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SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE AND GAUSS MAPS

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Abstract

The aim of this paper is to study the relations between submanifolds with parallel mean curvature and their Gauss maps, and in virtue of them we obtain three main theorems. They show some geometrical restrictions to a submanifold of Euclidean space E^n with parallel mean curvature.

§ 1. Local formulas

Let M and N be smooth connected oriented Riemannian manifolds of dimension m and n respectively. We denote their metrics by dS_M^2 and dS_N^2 respectively. We choose local fields of orthonormal frames in M and N respectively such that, locally, we have

$$dS_M^2=\omega_1^2+\cdots+\omega_m^2=\sum_i (\omega_i)^2,$$
$$dS_N^2=\tilde{\omega}_1^2+\cdots+\tilde{\omega}_n^2=\sum_\alpha (\tilde{\omega}_\alpha)^2, \tag{1}$$

where ω_i and $\tilde{\omega}_\alpha$ are the fields of dual frames in M and N respectively. We shall make use of the following convention on the ranges of indices:

$$1\leqslant i,\ j,\ k,\ \cdots,\ \leqslant m;\ 1\leqslant \alpha,\ \beta,\ \gamma,\ \cdots,\ \leqslant n,$$

and we shall agree that repeated indices are summed over the respective ranges. Then the structure equations of M and N are given respectively by

$$\begin{cases} d\omega_i=\Sigma\,\omega_j\wedge\omega_{ji}, \\ d\omega_{ij}=\Sigma\,\omega_{ik}\wedge\omega_{kj}+\Omega_{ij}, \\ \Omega_{ij}=-\dfrac{1}{2}\,\Sigma\,R_{ijkl}\omega_k\wedge\omega_l, \\ R_{ij}=\Sigma\,R_{kikj}, \\ R=\Sigma R_{ii}, \end{cases} \qquad (2) \qquad \begin{cases} d\tilde{\omega}_\alpha=\Sigma\,\tilde{\omega}_\beta\wedge\tilde{\omega}_{\beta\alpha}, \\ d\tilde{\omega}_{\alpha\beta}=\Sigma\,\tilde{\omega}_{\alpha\gamma}\wedge\tilde{\omega}_{\gamma\beta}+\tilde{\Omega}_{\alpha\beta}, \\ \tilde{\Omega}_{\alpha\beta}=-\dfrac{1}{2}\,\Sigma\,\tilde{R}_{\alpha\beta\gamma\delta}\tilde{\omega}_r\wedge\tilde{\omega}_\delta, \\ \tilde{R}_{\alpha\beta}=\Sigma\,\tilde{R}_{\gamma\alpha\gamma\beta}, \\ \tilde{R}=\Sigma\,\tilde{R}_{\alpha\alpha}, \end{cases} \qquad (\tilde{2})$$

where ω_{ij}, R_{ijkl}, R_{ij}, $R(\tilde{\omega}_{\alpha\beta}$, $\tilde{R}_{\alpha\beta\gamma\delta}$, $\tilde{R}_{\alpha\beta}$, $\tilde{R})$ are the connection form, the curvature tensor, the Ricci tensor and the scalar curvature of $M(N)$ respectively.

Let $f: M\to N$ be a smooth map. Then we have

$$f^*\tilde{\omega}_\alpha=\Sigma a_{\alpha i}\omega_i. \tag{3}$$

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For simplicity we still denote $f^*\tilde{\omega}_\alpha$ by $\tilde{\omega}_\alpha$, and $f^*\tilde{\omega}_{\alpha\beta}$ by $\tilde{\omega}_{\alpha\beta}$ similarly. By taking exterior differentiation of (3), we obtain

$$\sum Da_{\alpha i}\wedge\omega_i=0 \tag{4}$$

where

$$Da_{\alpha i}\overset{\text{def.}}{=\!=}da_{\alpha i}+\sum a_{\alpha j}\omega_{ji}+\sum a_{\beta i}\tilde{\omega}_{\beta\alpha}=\sum a_{\alpha ij}\omega_j. \tag{5}$$

From (4), by Cartan's lemma we have

$$a_{\alpha ij}=a_{\alpha ji}. \tag{6}$$

The energy density of f, $e(f)$, is defined by

$$e(f)=\frac{1}{2}\sum_{\alpha,i}(a_{\alpha i})^2. \tag{7}$$

A map f is said to be a harmonic map if and only if

$$\sum a_{\alpha ii}=0. \tag{8}$$

f is said to be totally geodesic map if and only if

$$a_{\alpha ij}=0. \tag{9}$$

By exterior differentiating (5) and using the structure equations (2) and (2^2), we obtain

$$\sum Da_{\alpha ij}\wedge\omega_j=\sum a_{\alpha k}\Omega_{ki}+\sum a_{\beta i}\tilde{\Omega}_{\beta\alpha}, \tag{10}$$

here

$$Da_{\alpha ij}\overset{\text{def.}}{=\!=}da_{\alpha ij}+\sum a_{\alpha kj}\omega_{ki}+\sum a_{\alpha ik}\omega_{kj}+\sum a_{\beta ij}\tilde{\omega}_{\beta\alpha}. \tag{11}$$

Set

$$Da_{\alpha ij}=\sum a_{\alpha ijk}\omega_k. \tag{12}$$

From (10), we have

$$a_{\alpha ijk}-a_{\alpha ikj}=-\sum a_{\alpha l}R_{likj}-\sum a_{\beta i}a_{jk}a_{\delta j}\tilde{R}_{\beta\alpha\gamma\delta}. \tag{13}$$

Then we may calculate the Laplacian of $a_{\alpha i}$ as follows

$$\begin{aligned}\Delta a_{\alpha i}&=\sum a_{\alpha ijj}=\sum a_{\alpha jij}=\sum a_{\alpha jji}-\sum a_{\alpha l}R_{ljji}-\sum a_{\beta j}a_{rj}a_{\delta i}\tilde{R}_{\beta\alpha\gamma\delta}\\&=\sum a_{\alpha jji}+\sum R_{li}a_{\alpha l}-\sum\tilde{R}_{\beta\alpha\gamma\delta}a_{\beta j}a_{\gamma j}a_{\delta i}.\end{aligned} \tag{14}$$

When f is a harmonic map, from (8) we obtain

$$\Delta a_{\alpha i}=\sum R_{li}a_{\alpha l}-\sum\tilde{R}_{\beta\alpha\gamma\delta}a_{\beta j}a_{\gamma j}a_{\delta i}, \tag{15}$$

and then we have

$$\begin{aligned}\Delta e(f)&=\frac{1}{2}\Delta\Big(\sum_{\alpha,i}(a_{\alpha i})^2\Big)=\sum_{\alpha,i,j}(a_{\alpha ij})^2+\sum a_{\alpha i}\Delta a_{\alpha i}\\&=|\beta(f)|^2+\sum R_{li}a_{\alpha l}a_{\alpha i}-\sum\tilde{R}_{\beta\alpha\gamma\delta}a_{\beta j}a_{\gamma j}a_{\delta i}a_{\alpha i},\end{aligned} \tag{16}$$

where $\beta(f)$ is the second fundamental form of f and

$$|\beta(f)|^2=\sum_{\alpha,i,j}(a_{\alpha ij})^2. \tag{17}$$

§ 2. Submanifolds of E^n with parallel mean curvature

In this section we consider the case that M^m is immersed in E^n, i. e., $f:M^m\hookrightarrow E^n$

is an isometrically immersion. In this case, we can choose a local field of orthonormal frames $e_1, \cdots, e_n$ in E^n such that, restricted to M, the vectors $e_1, \cdots, e_m$ are tangent to M and the remaining vectors $e_{m+1}, \cdots, e_n$ are normal to M. Then we have

$$dS_M^2 = \Sigma(\omega_i)^2,\ dS_{E^n}^2 = \Sigma(\tilde{\omega}_a)^2,\ f^*\tilde{\omega}_i = \omega_i,\ f^*\tilde{\omega}_r = 0,\ f^*\tilde{\omega}_{ij} = \omega_{ij}, \tag{18}$$

here the indices $\gamma, s, t, \cdots$ run over the range $m+1, \cdots, n$.

In this case we replace the nuclear letter a in the formulas of the previous section by nuclear letter h. Because E^n is flat, so from (5), we have

$$\tilde{\omega}_{ir} = \Sigma\, h_{irk}\omega_k, \tag{19}$$

$$h_{irk} = h_{kri}. \tag{20}$$

It is well-known that h_{irk} is called the second fundamental tensor of M. When $h_{irk}=0$, M is said to be a totally geodesic submanifold.

Now the formulas (11) and (12) become

$$Dh_{irk} = dh_{irk} + \Sigma\, h_{jrk}\omega_{ji} + \Sigma\, h_{irj}\omega_{jk} + \Sigma\, h_{isk}\tilde{\omega}_{sr} = \Sigma\, h_{irkj}\omega_j. \tag{21}$$

From (13) we obtain

$$h_{irjk} = h_{irkj}. \tag{22}$$

From this, together with (20), we know that h_{irjk} is symmetric with respect to the indices i, j, k. The vector

$$H = \frac{1}{m}\, \Sigma h_{iri} e_r \tag{23}$$

is called the mean curvature vector of M in E^n, and when it vanishes identically, i. e., $\Sigma h_{iri} = 0$, M is said to be a minimal submanifold.

A submanifold M is said to be a submanifold with parallel mean curvature, if the mean curvature vector H is parallel in the normal bundle. That means

$$DH = \frac{1}{m} \sum h_{irik}\omega_k e_r = 0. \tag{24}$$

Hence M is a submanifold with parallel mean curvature if and only if the following condition is satisfied

$$\sum h_{irik} = 0. \tag{25}$$

Let g be the Gauss map from M^m into the Grassman manifold $G_{(m,n-m)}$.

As we know from [1], if θ_{ir} and $\theta_{js,tr}$ denote the metric form and connection form of $G_{(m,n-m)}$ respectively, we have

$$g^*\theta_{ir} = \tilde{\omega}_{ir},$$
$$g^*\theta_{js,ir} = \tilde{\omega}_{js,ir} = -\delta_{rs}\omega_{ij} - \delta_{ij}\tilde{\omega}_{rs}. \tag{26}$$

Noting that $\tilde{\omega}_{ir} = \sum h_{irk}\omega_k$ and $G_{(m,n-m)}$ is of dimension $m(n-m)$, according to (3), we just obtain

$$a_{irk} = h_{irk}, \tag{27}$$

for Gauss map g.

In the same way, from (5) we have

$$\sum a_{irjk}\omega_k = da_{irj} + \sum a_{irk}\omega_{kj} + \sum a_{ksj}\tilde{\omega}_{ks,ir} = dh_{irj} + \sum h_{irk}\omega_{kj} + \sum h_{ksj}(-\delta_{ki}\tilde{\omega}_{rs} - \delta_{rs}\omega_{ik})$$
$$= dh_{irj} + \sum h_{irk}\omega_{kj} + \sum h_{krj}\omega_{ki} + \sum h_{isj}\tilde{\omega}_{sr} = \sum h_{irjk}\omega_k.$$

So,

$$a_{irjk}=h_{irjk}. \tag{28}$$

From (8), we knowa that g is harmonic if and only if $\sum a_{irjj}=0$, and it is equivalent to $\sum h_{jrji}=0$ because of (28) and (22), i. e., M^m is a submanifold with parallel mean curvature. Thus we have again proved the theorem obtained by Ruh and Vilms[2].

Theorem (Ruh & Vilms). *Gauss map g of animmersed submanifold of M of E^n is harmonic if and only if M is a submanifold with parallel mean curvature.*

§ 3. Main results

Throughout this section we assume that $M^m \hookrightarrow E^n$ is a submanifold immersed in E^n. we denote the square of the length of the second fundamental form by S, i. e., $S=\sum_{i,r,j}(h_{irj})^2$, and denote the infimum of the Ricci curvature of M at point x by $Q(x)$.

The following result[3] is well-known.

Lemma. *The sectional curvature of Grassman manifold $G_{(m,n-m)}$, $R^{G_{(m,n-m)}}_{iem}$, satisfies the inequality*

$$0\leqslant R^{G_{(m,n-m)}}_{iem}\leqslant 2.$$

Using this Lemma we can prove the following

Theorem 1. *Let $M^m \hookrightarrow E^n$ be a compact submanifold with parallel mean curvature. Then*

$$\int_{M^m} S(Q-2S)\,dV_M\leqslant 0. \tag{29}$$

Proof By Ruh-Vilms theorem we know that Gauss map g is harmonic. Then from (16) we have

$$\Delta e(g)=|\beta(g)|^2+\sum R^M_{li}a_{jrl}a_{jri}-\sum R^{G_{(m,n-m)}}_{lrisjtkq}a_{lrl'}a_{isj'}a_{jtl'}a_{kqj'}. \tag{30}$$

By the Iemma and (27), we obtain

$$\Delta e(g)\geqslant|\beta(g)|^2+QS-2\sum_{i',j'}\sum_{\substack{lr\\is}}(a_{lri'})^2(a_{isj'})^2=|\beta(g)|^2+QS-2S^2. \tag{31}$$

Aecause M^m is compact, integrating (31) over M^m, and applying Green's theorem, we obtain

$$0\geqslant\int_{M^m}|\beta(g)|^2dV_M+\int_{M^m}S(Q-2S)\,dV_M.$$

Then

$$\int_{M^m}S(Q-2S)\,dV_M\leqslant 0.$$

From the proof it is easy to obtain the following

Corollary 1. *Let $M^m \hookrightarrow E^n$ be a compact submanifold with parallel mean* *curvature. If the equality holds in* (29), *then Gauss map* g *is totally geodesic.*

Theorem 2. *Let* $M^m \hookrightarrow E^n$ *be a submanifold with parallel mean curvature. If* Q *is not less than* $2S$ *everywhere and* Q *is larger than* $2S$ *at a point* $p \in M^m$, *then* M^m *must be noncompact.*

Proof From (31) we conclude that $e(g)$ is a subharmonic function on M^m. If M^m is compact, then $e(g) \equiv$const. and $\beta(g) = 0$. Now we consider the formula (30) at the point p. We can easily obtain $e(g)(p) = 0$. So $e(g) \equiv 0$ and it means that g is a constant map, i. e., $h_{i r j} = 0$. Hence M^m is a totally geodesic submanifold of E^n and, consequently, must be noncompact. It is a contradiction.

This theorem is a non-existence theorem of some type of submanifold with parallel mean curvature in E^n.

Theorem 3. *Let* M^m *be a complete noncompact submanifold immersed in* E^n *with parallel mean curvature. If* Q *is not less than* $2S$ *everywhere and Gauss map* g *has finite energy, then* M^m *must be* E^m.

Proof From Ruh-Vilms theorem, it follows that Gauss map g is harmonic. Using Schwarz inequality, we obtain easily

$$|\nabla e(g)|^2 \leqslant 2e(g)\,|\beta(g)|^2.$$

On the other hand, because Q is not less than $2S$ everywhere, hence from (31) we have

$$\Delta e(g) - |\beta(g)|^2 \geqslant 0.$$

Hence

$$\Delta \sqrt{e(g)} = \frac{\Delta e(g)}{2\sqrt{e(g)}} - \frac{|\nabla e(g)|^2}{4e(g)^{3/2}} \geqslant \frac{\Delta e(g) - |\beta(g)|^2}{2\sqrt{e(g)}} \geqslant 0.$$

Because g has finite energy, i. e., $E(g) = \int_M e(g)\,dV_M < +\infty$,it together with (32), means that $\sqrt{e(g)}$ is a non-negative L^2-integrable subharmonic function on M^m

In [4], S. T. Yau has shown that every non-negative L^2-integrable subharmonic function on a complete Riemannian manifold must be a constant. Applying this to $\sqrt{e(g)}$,we conclude that $e(g)$ is a constant.

On the other hand, because $Q \geqslant 2S \geqslant 0$, M^m is a complete noncompact manifold with non-negative Ricci curvature. By a theorem obtaind by S. T. Yau in [4], the volume of M^m is infinite. This forces the constant $e(g)$ to be zero and g to be a constant map. Hence M^m is *a* totally geodesic submanifold of E^n and the theorem is proved.

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