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ON THE SUBGROUP LATTICE CHARACTERIZATION OF FINITE SIMPLE GROUPS OF LIE TYPE

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Abstract

Any finite simple group of Lie type is proved to be determined by its subgroup lattice up to isomorphism.

By the subgroup lattice $L(G)$ of a group G is meant the set of all subgroups of G partially ordered by inclusion. A group G^* is said to be lattice-isomorphic to G if $L(G) \cong L(G^*)$, i. e., there exists an order-preserving bijective mapping μ of $L(G)$ onto $L(G^*)$.

In general, the lattice-isomorphism $L(G) \cong L(G^*)$ of two groups G and G^* does not imply $G \cong G^*$. But such an implication was expected in case that G is a non-abelian finite simple group[2]. Although this conjecture has been verified in several cases (for example, the case of alternating groups), there is up to the present no proof in general. In this paper we shall prove the following result which will lead to the settlement of the matter, providing the classifying of finite simple groups is completed.

Theorem 1. *Let G be a finit esimple Chevalley group or a finite simple twisted group and G^* be a group with $L(G) \cong L(G^*)$, then $G \cong G^*$.*

The cases $G = A_1(q)$, ${}^2A_2(q^2)$, ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$ are excluded in the following exposition of the proof. And the following known theorems will be quoted:

The lattice-isomorphic image of a finite non-abelian p-group is also a p-group[2].

The lattice-isomorphic image of a finite non-abelian simple group is also a simple group[3].

It is well-known that the group G is a group with a BN-pair, where B is the normalizer of a Sylow p-subgroup of G, p being the characteristic of the ground field under consideration.But under our restriction a Sylow p-subgroup of G is non-abelian and is generated by its minimal subgroups, it is not difficult to prove on basis of a theorem of R. Baer[2] cited above that the lattice-isomorphism sends the set $\mathrm{Syl}_p(G)$ of all Sylow p-subgroups of G into the corresponding set $\mathrm{Syl}_p(G^*)$ of G^*. Further, in any finite group X the normalizer $N_X(P)$ of the Sylow p-subgroup P can be characterized in latticetheoretic terms as the greatest subgroup of X containing P but

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not containing other subgroups of $\mathrm{Syl}_p(X)$, so we see the mapping μ also sends the set of normalizers of Sylow p-subgroups of G into the corresponding set in G^*. Thus if we denote by i_g the inner automorphism of G^* induced by the element g and regard it as a permutation on the set of normalizers of Sylow p-subgroups of G^*, then $I_g=\mu^{-1}i_g\mu$ is a permutation on the set of normalizers of Sylow p-subgroups, and hence is a permutation on the set Ω of all parabolic subgroups of G. Since, by a theorem of M. Suzuki[3], G^* is simple, the mapping $g\mapsto I_g$ is an embedding of G^* into the symmetric group on Ω. On the other hand, according to J. Tits[4], a building $\Delta(G)$ with the reverse of set-theoretical inclusion as its partial ordering can be defined on Ω. Since each I_g preserves set-theoretical inclusion, it can be regarded as an automorphism of this building. Further, when g lies in $\mu(xBx^{-1})$, the building automorphism I_g fixes all faces of the chamber xBx^{-1} and is therefore type-preserving. But G^* is generated by all subgroups of the form $\mu(xBx^{-1})$, we see $g\mapsto I_g$ is an embedding of G^* into the group Spe $\Delta(G)$ of all type-preserving automorphisms of $\Delta(G)$.

A fact of vital importance at this moment is a theorem which was originally derived by J. Tits[4] in terms of the theory of algebraic groups and of which we shall give an direct proof in the remaining part of the present paper. According to this theorem, the group Spe $\Delta(G)$ of type-preserving automorphisms of the building $\Delta(G)$ is isomorphic to the group Spe G generated by all inner, diagonal and field automorphisms of the group G, and so we may regard the group G^* as a subgroup of Spe G. But the group Spe G has a normal series

$$G\cong \mathrm{Inn}\ G\lhd \mathrm{Spe}\ G$$

with solvable factor group Spe G/Inn G and the subgroup G^* is non-abelian simple, so we must have $G^*\cong G$ as was to be shown.

Now, after doing all this, our main concern is to establish the isomorphism between Spe $\Delta(G)$ and Spe G, i. e., to prove

Theorem 2. *Let G be a Chevalley group or a twisted group over a finite field, $G\neq A_1(q)$, ${}^2A_2(q^2)$, ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, then each element σ of the group* Spe G *generated by all inner, diagonal and field automorphisms of G induces a type-preserving automorphism σ^* of the building $\Delta(G)$ and $\sigma\mapsto\sigma^*$ is an isomorphism of Spe G onto* Spe $\Delta(G)$.

The exposition of our proof will be given only for the case that G is a Chevalley group over F_q, the twisted cases can be treated similarly. We note by the way that our method can be also applied to determine the full automorphism group Aut $\Delta(G)$ of $\Delta(G)$.

The notation about the Chevalley group G follows R. Carter[1]: B is the semi-direct product of U by H, where $U=\langle X_r;\ r\in\Phi^+\rangle$ is the unipotent subgroup generated by the positive root subgroups and H is the diagonal subgroup, while N is the monomial subgroup and $W \cong N/B \cap N = N/H$ is the Weyl group with the fundamental reflections $\{w_r;\ r \in \Pi\}$ as generating involutions. A subgroup containing B is of the form $P_J = \langle B,\ n_r;\ r \in J \rangle$, where $J \subseteq \Pi$ and n_r is the image of w_r in W under the natural homomorphism, while a parabolic subgroup is a conjugate gP_Jg^{-1} of some P_J. The set of apartments of $\Delta(G)$ is $\mathscr{A} = \{g\Sigma_0 g^{-1};\ g \in G\}$, where $\Sigma_0 = \{nP_Jn^{-1};\ n \in N,\ J \subseteq \Pi\}$. For the sake of convenience we shall denote gP_Jg^{-1} by ${}^gP_J$ and, similarly, $g\Sigma_0 g^{-1}$ by ${}^g\Sigma_0$.

The distance between the chamber ${}^gB$ and ${}^{gun}wB$ is $l(w)$, where $u \in U$, w is the image of n_W in W and $l(w)$ is the shortest length while w is expressed in the generators w_r, $r \in \Pi$. In particular, the diameter of $\Delta(G)$ is $l(w_0)$, where w_0 is the unique element of W sending Φ^+ to Φ^-, while the chambers opposite to ${}^gB$ are ${}^{gun_0}B$, n_0 mapping to w_0. $u \mapsto {}^{gu}\Sigma_0$ is a 1–1 correspondence between the elements of U and the apartments containing the chamber ${}^gB$, while the set $\{{}^{gx_r(a)}\Sigma_0;\ a \in F_q\}$ exhausts all apartments containing simultaneously ${}^gB$ and ${}^{gn_0}P_{\{s\}}$, where $s \in \Pi$ and $r = w_0(-s)$. The type of any face ${}^gP_J$ is the same as P_J, and ${}^{un}P_J$ is the unique face of ${}^u\Sigma_0$ which goes over to a given ${}^nP_J \in \Sigma_0$ under the retraction $\mathrm{retr}_{\Sigma_0, B}$.

Now we state the proof of the proposition.

The only difficulty to surmount is to show that every $\tau \in \mathrm{Spe}\ \Delta(G)$ can be induced by some $\sigma \in \mathrm{Spe}\ G$.

Suppose $\tau(B) = {}^{g_1}B$, then $\tau_1 = i_{g_1}^{*-1}\tau$ fixes B.

τ_1 permutes the chambers opposite to B among themselves, hence $\tau_1({}^{n_0}B) = {}^{u_1n_0}B$ for some $u_1 \in U$. Now $\tau_2 = i_{u_1}^{*-1}\tau_1$ fixes B and ${}^{n_0}B$. Thus τ_2 stabilizes the apartment $\Sigma_0 = \Sigma(B, {}^{n_0}B)$ and fixes all faces ${}^nP_J$ of Σ_0.

For any $r \in \Pi$, the automorphism τ_2 fixes B and ${}^{n_0}P_{\{W_0(-r)\}}$, and so it induces a permutation on the set $\{{}^{x_r(a)}\Sigma_0;\ a \in F_q\}$ of all apartments containing B and ${}^{n_0}P_{\{W_0(-r)\}}$. In particular, $\tau_2({}^{x_r(1)}\Sigma_0) = {}^{x_r(t_r)}\Sigma_0$, $t_r \in F_q^*$, and the function $\chi(r) = t_r$ can be extended to an F_q-character χ of the free additive group generated by the fundamental roots. Let $h(\chi)$ be the element of $\hat{H}$ determined by this character and let d be the automorphism $x \mapsto h(\chi)xh(\chi)^{-1}$ of G, then $\tau_3 = d^{*-1}\tau_2$ fixes all faces of the apartments Σ_0 and $\{{}^{x_r(1)}\Sigma_0;\ r \in \Pi\}$.

Now let us make a closer analysis of the automorphism τ_3. Since this automorphism fixes B, it induces a permutation on the set $\{{}^u\Sigma_0;\ u \in U\}$ of all apartments containing B. By putting $\tau_3({}^u\Sigma_0) = {}^{\varphi(u)}\Sigma_0$ we get a permutation φ: $u \mapsto \varphi(u)$ of the elements of U. τ_3 preserves the image of every face under the retraction $\mathrm{retr}_{\Sigma_0, B}$, hence it maps ${}^{un}P_J$ to ${}^{\varphi(u)n}P_J$. Further, we know that any $r \in \Phi^+$ can be written as $r = \mathrm{w}(r')$ with $r' \in \Pi$ and $w \in W$. Since τ_3 sends the pair ${}^uw_B$, ${}^{unwn_0}P_{\{w_0(-r')\}}$ to the pair ${}^{\varphi(u)nw}B$, ${}^{\varphi(u)nwn_0}P_{\{w_0(-r')\}}$, it maps the set $\{{}^{unwx_{r'}(a)}\Sigma_0;\ a \in F_q\} = \{{}^{ux_r(a)}\Sigma_0;\ a \in F_q\}$ of all apartments containing the former pair to the set $\{^{\varphi(u)x_r(a)}\Sigma_0;\ a\in F_q\}$ of all apartments containing the latter. In other words, we have $\varphi(ux_r(a))=\varphi(u)x_r(b)$ and $b\neq 0$ iff $a\neq 0$. Hence, we have $\varphi(u\prod_i x_{r_i}(a_i))=\varphi(u)\prod_i x_{r_i}(b_i)$, and $\varphi(\prod_i x_{r_i}(a_i))=\prod_i x_{r_i}(b_i')$, where b_i, b_i' are non-zero iff $a_i\neq 0$.

Given $r\in\Phi^+$, we can define a permutation f_r of the elements of F_q by

$$\varphi(x_r(a))=x_r(f_r(a)).$$

On the other hand, given any ordering of the roots compatible with the height function, every element $u\in U$ can be uniquely expressed in the form $u=\prod_{r\in\Phi^+}x_r(a_r)$ with the roots arranged in increasing order. Applying the function φ, we have $\varphi(u)=\prod_{r\in\Phi^+}x_r(b_r)$. But for any given $s\in\Phi^+$, we have $u=x_s(a_s)u_s$ and

$$u_s=x_s(a_s)^{-1}\prod_{r\in\Phi^+}x_r(a_r)$$

can be brought to the canonical form $\prod_{r\in\Phi^+}x_r(a_r^*)$ with $a_s^*=0$. So we have

$$\varphi(u)=x_s(f_s(a_s))\prod_{r\in\Phi^+}x_r(b_r^*)\ \text{with}\ b_s^*=0.$$

Bringing this expression of $\varphi(u)$ to the canonical form and comparing it with the former expressi on of $\varphi(u)$, we find that $b_s=f_s(a_s)$. Thus we have the important equality $\varphi(\prod_{r\in\Phi^+}x_r(a_r))=\prod_{r\in\Phi^+}x_r(f_r(a_r))$.

As a further step, we show that under the restriction $G\neq A_1(q)$, the function does not dependon r and is an automorphism of the field F_q. To start with, let r, s be any pair of non-orthogonal fundamental roots with $N_{r,s}=-1$. By Chevalley's commutator formula, we have for any a, $b\in F_q^*$

$$x_s(b)x_r(a)=x_r(a)x_s(b)x_{r+s}(ab)\cdots.$$

Here the ordering of the roots on both sides are distinct but are both compatible with the height function. By applying the function φ we have

$$x_s(f_s(b))x_r(f_r(a))=x_r(f_r(a))x_s(f_s(b))x_{r+s}(f_{r+s}(ab))\cdots.$$

But

$$x_s(f_s(b))x_r(f_r(a))=x_r(f_r(a))x_s(f_s(b))x_{r+s}(f_r(a)f_s(b))\cdots,$$

so we have $f_{r+s}(ab)=f_r(a)f_s(b)$. Since $f_r(1)=f_s(1)=1$, we have $f_r(a)=f_{r+s}(a)=f_s(a)$ and $f_r(ab)=f_r(a)f_r(b)$. By applying φ to the equality

$$x_s(a)x_r(1)x_{r+s}(b)=x_r(1)x_s(a)x_{r+s}(a+b)\cdots,$$

we have

$$x_s(f_s(a))x_r(1)x_{r+s}(f_{r+s}(b))=x_r(1)x_s(f_s(a))x_{r+s}(f_{r+s}(a+b))\cdots.$$

But

$$x_s(f_s(a))x_r(1)x_{r+s}(f_{r+s}(b))=x_r(1)x_s(f_s(a))x_{r+s}(f_s(a)+f_{r+s}(b))\cdots,$$

so we have $f_{r+s}(a+b)=f_s(a)+f_{r+s}(b)$, and hence $f_r(a+b)=f_r(a)+f_r(b)$. Thus f_r is an automorphism of F_q. From the connectedness of the Dynkin diagram of the associated simple Lie algebra we see that $f_r=f$ for all $r_i\in\Pi$. Now we prove by induction on the height of r' that $f_r=f_{r'}$ for any $r'\in\Phi^+$. As a matter of fact, if $r'\in\Phi^+\backslash\Pi$, then there exists $r''\in\Phi^+$ such that $r'-r''\in\Phi^+$. Let $r_0=r'-mr''$ begin the r''-chain of roots through r', then $r'=r_0+mr''$ and $M_{r_0,r',m}=\pm1$, hence

$$x_{r''}(1)x_{r_0}(a)=x_{r_0}(a)x_{r''}(1)\cdots x_r(\eta a),$$

where $a\in F_q$, $\eta=(-1)^m M_{r_0,r'',m}=\pm1$. Since $f_r=f_{r_0}=f_{r''}$ by the induction hypothesis, we have by applying the function φ

$$x_{r''}(1)x_{r_0}(f_r(a))=x_{r_0}(f_r(a))x_{r''}(1)\cdots x_{r'}(f_{r'}(\eta a)).$$

But

$$x_{r''}(1)x_{r_0}(f_r(a))=x_{r_0}(f_r(a))x_{r''}(1)\cdots x_{r'}(\eta f_r(a)),$$

and so $f_{r'}(\eta a)=\eta f_r(a)=f_r(\eta a)$, hence $f_{r'}=f_r$ as desired. Thus f_r can be regarded as a field automorphism f of G and we have $f^{*-1}\tau_3=1$, i.e., $\tau=(i_{g_1}i_{u_1}df)^*$. This completes the proof of Theorem 2.

The cases excluded from our proof are those in which $W\cong N/B\cap N$ is generated by only one involution. Hence, in these cases, $\Delta(G)$ has only one face beyond the chambers and Spe $\Delta(G)=$ Aut $\Delta(G)$ is the symmetric group on the set of all chambers. However, Theorem 1 is still true for them. In particular, a proof for $A_1(q)\cong\mathrm{PSL}(2,q)$ and ${}^2A_2(q^2)\cong\mathrm{PSU}(3,q^2)$ can be found in [5, 6].

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