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FIXED POINT THEOREMS OF GENERALIZED CONTRACTIVE TYPE MAPPINGS (II)

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Abstract

In this paper the author obtains several new fixed point theorems for generalized contractive type mappings by means of Kwapisz's contractive gauge function and then proves that lots of contractive type mappings are topologically equivalent to Banach contraction with given contractive constant $C\in[0, 1)$.

§ 1. Introduction

In the paper[1], we have discussed the existence of the fixed point for several classes of generalized contractive type mappings.

After [1], in papers [2—5] we prove again a lot of new fixed point theorems with the help of contractive gauge function which is, by definition, a real valued function Φ: $[0, \infty)\to[0, \infty)$ satisfying the following conditions:

(A_1) Φ is nondecreasing.

(A_2) $\lim\limits_{n\to\infty}\Phi^n(t)=0$, $\forall t>0$, where Φ^n denotes n-th iteration of Φ.

(A_3) $\lim\limits_{t\to\infty}(t-\Phi(t))=\infty$.

Our results unify and extend some recent results obtained by pal; Maiti[6], Fisher[7], Rhoades[8], Ciric[9], Das; Naik[10] and others.

Recently Kwapisz[11] also proves some fixed point theorems with the help of other contractive gauge function Φ: $[0, \infty)\to[0, \infty)$ satisfying the following conditions:

(B_1) Φ is nondecreasing and continuous from the right.

(B_2) For any $q\in[0, \infty)$ there exists a maximal solution $\mu(q)$ of the equation $t=\Phi(t)+q$, $t\geqslant 0$, which satisfies $\mu(0)=0$.

The purpose of this paper is to obtain some new results for generalized contractive type mappings by means of Kwapisz's contractive gauge function.

§ 2. Orbitally Contractive mappings and Quasi-Contractive mappings

Let (X, d) be a nonempty metric space and f be a self mapping on X. For each

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$x\in X$, let $O_f(x, 0, \infty)=\{x_0=x, x_1=f(x_0), \cdots, x_{n+1}=f(x_n), \cdots\}=\{x_n\}_{n\in\omega}$ denote the orbit of f at x, where ω is the set of all nonnegative integers. For $i, j\in\omega$, $j>i$, write

$$O_f(x, i, j)=\{x_i, x_{i+1}=fx_i, \cdots, x_j\}.$$

$\delta(A)=\sup\{d(x, y): x, y\in A\}$ is the diameter of $A\subset X$.

Lemma 1.[11] *Let Φ: $[0, \infty)\to[0, \infty)$ satisfy the conditions (B_1) and (B_2). If $\mu(q)$ is the maximal solution of the equation $t=\Phi(t)+q$ for any given $q\in[0, \infty)$, then the following inequality*

$$p\leqslant\Phi(p)+q, \quad p\in[0, \infty)$$

implies $p\leqslant\mu(q)$.

Theorem 1. *Let f be a continuous self mapping on a complete metric space (X, d). Suppose that Φ: $[0, \infty)\to[0, \infty)$ satisfies the conditions (B_1) and (B_2). If there exists $p\in\omega$, $p\geqslant1$, such that*

$$d(f^px, f^{p+k}x)\leqslant\Phi(\delta(O_f(x, O, p+k))) \tag{1}$$

for all $x\in X$ and all $k\in\omega$, then the orbit $\{x_n\}_{n\in\omega}$ of f at x for each $x\in X$ converges to a fixed point x^* of f.

Proof Let x be an arbitrary point in X. $\{x_n\}_{n\in\omega}=O_f(x, 0, \infty)$. For all $i, j\in\omega$ $j>i\geqslant p$, $i=np+l$, $0\leqslant l<p$, by (1) we have

$$\begin{aligned} d(x_i, x_j)&=d(f^px_{(n-1)p+l}, f^{j-(n-1)p-l}x_{(n-1)p+l})\\ &\leqslant\Phi(\delta(O_f(x_{(n-1)p+l}, 0, j-(n-1)p-l)))\\ &=\Phi(\delta(O_f(x, (n-1)p+l, j))). \end{aligned} \tag{2}$$

Since Φ is nondecreasing, By (2) we have

$$\delta(O_f(x, np+l, \infty))\leqslant\Phi(\delta(O_f(x, (n-1)p+l, \infty))). \tag{3}$$

But

$$\begin{aligned} \delta(O_f(x, (n-1)p+l, \infty))\leqslant&\,\delta(O_f(x, (n-1)p+l, np+l))\\ &+\delta(O_f(x, np+l, \infty)). \end{aligned} \tag{4}$$

Then it follows from (3) and (4) that

$$\begin{aligned} \delta(O_f(x, (n-1)p+l, \infty))\leqslant&\,\delta(O_f(x, (n-1)p+l, np+l))\\ &+\Phi(\delta(O_f(x, (n-1)p+l, \infty))). \end{aligned} \tag{5}$$

Now letting $n=1$ and $l=0$ in (5) we obtain

$$\delta(O_f(x, 0, \infty))\leqslant\delta(O_f(x, 0, p))+\Phi(\delta(O_f(x, 0, \infty))). \tag{6}$$

Let $M=\delta(O_f(x, 0, p))$ and suppose that $\mu(M)$ is the maximal solution of equation $t=\Phi(t)+M$, then it follows from Lemma 1 and (6) that

$$\delta(O_f(x, 0, \infty))\leqslant\mu(M)\overset{\text{def.}}{=\!=}u_0<\infty. \tag{7}$$

Letting $n=1$ in (3) and noting (7) we have

$$\begin{aligned} \delta(O_f(x, p+l, \infty))&\leqslant\Phi(\delta(O_f(x, l, \infty)))\\ &\leqslant\Phi(\delta(O_f(x, 0, \infty)))\leqslant\Phi(u_0)\overset{\text{def.}}{=\!=}u_1. \end{aligned}$$

By induction it is easy to obtain

$$\delta(O_f(x,\ np+l,\ \infty))\leqslant\Phi(u_{n-1})\overset{\text{def}}{=\!=}u_n,\ n=1,\ 2,\ \cdots. \tag{8}$$

Then

$$u_1=\Phi(u_0)\leqslant\Phi(\mu(M))+M=\mu(M)=u_0,$$
$$u_2=\Phi(u_1)\leqslant\Phi(u_0)=u_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_n=\Phi(u_{n-1})\leqslant\Phi(u_{n-2})=u_{n-1},\ n=1,\ 2,\ \cdots.$$

Let $t_0=\lim\limits_{n\to\infty}u_n$. By the continuity of Φ from the right we obtain

$$t_0=\lim_{n\to\infty}u_m=\lim_{n\to\infty}\Phi(u_{m-1})=\Phi(\lim_{n\to\infty}u_{m-1})=\Phi(t_0)\leqslant\lim_{n\to\infty}u_{m-1}=t_0,$$

and hence $t_0=\Phi(t_0)$. It follows from Lemma 1 and (B_2) that $t_0=0$. Then from (8) we obtain

$$\lim_{n\to\infty}\delta(O_f(x,\ np+l,\ \infty))\leqslant\lim_{n\to\infty}u_n=0,\quad \forall 0\leqslant l\leqslant p, \tag{9}$$

which implies that $\{x_n\}_{n\in\omega}$ is a Cauchy sequence in complete metric space $(X,\ d)$ and hence $x_n\to x^*\in X$. By the continuity of f it follows immediately that x^* is a fixed point of f.

Remark 1. Let $p=1$ and $\Phi(t)=\alpha t,\ \alpha\in[0,\ 1)$, in Theorem 1. Obviously Φ: $[0,\ \infty)\to[0,\ \infty)$ satisfies the conditions (B_1) and (B_2). Then we obtain, as a special case, the main result of pal, Maiti[6].

Theorem 2. *Let f be a continuous self mapping on a complete metric space $(X,\ d)$. Suppose that Φ: $[0,\ \infty)\to[0,\ \infty)$ satisfies the conditions (B_1) and (B_2). If there exists $p,\ q\in\omega,\ p,\ q\geqslant1$ such that*

$$d(f^px,\ f^qy)\leqslant\Phi(\delta(O_f(x,\ o,\ p)\cup O_f(y,\ o,\ q))), \tag{10}$$

for all $x,\ y\in X$, then f has a unique fixed point x^* and for each $x\in X$ the orbit $\{x_n\}_{n\in\omega}$ of f at x converges to x^*.

Proof Without loss of generality we may assume $p\geqslant q$ in (10). For any $x\in X$, Let $\{x_n\}_{n\in\omega}=O_f(x,\ 0,\ \infty)$. Putting $y=f^{p-q+k}x$ for all $k\in\omega$ in (10), we have

$$\begin{aligned}d(f^px,\ f^{p+k}x)&\leqslant\Phi(\delta(O_f(x,\ 0,\ p)\cup O_f(x_{p-q+k},\ 0,\ q)))\\&\leqslant\Phi(\delta(O_f(x,\ 0,\ p+k)))\end{aligned} \tag{11}$$

for all $x\in X$ and all $k\in\omega$. Theorem 1 yields the conclusion that $\{x_n\}_{n\in\omega}$ converges to a fixed point x^* of f for each $x\in X$.

Now assume that y^* also is a fixed point of f. Then

$$d(x^*,\ y^*)=d(f^px^*,\ f^qx^*)\leqslant\Phi(d(x^*,\ y^*)).$$

It follows from Lemma 1 and (B_2) that $d(x^*,\ y^*)=0$, that is $x^*=y^*$ and hence x^* is the unique fixed point of f.

Remark 2. Letting $\Phi(t)=\alpha t,\ \alpha\in[0,\ 1)$, in Theorem 2, we find that the Theorem 2 of Fisher[7] is the special case of our Theorem 2.

Either let p(or q)$=1$ or strengthen the contractive type condition (9) in Theorem 2. Then the continuity of f isn't necessary. Using the same argument as in the proof of Theorem 3 and 4 in [3]. we may establish the following Theorems 3 and 4.

Theorem 3. Let f be a self mapping on a complete metric space (X, d). Suppose that Φ: $[0, \infty)\to[0, \infty)$ satisfies (B_1) and (B_2). If there exists $p, q\in\omega$, $p, q\geqslant 1$, p(or q) $=1$, such that

$$d(f^p x, f^q y)\leqslant\Phi(\delta(O_f(x, 0, p)\cup O_f(y, 0, p)))$$

for all $x, y\in X$, then f has a unique fixed point x^* and $\{x_n\}_{n\in\omega}$ converges to x^* for each $x\in X$.

Remark 3. The Theorem 3 of Fisher[7] is a special case of our Theorem 3.

Theorem 4. *Let f be a self mapping on a complete metric space* (X, d), *suppose that* Φ: $[0, \infty)\to[0, \infty)$ *satisfies* (B_1) *and* (B_2). *If there exists* $p, q\in\omega$, $p, q\geqslant 1$ *such that*

$$d(f^p x, f^q y)\leqslant\Phi(\max\{d(x, y), d(x, f^p x), d(y, f^q y), d(x, f^q y), d(y, f^p x)\})$$

for all $x, y\in X$, *then f has a unique fixed point* x^* *and for each* $x\in X$ $\{x_n\}_{n\in\omega}$ *converges to* x^*.

Remark 4. Theorem 4 unifies and extends the main results of Rhoades[8] and ciric[9].

§ 3. Some common fixed point theorems of generalized contractive mapping

In this section we generalize the recent results of Das, Naik[10]. Let N denotes the set of all positive integers.

Theorem 5. *Let f be a continuous self mapping on a complete metric space* (X, d) *and* $\{g_n\}_{n\in N}$ *be a sequence of self mappings on X such that* $g_n f=f g_n$ *and* $g_n^{m_n}(X)\subset f(X)$, $\forall n\in N$, *where* $\{m_n\}_{n\in N}$ *is a sequence of positive integers. If there exists* Φ: $[0, \infty)\to[0, \infty)$ *satisfying* (B_1) and (B_2) *such that*

$$d(g_i^{m_i}x, g_j^{m_j}y)\leqslant\Phi(\max\{d(fx, fy), d(fx, g_i^{m_i}x), d(fy, g_j^{m_j}y), d(fx, g_j^{m_j}y), d(fy, g_i^{m_i}x)\}) \quad (12)$$

for all $x, y\in X$ *and for all* $i, j\in N$, $i\neq j$, *then f and* $\{g_n\}_{n\in N}$ *have a unique common fixed point* $fy^*(=g_n^{m_n}y^*, \forall n\in N)$ *and for each* $x_0\in X$ $g_{n+1}^{m_{n+1}}fx_n\to fy^*$.

Proof Take any $x_0\in X$. Since $g_n^{m_n}(X)\subset f(X)$, $\forall n\in N$, we may define a sequence $\{x_n\}_{n\in N}$ in X such that $g_{n+1}^{m_{n+1}}x_n=fx_{n+1}=y_n$, $\forall n\in\omega$.

Now we prove that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence. For any $i, j\in N$, $j>i$, by (12) we have

$$d(y_i, y_j)=d(g_{i+1}^{m_{i+1}}x_i, g_{j+1}^{m_{j+1}}x_j)\leqslant\Phi(\max\{d(fx_i, fx_j), d(fx_i, g_{i+1}^{m_{i+1}}x_i), d(fx_j, g_{j+1}^{m_{j+1}}x_j), d(fx_i, g_{j+1}^{m_{j+1}}x_j), d(fx_j, g_{i+1}^{m_{i+1}}x_i)\})\leqslant\Phi(\max\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), d(y_{i-1}, y_j), d(y_{j-1}, y_i)\}). \quad (13)$$

It follows from (13) that

$$\delta(O(y_i,\ j))\leqslant\Phi(\delta(O(y_{i-1},\ j))),\ \forall i,\ j\in N,\ j\geqslant i, \tag{14}$$

where $O(y_i,\ j)=\{y_i,\ y_{i+1},\ \cdots,\ y_j\}$, then we have

$$\delta(O(y_i,\ \infty))\leqslant\Phi(\delta(O(y_{i-1},\ \infty))). \tag{15}$$

From (15) we obtain

$$\begin{aligned}\delta(O(y_{i-1},\ \infty))&\leqslant d(y_{i-1},\ y_i)+\delta(O(y_i,\ \infty))\\ &\leqslant d(y_{i-1},\ y_i)+\Phi(\delta(O(y_{i-1},\ \infty))).\end{aligned} \tag{16}$$

putting $i=1$ in (16), we have

$$\delta(O(y_0,\ \infty))=d(y_0,\ y_1)+\Phi(\delta(O(y_0,\ \infty))). \tag{17}$$

Suppose that $\mu(d(y_0,\ y_1))$ is the maximal solution of the equation $t=\Phi(t)+d(y_0,\ y_1)$ in $[0,\ \infty)$, then it follows from Lemma land (17) that

$$\delta(O(y_0,\ \infty))\leqslant\mu(d(y_0,\ y_1))\overset{\text{def.}}{=\!=}u_0. \tag{18}$$

Then putting $i=1$ in (15), we obtain

$$\delta(O(y_1,\ \infty))\leqslant\Phi(\delta(O(y_0,\ \infty)))\leqslant\Phi(u_0)\overset{\text{def.}}{=\!=}u_1.$$

By induction we have

$$\delta(O(y_i,\ \infty))\leqslant\Phi(u_{i-1})\overset{\text{def.}}{=\!=}u_i,\ i\in N. \tag{19}$$

Then

$$\begin{aligned}&u_1=\Phi(u_0)\leqslant\Phi(u_0)+d(y_0,\ y_1)=u_0,\\ &u_2\leqslant\Phi(u_1)\leqslant\Phi(u_0)=u_1,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &u_i\leqslant\Phi(u_{i-1})\leqslant\Phi(u_{i-2})=u_{i-1},\ i\in N.\end{aligned}$$

By the continuity of Φ from the right, we obtain

$$\lim_{i\to\infty}u_i\leqslant\Phi(\lim_{i\to\infty}u_{i-1})\leqslant\lim_{i\to\infty}u_{i-1}. \tag{20}$$

Letting $t_0=\lim\limits_{i\to\infty}u_i$, it follows from (20) that $t_0=\Phi(t_0)$. By Lemma 1 and (B_2) it follows that $t_0=0$, hence

$$\lim_{i\to\infty}\delta(O(y_i,\ \infty))\leqslant\lim_{i\to\infty}u_i=0,$$

which imply that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence in the complete metric space $(X,\ d)$ and so it converges to $y^*\in X$. since f is continuous, It follows that $\{fy_n\}_{n\in\omega}$ also converges to fy^*.

Now we prove that $g_j^{m_j}y^*=fy^*$, $\forall j\in N$. since $fy_n\to fy^*$, and

$$g_{n+2}^{m_{n+2}}y_n=g_{n+2}^{m_{n+2}}fx_{n+1}=fg_{n+2}^{m_{n+2}}x_{n+1}=fy_{n+1},$$

hence for any given $\varepsilon>0$ there exists $n_0\in N$ such that

$$d(fy_n,\ fy^*)\leqslant\frac{\varepsilon}{2},\ d(fy_n,\ fy_{n+1})<\varepsilon,$$

whenever $n\geqslant n_0$. By (12) we obtain

$$d(fy_{n+1}, g_j^{m_j}y^*) = d(g_{n+2}^{m_{n+2}}y_n, g_j^{m_j}y^*)$$
$$\leqslant \Phi(\max\{d(fy_n, fy^*), d(fy_n, fy_{n+1}), d(fy^*, g_j^{m_j}y^*),$$
$$d(fy_n, g_j^{m_j}y^*), d(fy^*, fy_{n+1})\})$$
$$\leqslant \Phi\left(\max\left\{\frac{\varepsilon}{2}, \varepsilon, d(fy^*, g_j^{m_j}y^*), \frac{\varepsilon}{2}+d(fy^*, y_j^{m_j}g^*), \frac{\varepsilon}{2}\right\}\right)$$
$$\leqslant \Phi(\varepsilon+d(fy^*, g_j^{m_j}y^*)), \forall n\geqslant n_0. \tag{21}$$

Let $n\to\infty$ in (21) and then let $\varepsilon\to 0$. We have

$$d(fy^*, g_j^{m_j}y^*)\leqslant \Phi(d(fy^*, g_j^{m_j}y^*)).$$

Thus Lemma 1 and (B_2) yield $d(fy^*, g_j^{m_j}y^*)=0$, and that is $g_j^{m_j}y^*=fy^*$, $\forall j\in N$.

Now for any $i\in N$ we have

$$d(g_i^{m_i}fy^*, fy^*) = d(g_i^{m_i}fy^*, g_j^{m_j}y^*)$$
$$\leqslant \Phi(\max\{d(ffy^*, fy^*), d(ffy^*, g_i^{m_i}fy^*), d(fy^*, g_j^{m_j}y^*),$$
$$d(ffy^*, g_j^{m_j}y^*), d(fy^*, g_i^{m_i}fy^*)\})$$
$$= \Phi(\max\{d(g_i^{m_i}fy^*, fy^*), 0, 0, d(g_i^{m_i}fy^*, fy^*), d(g_i^{m_i}fy^*, fy^*)\})$$
$$= \Phi(d(g_i^{m_i}fy^*, fy^*)), \tag{22}$$

which implies $d(g_i^{m_i}fy^*, fy^*)=0$ and hence $g_i^{m_i}fy^*=fy^*$, $\forall i\in N$. Since $ffy^*=g_i^{m_i}fy^*=fy^*$, hence fy^* also is a fixed point of f and so fy^* is a common fixed point of $\{g_n^{m_n}\}_{n\in N}$ and f.

Now for any fixed $i\in N$, suppose that x^* also is a common fixed point of $g_i^{m_i}$ and f, then by (12) we obtain

$$d(x^*, fy^*) = d(g_i^{m_i}x^*, g_j^{m_j}fy^*)$$
$$\leqslant \Phi(\max\{d(x^*, fy^*), 0, 0, d(x^*, fy^*), d(x^*, fy^*)\})$$
$$= \Phi(d(x^*, fy^*))$$

which implies $x^*=fy^*$ and hence the common fixed point of $g_i^{m_i}$ and f is unique. But since $g_i^{m_i}fy^*=fy^*$ and $f\,fy^*=fy^*$ imply $g_i^{m_i}g_ifу^*=g_ifу^*$ and $fg_i\,fy^*=g_ify^*$ respectively, g_ify^* is also a common fixed point of $g_i^{m_i}$ and f. By the uniqueness of the common fixed point of $g_i^{m_i}$ and f we obtain $g_ify^*=fy^*$. Since $i\in N$ is arbitrary, fy^* is a unique common fixed point of $\{g_n\}_{n\in\omega}$ and f.

Putting $m_n=1$, $\forall n\in N$ we obtain

Corollary 1. *Let f be a continuous self mapping on a complete metric space (X, d) and $\{g_n\}_{n\in N}$ be a sequence of self mappings on X such that $g_nf=fg_n$* and $g_n(X)\subset f(X)$, *$\forall n\in N$. If there exists Φ: $[0, \infty)\to[0, \infty)$ satisfying (B_1) and (B_2) such that*

$$d(g_ix, g_jy)\leqslant \Phi(\max\{d(fx, fy), d(fx, g_ix), d(fy, g_jy),$$
$$d(fx, g_jy), d(fy, g_ix)\})$$

for all $x, y\in X$ and all $i, j\in N$, $i\neq j$, then f and $\{g_n\}_{n\in N}$ have a unique common fixed point $fy^*(=g_ny^*)$ and for each $x_0\in X$, $g_{n+1}fx_n\to fy^*$.

Corollary 2. *Let f be a continuous self mapping on a complete metric space (X, d), g be a self mapping on X such that $gf=fg$, $g(X)\subset f(X)$. If there exists*

Φ: $[0, \infty)\to[0, \infty)$ satisfying (B_1) and (B_2) such that

$$d(gx, gy)\leqslant\Phi(\max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\})$$

for all $x, y\in X$, then f and g have a unique common fixed point fy^* and for each $x_0\in X$ $\{gfx_n\}_{n\in\omega}$ converges to fy^*.

Remark 5. Obviously, the Theorem 2.1 of Das, Naik[10] and the main result of Ranganathan[12] are the special cases of Corollary 2.

Corollary 3. *Let $\{g_n\}_{n\in N}$ be a, sequence of self mappings on a complete metric space (X, d). If there exists a sequence $\{m_n\}_{n\in N}$ of positive integers and Φ: $[0,\infty)\to[0,\infty)$ satisfying (B_1) and (B_2) such that*

$$d(g_i^{m_i}x, g_j^{m_j}y)\leqslant\Phi(\max\{d(x, y), d(x, g_i^{m_i}x), d(y, g_j^{m_j}y) d(x, g_j^{m_j}y), d(y, g_i^{m_i}x)\})$$

for all $x, y\in X$ and all $i, j\in N$, $i\neq j$, then $\{g_n\}_{n\in N}$ has a unique common fixed point y^* and for each $x_0\in X$ $\{g_{n+1}^{m_{n+1}}x_n\}_{n\in\omega}$ converges to y^*.

Proof Let $f=I$ (identical mapping) in Theorem 5. The conclusion of Corollary 3 follows.

Using a similar argument as in the proof of Theorem 5 we can establish

Theorem 6. *Let f be a self mapping on a complete metric space (X, d) and f^m be continuous, where m is a fixed positive integer. Suppose that $\{g_n\}_{n\in N}$ is a sequence of self mappings on $f^{m-1}(X)$ into X satisfying $g_nf=fg_n$ and $g_n^{m_n}(f^{m-1}(X))\subset f^m(X)$, $\forall n\in N$, where $\{m_n\}_{n\in N}$ is a sequence of positive inteyers. If there exists Φ: $[0, \infty)\to[0, \infty)$ satisfying (B_1) and (B_2) such that (12) holds for all $x, y\in X$ and all $i, j\in N$, $i\neq j$, then f and $\{g_n\}_{n\in N}$ have a unique common fixed point.*

Remark 6. Theorem 6 improves and generalizes the Theorems 3.1 and 4.1 of Das, Naik[10].

§ 4. The relations between some contractive type mappings

Lemma 2.[13, 14, 15] *Let f be a continuous self mapping on a metric space (X, d) with the following properties:*

(i) *f has a unique fixed point x^*.*

(ii) *For each $x\in X$ the sequence of iterations $\{f^n(x)\}_{n\in\omega}$ converges to x^*.*

(iii) *There exists an open neighborhood U of x^* with the property that given any open set V containing x^* there exists an integer n_0 such that $n\geqslant n_0$ implies $f^n(U)\subset V$.*

Then for an arbitrary $C\in[0, 1)$ there exists a metric d^* on X topologically equivalent to d such that f is a Banach contraction mapping under d^* with the Lipschitz constant C.

Theorem 7. *Let f be a continuous self mapping on a complete metric space (X, d).*

If one among the following contractive type conditions holds:

(I) (Banach) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta d(x, y)$$

for all $x, y\in X$;

(II) (*Kannan*[16]) *there exists* $\beta\in\left(0, \frac{1}{2}\right)$ *such that*

$$d(fx, fy)\leqslant\beta[d(x, fx)+d(y, fy)]$$

for all $x, y\in X$;

(III) (*Bianchini*[17]) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta\max\{d(x, fx), d(y, fy)\}$$

for all $x, y\in X$;

(IV) (*Reich*[18]) *there exists nonnegative real numbers* $a, b, c, a+b+c<1$, *such that*

$$d(fx, fy)\leqslant ad(x, fx)+bd(y, fy)+cd(x, y)$$

for all $x, y\in X$;

(V) (*Roux; Socrdi*[19]) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta\max\{d(x, fx), d(y, fy), d(x, y)\}$$

for all $x, y\in X$;

(VI) (*Chatterjea*[20]) *there exists* $\beta\in\left[0, \frac{1}{2}\right)$ *such that*

$$d(fx, fy)\leqslant\beta[d(x, fy)+d(y, fx)]$$

for all $x, y\in X$;

(VII) (*Hardy; Roges*[21]) *there exist nonnegative real numbers* a_1, a_2, a_3, a_4, a_5, $\sum_{i=1}^{5}a_i<1$, *such that*

$$d(fx, fy)\leqslant a_1d(x, y)+a_2d(x, fx)+a_3d(y, fy)$$
$$+a_4d(x, fy)+a_5d(y, fx)$$

for all $x, y\in X$;

(VIII) (*Zamfirescu*[22], *Massa*[23]) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta\max\left\{d(x, y), \frac{1}{2}[d(x, fx)+d(y, fy)], \frac{1}{2}[d(x, fy)+d(y, fx)]\right\}$$

for all $x, y\in X$;

(IX) (*Ciric*[24]) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta\max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy)+d(y, fx)]\right\}$$

for all $x, y\in X$;

(X) (*Ciric*[9]) *there exists* $\beta\in[0, 1)$ *such that*

$$d(fx, fy)\leqslant\beta\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for all $x, y\in X$;

(XI) (*Rhoades*[8]) *there exist* $\beta\in[0, 1)$ *and* $p, q\in N$ *such that*

$$d(f^p x, f^q y)\leqslant\beta\max\{d(x, y), d(x, f^p x), d(y, f^q y), d(x, f^q y), d(y, f^p x)\}$$

for all $x, y\in X$;

(XII) (*Fisher*[7]) *there exist* $\beta\in[0, 1)$ *and* $p, q\in N$ *such that*

$$d(f^p x, f^q y)\leqslant\beta\max\{d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y):$$
$$0\leqslant r, r'\leqslant p; 0\leqslant s, s'\leqslant q\};$$

(XIII) (Ding xie-ping) *there exists* $p, q\in N$ *and* Φ_1: $[0, \infty)^{(p+1)(q+1)+(p+1)^2+(q+1)^2}\rightarrow[0, \infty)$ *are nondecreasing and continuous from the right in each cooridinate variable and* $\Phi(t)=\Phi_1(t, t, \cdots, t)$ *satisfies* (B_1) *and* (B_2) *such that*

$$d(f^p x, f^q y)\leqslant\Phi_1(d(f^r x, f^s y), d(f^r x, f^{r'} x), d(f^s y, f^{s'} y):$$
$$(0\leqslant r, r'\leqslant p; 0\leqslant s, s'\leqslant q)$$

for all $x, y\in X$;

(XIV) (*Ding xie-ping*) *there exist* $p, q\in N$ *and* $\Phi:[0, \infty)\rightarrow[0, \infty)$ *satisfying* (B_1) *and* (B_2) *such that*

$$d(f^p x, f^q y)\leqslant\Phi(\delta(O_f(x, 0, p)\cup O_f(y, 0, q)))$$

for all $x, y\in X$;

then the following conclusions hold:

(i) f *has a unique fixed point* x^* *in* X;

(ii) *for each* $x\in X$ *the sequence of iterations* $\{f^n(x)\}_{n\in\omega}$ *converges to* x^*;

(iii) *U-uniform convergence*: *there exists some neighborhood* U *of* x^* *such that*

$$\lim_{n\to\infty} f^n(U)=\{x^*\},$$

this means that for any open set V *containing* x^* *there exists an integer* n_0 *such that* $n\geqslant n_0$ *implies* $f^n(U)\subset V$;

(iv) *stability of the fixed point* x^*: *for any neighborhood* W *of* x^* *there exists some neighborhood* V *of* x^* *such that each* $x\in V$ *implies* $f^n(x)\in W$, $\forall n\in\omega$;

(v) *for an arbitrary* $c\in[0, 1)$ *there exists a metric* d^* *topologically equivalent to* d *such that* f *is a Banach contraction mapping under* d^* *with the Lipschitz constant* c.

Proof It is easy to see that the contractive type conditions (I)—(XIII) are all the special cases of the contractive type condition (XIV). Thus we only need to prove the conclusions of Theorem 7 for f satisfying the contractive type condition (XIV).

In fact, we have proved, in Theorem 2, the conclusions (i) and (ii) of the Theorem 7. Now we prove the conclusion (iii) of this theorem.

Since f, and so $f^2, \cdots, f^p$, are continuous and x^* is a unique fixed point of f, hence for a given positive number $\frac{M}{2}>0$ there exists a positive number η, $0<\eta\leqslant\frac{M}{2}$

such that

$$d(f^i x,\ x^*) = d(f^i x,\ f^i x^*) \leqslant \frac{M}{2}$$

for all $x \in U = \{x \mid d(x,\ x^*) < \eta\}$ and all $i \in \{0,\ 1,\ \cdots p\}$. Thus for all $i,\ j \in \{0,\ 1,\ \cdots p\}$ and all $x,\ y \in U$ we have

$$d(f^i x,\ f^j y) \leqslant d(f^i x,\ x^*) + d(f^j y,\ x^*) \leqslant \frac{M}{2} + \frac{M}{2} = M,$$

and so

$$\sup_{x \in U} \delta(O_f(x,\ 0,\ p)) \leqslant M. \tag{23}$$

Since the contractive type condition (XIV) implies the orbitally contractive type condition (1) in Theorem 1.

An analysis of the proof of Theorem 1 shows that when we take the supremum in the inequlities (3)—(9) of Theorem 1 for all $x \in U$, these inequalities still hold, where M satisfies (23). Then it follows from (9) that

$$\lim_{n\to\infty} \sup_{x \in U} \delta(O_f(x,\ np+l,\ \infty)) = 0,\ \forall 0 \leqslant l \leqslant p. \tag{24}$$

Now we fix the open neighborhood $U = \{x \mid d(x,\ x^*) < \eta\}$ of x^*, then for any given $\varepsilon > 0$, (24) implies that there exists $n_0 \in N$ such that

$$d(x_i,\ x_j) = d(f^i x,\ f^j x) \leqslant \frac{\varepsilon}{2} \tag{25}$$

for all $x \in U$ and all $i,\ j \in N,\ j > i \geqslant n_0$.

Letting $j \to \infty$ in (25) we obtain

$$d(x_i,\ x^*) = d(f^i x,\ x^*) \leqslant \frac{\varepsilon}{2} \tag{26}$$

for all $x \in U$ and all $i \geqslant n_0$, then the diameter of $f^i(U)$ satisfy

$$\delta(f^i(U)) = \sup_{x,y \in U} d(f^i x,\ f^i y) \leqslant \sup_{x,y \in U} [d(f^i x,\ x^*) + d(f^j y,\ x^*)]$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{27}$$

for all $i \geqslant n_0$. (27) means that $\delta(f^i(U)) \to 0$, as $i \to \infty$. Thus $f^i(U)$ is squeezed into any neighborhood of x^*, as i is large enough and the proof of conclusion (iii) is completed.

According to the Lemma 2 the conclusion (V) of this theorem is also true.

Following the Lemma 2.1 of Олойцев[14] the conclusion (iv) is again true and hence the proof of this theorem is completed.

Remark. Theorem 7 improves and extends some main results in [7—9] and [16—24] and strengthens their conclusions. Theorem 7 shows in essential that under a suitable metric d^* toplogically equivalent to d, these mappings in this theorem are topologically equivalent to each other.

Clearly, by Theorem 7 we have

Theorem. 8. *Let f be a continuous self mapping on a complete metric space $(X,\ d)$. If f satisfies one among the contractive type conditions (I)—(XIV) in Theorem*

7, then there exists a metric d^* on X topologically equivalent to d such that under d^* f is an Edelstein contractive mapping[25] *(For all $x, y \in X$, $d(fx, fy) < d(x, y)$).*

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