure on Ω to a quasi-additive functional or doesn't converges in measure on any set $E\in\mathscr{B}$ with positive measure.*

Let R be the real number field, $\mathscr{B}$ the Borel σ-field in R. Setting $(R_i,\ \mathscr{B}_i)=(R,\ \mathscr{B})$ $i=1,\ 2,\ \cdots$, we denote the product measure space by $(R^\infty,\ \mathscr{B}^\infty)$, where $R^\infty=\prod\limits_{i=1}^{\infty}R_i$, $\mathscr{B}^\infty=\bigotimes\limits_{i=1}^{\infty}\mathscr{B}_i$. Let $q(x_1,\ x_2,\ \cdots)$ be a mapping of R^∞ to $\bar{R}_+$. If $q(x)=q(-x)$ for every $x=(x_i)\in R^\infty$, then we say that q is an even. $q(x)$ is called symmetric if

$$q(\pm x_1,\ \pm x_2,\ \cdots,\ \pm x_n,\ \cdots)=q(x_1,\ x_2,\ \cdots,\ x_n,\ \cdots)$$

for all choices of signs $\pm$ and all $x\in R^\infty$. $q(x)$ is subadditive if $q(x+y)\leqslant q(x)+q(y)$ for any $x,\ y\in R^\infty$.

Theorem 3. *Let $\Omega=(G,\ \mathscr{B},\ \mu)$ satisfy the assumptions of Theorem 1, $q(x)$ be an even subadditive measurable mapping of $(R^\infty,\ \mathscr{B}^\infty)$ to $\bar{R}_+$. If $\{f_i(g)\}$ is an arbitrary sequence of quasi-additive functional, then*

$$\mu\{g\,|\,\sup_n q(f_1(g),\ f_2(g),\ \cdots,\ f_n(g),\ 0,\ 0,\ \cdots)<\infty\}=0 \text{ or } 1.$$

Proof It is convenient to set $\Pi_n(x)=(x_1,\ \cdots,\ x_n,\ 0\cdots)$ and

$$q_n(x)=q(\Pi_n x)=q(x_1,\ \cdots,\ x_n,\ 0,\ 0\cdots).$$

$f(g)=(f_n(g))$ may be regarded as R^∞-valued random variable. Let

$$A=\{g\,|\,\sup_n q_n(f(g))<\infty\}.$$

If $\mu(A)=0$, then the theorem holds. If $\mu(A)>0$, then there exists a measurable set $A'\subset A$ such that $\mu(A')>0$ and $\sup\limits_n q_n(f(g))<M$ for $g\in A'$, where M is some constant.

Now, in order to apply the Lemma 1, first of all, we point out that $q_n(f(g))$ and $q_n(\tilde{f}(h))$ is a pair of quasi-convex functions on Ω *w. r. t.* $\mathfrak{G}$. In fact, for every $h\in\mathfrak{G}$ we have

$$q_n(\tilde{f}(h))\leqslant q_n(\tilde{f}(h)+f(g))+q_n(f(g))=q_n(f(hg))+q_n(f(g))$$

for almost all $g\in G$. By Lemma 1, there exists a positive constant C and a neighborhood $V_{A'}$ of I in $(\mathfrak{G},\ \mathscr{T})$ such that

$$\sup_{h\in V_{A'}} q_n(\tilde{f}(h))\leqslant C\int_{A'} q_n(f(g))d\mu(g)\leqslant CM\mu(A').$$

Hence, for any $h\in V_{A'}$, $\sup\limits_n q_n(\tilde{f}(h))\leqslant CM\cdot\mu(A)<\infty$. Moreover, because $(\mathfrak{G},\ \mathscr{T})$ is connected and $q_n(\tilde{f}(h))$ is convex, we get $\sup\limits_n q_n(\tilde{f}(h))<\infty$ for all $h\in\mathfrak{G}$.

Simultaneously, we have

$$\sup_n q_n(f(hg)) \leqslant \sup_n q_n(f(g)) + \sup_n q_n(\tilde{f}(h))$$

for almost all $g \in G$. Similar to the proof of Theorem 2, we can show that A is a quasi-invariant set. Since μ is ergodic w. r. t. $\mathfrak{G}$ and $\mu(A)>0$, we have $\mu(A)=0$ and the Theorem is thus proved.

Corollary 1. *Let $\{f_i(g)\}$ $i=1, 2, \cdots$ be a sequence of quasi-additive functionals,* $0<p<\infty$, then

$$\mu\left\{g \middle| \sum_1^\infty |f_i(g)|^p < \infty\right\} = 0 \text{ or } 1.$$

Corollary 2. *Under the assumptions of Corollary 1, we have*

$$\mu\left\{g \middle| \sup_n \left|\sum_1^n f_i(g)\right| < \infty\right\} = 0 \text{ or } 1.$$

Corollary 3. *Let $\|\cdot\|$ be a measurable norm on $(R^\infty, \mathscr{B}^\infty)$. If $\{f_i(g)\}$ $i=1, 2, \cdots$, is an arbitrary sequence of quasi-additive functionals, then*

$$\mu\{g \mid \sup_n \|\Pi_n f(g)\| < \infty\} = 0 \text{ or } 1$$

where $f(g) = (f_1(g), f_2(g), \cdots, f_n(g), \cdots)$.

Similarly, we may introduce the quasi-additive operator. Let $(E, \|\cdot\|)$ be a normed linear space, $(E, \mathscr{B})$ be measurable linear space, and $F(g)$ be E-valued measurable function. If for each $h \in \mathfrak{G}$, there exists an element $\tilde{F}(h) \in E$ and μ-null set E_h such that

$$F(hg) = F(g) + \tilde{F}(h)$$

for every $g \overline{\in} E_h$, then we say that $F(g)$ is an E-valued quasi-additive operator on Ω w. r. t. $\mathfrak{G}$. The proof of the following theorem is similar to that of Theorem 1, so we omit its proof.

Theorem 4. *Let $\Omega = (G, \mathscr{B}, \mu)$ satisfy the assumptions of Theorem 1, $(E, \|\cdot\|)$ be a normed linear space, $(E, \mathscr{B})$ be a measurable linear space. Let $\|\cdot\|_1$ be a measurable norm on $(E, \mathscr{B})$, and B be the completion of E in norm $\|\cdot\|_1$. If $\{F_i(g)\}$ is a sequence of E-valued quasi-additive operators, then*

$$\mu\left\{g \middle| \sum_{i=1}^n F_i(g) \text{ converges in } B\right\} = 0 \text{ or } 1.$$

If, in addition, $q(x_1, x_2, \cdots)$ is an even subadditive measurable functional from $(E^\infty, \mathscr{B}^\infty)$ to $\bar{R}_+$, then

$$\mu\{g \mid \sup_n q(F_1(g), \cdots, F_n(g), 0, 0, \cdots) < \infty\} = 0 \text{ or } 1.$$

Corollary 1. *Let $\{f_i(g)\}$ be a sequence of quasi-linear functionals. If $\|\cdot\|_1$ is a measurable semi-norm, then for any sequence $\{e_n\}$ in E*

$$\mu\left\{g \middle| \sup_n \left\|\sum_1^n f_i(g) e_i\right\|_1 < \infty\right\} = 0 \text{ or } 1.$$

Corollary 2. *Let $\{f_i(g)\}$ be a sequence of quasi-linear functionals, $\|\cdot\|_1$ be a* *continuous semi-norm on* $(E, \|\cdot\|)$, *B be the completion of* $(E, \|\cdot\|_1)$, *then*

$$\mu\left\{g\left|\sum_{i=1}^{n} f_i(g)e_i \text{ converges in } B\right.\right\}=0 \text{ or } 1.$$

Especially, let $\Omega=(G, \mathscr{B}, \mu)$ *be an abstract Wiener measure* w. r. t. *Hilbert space H. for each* $x\in H$, *there exists a quasi-linear functional* $x(g)$ *on* Ω w. r. t. *H such that* $x(g)$ *induces linear functional* $\tilde{x}(y)$ *on H, and* $\tilde{x}(y)=(y, x)$ *for* $y\in H$. *Let* $\|\cdot\|_1$ *be a continuous semi-norm on Hilbert space H, and B be the completion of H in* $\|\cdot\|_1$, *then for any sequence* $\{e_n\}$ *in H*

$$\mu\left\{g\left|\sum_{i=1}^{n} e_i(g)e_i \text{ converges in } B\right.\right\}=0 \text{ or } 1.$$

Thus, we see that it is natural to introduce the notion of the measurable semi-norm in the sense of Gruss in oder to discuss the σ-additivity of Gaussian cylinder probability.

Finally, let us discuss the zero-one law for the stochastic boundedness of the sequence of the random variables (measurable function). Let $\Omega=(G, \mathscr{B}, \mu)$ be a probability measure space and S denote the space of all real valued measurable functions on Ω. The S equipped with the topology of convergence in measure is a topological vector space. Then each element in S may be regarded as a random variable. If a subset M of the radom variables is a bounded set in S, then A is called stochastic bounded. It is easy to verify that a set $M\subset S$ is stochastic bounded iff for arbitrary $\varepsilon>0$ there exists a positive constant t such that

$$\mu\{g\,|\,|f(g)|>t\}<\varepsilon$$

for every $f\in M$.

Let $M\subset S$ and $A\in\mathscr{B}$ with $\mu(A)>0$. If we restrict M to A and it is bounded in measure on Ω_A, then, for convenience, we say that M is stochastic bounded on A. It is clear that the stochastic boundedness is weaker than the boundedness almost everywhere.

Theorem 5. *Let* $\Omega=(G, \mathscr{B}, \mu)$ *satisfy the assumptions of Theorem* 1, $\{F_i(g)\}$ *be a sequence of E-valued quasi-additive operators,* $q(x)$ *be symmetric convex measurable mapping from* $(E^\infty, \mathscr{B}^\infty)$ *to* $\bar{R}_+$. *Let* $S_n(g)=q(\Pi_n F)=q(F_1(g), F_2(g), \cdots, F_n(g), 0, 0, \cdots)$. *If* $\{S_n(g)\}$ *is stochastic bounded on any A with* $\mu(A)>0$, *then* $\{S_n(g)\}$ *must be stochastic bounded on G, and we have*

$$\mu\{g\,|\sup_n S_n(g)<\infty\}=1.$$

Proof Let $t>0$ and define the stopping time w. r. t. $\{S_n(g)\}$, by

$$T(g)=\inf\{j\,|\,S_j(g)>t\}.$$

Then

$$\{g\,|\,T(g)\leqslant n\}=\{g\,|\max_{1\leqslant i\leqslant n} q(F_1(g), \cdots, F_i(g), 0, 0, \cdots)>t\}.$$

For $1\leqslant j\leqslant n$, we put

$$Z_{nj}=(F_1,\ \cdots,\ F_j,\ -F_{j+1},\ \cdots,\ -F_n,\ 0,\ 0\cdots),$$

and we have

$$Z_{jj}=\frac{1}{2}Z_{nj}+\frac{1}{2}Z_{nn}.$$

Since $q(x)$ is a convex function on E^∞, we have

$$S_j=q(Z_{jj})\leqslant\frac{1}{2}q(Z_{nj})+\frac{1}{2}q(Z_{nn})=\frac{1}{2}q(Z_{nj})+\frac{1}{2}S_n.$$

If $T(g)=j$, then, by the definition of T, $S_j>t$,

$$\text{and } \{g\in A\,|\,T(g)=j\}\subset\{g\in A\,|\,T(g)=j,\ S_n>t\}$$
$$\cup\{g\in A\,|\,T(g)=j,\ q(Z_{nj})>t\}.$$

Hence, $\mu_A(T=j)\leqslant\mu_A(T=j,\ S_n>t)+\mu_A(T=j,\ q(Z_{nj})>t)$. Since $q(x)$ is symmetric, then $q(Z_{nn})=q(Z_{nj})$, and

$$\mu_A(T=j)\leqslant 2\mu_A(T=j,\ S_n>t).$$

Summing over $j=1,\ \cdots,\ n$, we have

$$\mu_A\{g\,|\max_{1\leqslant i\leqslant n} q(F_1(g),\ \cdots,\ F_i(g),\ 0,\ 0,\ \cdots)>t\}$$
$$=\mu_A\{g\,|\,T\leqslant n\}\leqslant 2\mu_A(S_n>t),$$

by the assumption, $\{S_n\}$ is stochastic bounded on A, for $\varepsilon>0$ there exists some positive constant t such that $\mu_A(S_n>t)<\varepsilon$ for $n=1,\ 2,\ \cdots$. Hence,

$$\mu_A\{g\,|\sup_i q(F_1,\ \cdots,\ F_i,\ 0,\ 0,\ \cdots)<\infty\}=\mu(A)>0,$$

by Theorem 4, $\mu\{g\,|\sup_i q(F_1,\cdots,\ F_i,\ 0,0\cdots)<\infty\}=1$, and $\{S_n\}$ is stochastic bounded on the whole G, and the Theorem is proved.

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