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NONLINEAR BOUNDARY PROBLEMS WITH NONLOCAL BOUNDARY CONDITIONS

ZHENG SONGMU (郑宋穆)
(Institute of Mathematics, Fudan University)

Abstract

By means of the supersolution and subsolution method and monotone iteration technique, the following nonlinear elliptic boundary problem with the nonlocal boundary conditions

$$\begin{cases} Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u), \\ u|_\Gamma = \text{const (unknown)}, \ -\int \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(n, x_i)\,ds = 0 \end{cases}$$

is considerd. The sufficient conditions which ensure at least one solution are given. Furthermore, the estimate of the first nonzero eigenvalue for the following linear eigenproblem

$$\begin{cases} L\varphi = \lambda\varphi, \\ \varphi|_\Gamma = \text{const (unknown)}, \ -\int_\Gamma \sum_{i,j=1}^{n} a_{ij}\frac{\partial \varphi}{\partial x_j}\cos(n, x_i)\,ds = 0 \end{cases}$$

is obtained, that is

$$\lambda_1 \geqslant \frac{2\alpha}{nd^2}.$$

In this paper we consider the nonlinear boundary problem with the nonlocal boundary conditions as follows

$$(P)\begin{cases} Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u), \text{ in } \Omega \\ u|_\Gamma = \text{unk nown constant}, \ -\int_\Gamma \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)\,ds = 0. \end{cases}$$

We emphasize that the above problem which arises from many physical problems can not be reduced to the Dirichlet or the Neumann problem. The problem (P) with the special nonlinear term f which arises from plasma physics has been studied by several authors (see [2] and the referred papers). In this paper we apply supersolution and subsolution method to discuss existence of solutions of the problem (P).

Throughout this paper we always assume that $\Omega \subset R^n$ is a bounded domain with $C^{2+\mu}$ boundary Γ, $f(x, s)$ is a C^1 function defined in $\bar{\Omega} \times R$, $a_{ij} \in C^{1+\mu}$ and $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2$, $\alpha > 0$, $0 < \mu < 1$.

Remark 1. The problem with nonhomogeneous boundary condition

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$$-\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(x_i, n) ds = I \text{ (given constant)}$$

can be transformed into the problem (P) by means of substracting from u a function v which satisfies

$$\begin{cases} Lv+v=0 \text{ in } \Omega, \\ v|_\Gamma = \text{unknown constant}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial v}{\partial x_j} \cos(n, x_i) ds = I. \end{cases}$$

The existence and uniqueness of v is easily verified by Lax-Milgram theorem.

§ 1. Some Lemmas

First we prove the following lemma.

Lemma 1. *Let $c(x)$ be continuous in $\overline{\Omega}$, $c(x) \geqslant 0$, $c(x) \not\equiv 0$.*

If $u(x) \in C^2$ satisfies

$$\begin{cases} Lu+c(x)u \leqslant 0 (\geqslant 0), \text{ in } \Omega. \\ u|_\Gamma = \text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) ds \geqslant 0 (\leqslant 0). \end{cases}$$

Then $u \leqslant 0 (\geqslant 0)$ in Ω.

Proof By maximum principle $Lu+cu \leqslant 0$ implies that u can not attain positive maximum in Ω. If $\max_{\overline{\Omega}} u(x) > 0$, then positive maximum can only be attained on Γ. From theboundary condition $u=$const, u attain positive maximum on every point of Γ. Applying strong maximum principle, we have

$$\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) > 0$$

on every point of Γ which contradicts the boundary condition

$$-\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) ds \geqslant 0.$$

Thus the proof is completed.

Remark. In the case $c(x) \equiv 0$, if u satisfies $Lu \leqslant 0$, $u|_\Gamma = \text{const}$,

$$-\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) ds \geqslant 0,$$

then u must be constant. In fact, by $Lu \leqslant 0$, u can not attain its maximum in Ω unless u is constant. On the other hand, using the same argument as above, u can not attain its maximum on the boundary unless u is constant. Therefore we conclude that u must be constant.

Lemma 2. *For the linear boundary problem*

$$Lu+cu=F(x) \text{ in } \Omega, \tag{1.1}$$

$$u|_\Gamma = \text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) ds = 0, \tag{1.2}$$

where $c(x) \in C^\mu$ is a given function, $c(x) > 0$, $\forall$ $F(x) \in C^\mu$, the problem (1.1), (1.2) *admits a unique solution* $u_F \in C^{2+\mu}$,

$$\|u_F\|_{2+\mu} \leqslant \text{const } \|F\|_\mu.$$

Proof Introduce the space

$$H_c^1 = \{u \mid u \in H^1(\Omega),\ \gamma_0(u) = \text{const}\}, \tag{1.3}$$

where $H^1(\Omega)$ denotes the Sobolev space as usual, $\gamma_0(u)$ denotes the trace of u on Γ, thus H_c^1 is a Hilbert spact with H^1 norm. Define the weak solution for (1.1), (1.2) as follows:

$u \in H_c^1$ and

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + cuv \Big) dx = \int_\Omega Fv dx,\ \forall v \in H_c^1. \tag{1.4}$$

Applying Lax-Milgram theorem, $\forall\ F(x) \in C^\mu \subset L^2(\Omega)$ we have a unique solution $u_F \in H_c^1$ (see [1]). Denote $k = u_F|_\Gamma = \gamma_0(u_F)$ which is constant, thus $u_0 = u_F - k \in H_c^1$ is a weak solution for the following Dirichlet probem

$$\begin{cases} Lu + cu = F - ck, & (1.5) \\ u|_\Gamma = 0. & (1.6) \end{cases}$$

According to the regularity results for Dirichlet problem, we have $u_0 \in C^{2+\mu}$ and

$$\|u_0\|_{2+\mu} \leqslant \text{const} \|F - ck\|_\mu \leqslant \text{const}(\|F\|_\mu + |k|). \tag{1.7}$$

Thus $u_F \in C^{2+\mu}$ and

$$\|u_F\|_{2+\mu} \leqslant \text{const}(\|F\|_\mu + |k|). \tag{1.8}$$

By imbedding theorem, we have

$$\int_\Gamma |k|^2 ds \leqslant \text{const} \|u_F\|_{H^1}^2 \leqslant \text{const} \|F\|_{L^2}^2, \tag{1.9}$$

$$|k| \leqslant \text{const} \|F\|_{L^2} \leqslant \text{const} \|F\|_\mu. \tag{1.10}$$

Substituting (1.10) into (1.7) we obtain

$$\|u_F\|_{2+\mu} \leqslant \text{const} \|F\|_\mu, \tag{1.11}$$

that the proof is completed.

We now consider the linear eigenproblem

$$\begin{cases} L\varphi = \lambda\varphi, & (1.12) \\ \varphi|_\Gamma = \text{const},\quad -\displaystyle\int_\Gamma \sum_{i,j=1}^n a_{ij} \frac{\partial \varphi}{\partial x_j} \cos(n, x_i) ds = 0. & (1.13) \end{cases}$$

We have

Lemma 3. *For the eigenproblem* (1.12), (1.13) *there are the denumerable eigenvalues* $\{\lambda_i\}$ *such that* $\lambda_i \geqslant 0$, $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to +\infty$ *the eigenfunction space corresponding to* $\lambda_0 = 0$ *is spanned by* 1 *and the eigenfunctions corresponding to different eigenvalues orthogonize each other.*

Proof In fact, it is a consequence of the Riesz-Schauder theory for complete continuous operator. We leave the details to the readers.

Further for the first nonzero eigenvalueλ_1 we have the estimate bounded from below which only depends on the elliplicity constant α and the diameter d of $\overline{\Omega}$.

Theorem 1. *We have the estimates*

$$\lambda_1 \geqslant \frac{2\alpha}{nd^2}, \tag{1.14}$$

$$\lambda_1 \geqslant \mu_1, \tag{1.15}$$

where α is the ellipticity constant, d is the diameter, n is the dimension, μ is the least nonzero eigenvalue with Neumann boundary condition.

Proof Notice

$$\lambda_1 = \inf_{\varphi\in H_0^1,\ \int_\Omega \varphi\,dx=0} \frac{a(\varphi,\ \varphi)}{\|\varphi\|_{L^2}^2} \tag{1.16}$$

$$\mu_1 = \inf_{\varphi\in H_0^1,\ \int_\Omega \varphi\,dx=0} \frac{a(\varphi,\ \varphi)}{\|\varphi\|_{L^2}^2}, \tag{1.17}$$

where $a(\varphi,\ \varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial\varphi}{\partial x_i} \frac{\partial\varphi}{\partial x_j} dx$, (1.15) follows directly from $H_c^1 \subset H^1$. Let φ_1 be the eigenfunction corresponding to λ_1, then $\varphi_1 \in H_c^1, \int_\Omega \varphi_1 dx = 0, a(\varphi_1, \varphi_1) = \lambda_1 \|\varphi_1\|_L^2$. Let D be the cube in R^n with edges of length d containing Ω and

$$\gamma_0(\varphi_1) = \varphi_1|_\Gamma = k \text{ (constant)}.$$

Extendingφ_1 into D with k, denote the extension by $\tilde{\varphi}_1$, i. e.

$$\tilde{\varphi}_1 = \begin{cases} \varphi_1, \text{ in } \Omega, \\ k, \text{ in } D_1\backslash\Omega. \end{cases} \tag{1.18}$$

It is easy to proof $\tilde{\varphi}_1 \in H^1(D)$. Applying the Poincar'e inequality in the cube D, we obtain

$$\int_D \tilde{\varphi}_1^2 dx \leqslant \frac{1}{d^n}\left(\int_D \tilde{\varphi}_1 dx\right)^2 + \frac{nd^2}{2}\left(\int_D |\text{grad}\tilde{\varphi}_1|^2 dx\right), \tag{1.19}$$

hence

$$\begin{aligned} k^2\cdot\text{mes}(D-\Omega) + \int_\Omega \varphi_1^2 dx &\leqslant \frac{1}{d^n}\left(k\cdot\text{mes}(D-\Omega) + \int_\Omega \varphi_1 dx\right)^2 \\ &\quad + \frac{nd^2}{2}\int_\Omega |\text{grad }\varphi_1|^2 dx \\ &= \frac{1}{d^n}\cdot k^2\cdot(\text{mes}(D-\Omega))^2 + \frac{nd^2}{2}\int_\Omega |\text{grad }\varphi_1|^2 dx. \end{aligned} \tag{1.20}$$

Because

$$\text{mes}(D-\Omega) \leqslant \text{mes}(D) = d^n, \tag{1.21}$$

$$\alpha\int_\Omega |\text{grad }\varphi_1|^2 dx \leqslant a(\varphi_1,\ \varphi_1) = \lambda_1 \|\varphi_1\|_{L^2}^2, \tag{1.22}$$

from (1.20), (1.21), (1.22), we obtain

$$\lambda_1 \geqslant \frac{2\alpha}{nd^2}.$$

Thus the proof is completed.

§ 2. Main Results

We are now in position to discuss the problem (P).

Define. *Supersolution: If* $u^+\in C^2$ *Satisfies*

$$\begin{cases} Lu_+\geqslant f(x,\ u_+), & \text{in } \Omega, \\ u|_\Gamma=\text{const}, & -\int_\Gamma \sum_{j,j=1}^n a_{ij}\dfrac{\partial u_+}{\partial x_j}\cos(n,\ x_i)ds\leqslant 0, \end{cases} \tag{2.1}$$

then we call u_+ *the supersolution for the problem (P). Subsolution: If* $u_-\in C^2$ *satisfies*

$$\begin{cases} Lu_-\leqslant f(x,\ u_-), \\ u_-|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u}{\partial x_j}\cos(n,\ x_i)ds\geqslant 0, \end{cases} \tag{2.2}$$

then we call u_- *the subsolution for the problem (P).*

Applying the monotone iterative method to the problem (P), We have

Theorem 2. *If there are supersolution* $u_+(x)$ *and subsolution* $u_-(x)$ *satisfying* $u_-(x)\leqslant u_+(x)$ *in* $\bar{\Omega}$, *then the problem (P) admits at least one solution* $u(x)\in C^2$ *with* $u_-(x)\leqslant u(x)\leqslant u_+(x)$ *in* $\bar{\Omega}$.

Proof Because $\dfrac{\partial f}{\partial s}$ is bounded in $Q=\{x\in\bar{\Omega},\ \min_{\bar{\Omega}} u_-(x)\leqslant s\leqslant \max_{\bar{\Omega}} u_+(x)\}$, there exists a positive number c such that $\dfrac{\partial f}{\partial s}+c$ is positive in Q. Thus the problem (P) is equivalent to the problem (P′).

$$\text{(P)} \begin{cases} Lu+cu=f(x,\ u)+cu \text{ in } \Omega, & (2.3) \\ u|_\Gamma=\text{constant}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u}{\partial x_j}\cos(n,\ x_i)ds=0. & (2.4) \end{cases}$$

Obviously, u_+, u_- are also the supersolution and subsolution for the problem (P′) respectively.

Consider the linear problem

$$\begin{cases} Lv+cv=f(x,\ u)+cu, \text{ in } \Omega, \\ v=\text{const}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial v}{\partial x_j}\cos(n,\ x_i)ds=0. \end{cases} \tag{2.5}$$

By Lemma 2, $\forall u\in C^2$ we have a unique $v\in C^{2+\mu}$ satisfying (2.5).

Define.

$$v=Tu \tag{2.6}$$

and

$$u_1=Tu_+,\ v_1=Tu_-, \tag{2.7}$$

then u_1-u_+ *and* v_1-u_- *satisfy*

$$\begin{cases} L(u_1-u_+)+c(u_1-u_+)\leqslant(f(x,\ u_+)+cu_+)-(f(x,\ u_+)+cu_+)=0, \text{ in } \Omega, \\ (u_1-u_+)|_\Gamma=\text{const}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial(u_1-u_+)}{\partial x_j}\cos(n,\ x_i)ds\geqslant 0, \end{cases} \tag{2.8}$$

and

$$\begin{cases} L(v_1-u_-)+c(v_1-u_-)\geqslant 0, \text{ in } \Omega, \\ (v_1-u_-)\,|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial(v_1-u_-)}{\partial x_j}\cos(n, x_i)ds\leqslant 0, \end{cases} \tag{2.9}$$

respectively.

By Lemma 1 we have

$$u_1-u_+\leqslant 0, \ v_1-u_-\geqslant 0, \text{ in } \Omega. \tag{2.10}$$

On the other hand

$$(f(x, u_+)+cu_+)-(f(x, u_-)+cu_-)=\left(\frac{\partial f}{\partial u}\bigg|_{u_-+\theta(u_+-u_-)}+c\right)(u_+-u_-)\geqslant 0.$$

Because u_1-v_1 satisfies

$$\begin{cases} L(u_1-v_1)+c(u_1-v_1)\geqslant 0 \text{ in } \Omega, \\ (u_1-v_1)\,|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial(u_1-v_1)}{\partial x_j}\cos(n, x_i)ds\leqslant 0, \end{cases} \tag{2.11}$$

again by Lemma 1 we obtain

$$u_1-v_1\geqslant 0. \tag{2.12}$$

From (2.10), (2.11), (2.12), we have

$$u_-\leqslant v_1\leqslant u_1\leqslant u_+, \text{ in } \bar{\Omega}, \tag{2.13}$$

Let

$$v_{n+1}=Tv_n, \quad u_{n+1}=Tu_n. \tag{2.14}$$

Using the same argument as above, by induction it is easy to prove that

$$u_n, \ v_n\in C^{2+\mu}, n=1, 2, \cdots$$

and

$$u_-\leqslant v_1\leqslant v_2\cdots\leqslant v_n\leqslant\cdots\leqslant u_n\leqslant u_{n-1}\leqslant\cdots\leqslant u_1\leqslant u_+, \tag{2.15}$$

it follows that $\{u_n\}$ and $\{v_n\}$ are monotone bounded sequences which converge to the measurable function $\bar{u}$ and $\underline{u}$ respectively. Moreover we have $u_-\leqslant\underline{u}\leqslant\bar{u}\leqslant u_+$ in $\bar{\Omega}$. Hence $\bar{u}$ and $\underline{u}$ belong to L^p $\forall p$, $1<p<\infty$. By the Levi theorem u_n and v_n also converge to $\bar{u}$ and $\underline{u}$ in L^p respectively. On the other hand, it follows from (2.15) that $k_n=u_n|_\Gamma$ is a monotone bounded sequence. Let $w_n=u_n-k_n$, then w_n satisfies

$$\begin{cases} Lw_n+cw_n=f(x, u_{n-1})+cu_{n-1}-ck_n, \text{ in } \Omega, \\ w_n\,|_\Gamma=0. \end{cases} \tag{2.16}$$

By the L^p estimate for elliptic equation (see [4]), $\forall p$, $1<p<\infty$ we have

$$\begin{aligned} \|w_n-w_m\|_{H_p^2} &\leqslant \text{const}\|f(x, u_{n-1})+cu_{n-1}-ck_n-f(x, u_{m-1})-cu_{m-1}+ck_m\|_{L^p} \\ &\leqslant \text{const}\ ((M+c)\,\|u_{n-1}-u_{m-1}\|_{L^p}+c\cdot\text{mes}\,(\Omega)^{\frac{1}{p}}\,|k_n-k_m|) \\ &\to 0 \ (n, m\to\infty), \end{aligned}$$

where $M=\sup\limits_{x\in\bar{\Omega}, \min u_-\leqslant s\leqslant\max u_+}\left|\dfrac{\partial f}{\partial s}\right|$. This means tha t $\{w_n\}$ is the Cauchy sequence in H_p^2. Choose $p>$n, and by the imbedding theorem $\{w_n\}$ is also a Cauchy sequence in C^1. So is $u_n=w_n+k_n$. By Lemma 2, u_n is also the Cauchy sequence in $C^{2+\mu}$. Passing the limit, we conclude that $\bar{u}\in C^{2+\mu}$ is the solution for the problem (P). Using the same argument with $\{v_n\}$, thus the proof is completed.

Corollary. *For any solution $u(x)$ of the problem (P), $u_-(x)\leqslant u(x)\leqslant u_+(x)$, we have $\underline{u}(x)\leqslant u(x)\leqslant\bar{u}(x)$.*

We usually call $\bar{u}(x)$ the great solution and $\underline{u}(x)$ the least solution for the problem (P).

we can also discuss the multiplicity of solution by the method introduced by H. Amann (see [5]).

Remark 1. Using the same argument as the Theorem F in [5], if $\underline{u}$ and $\bar{u}$ are different, and the linearized problems corresponding to $\underline{u}$ and $\bar{u}$ have only zero solution, then we can conclude that the problem (P) has at least three solutions $\underline{u}(x)\leqslant u^*(x)\leqslant\bar{u}(x)$.

Remark 2. Using topological degree argument for T, it follows that if there are the supersolutions $u_1(x)$, $u_2(x)$ and subsolutions $v_1(x)$, $v_2(x)$ satisfying $v_1(x)<u_1(x)<v_2(x)<u_2(x)$ in $\bar{\Omega}$, then problem (P) has at least three solutions.

In what follows we put some assumptions on the behavior of f at infinity which will ensure that the problem admits at least one solution.

Condition (A_+). $\exists$ positive number s_+ and a bounded differentiable function $g_+(x, s)$ defined in $x\in\bar{\Omega}$, $s\in R$ such that for $x\in\bar{\Omega}$, $u>s_+$ we have

$$f(x, u)\leqslant g_+(x, u),\quad \int_\Omega g_+(x, u)dx\leqslant 0. \tag{2.17}$$

Condition (A_-). $\exists$ negative number s_- and a bounded differentiable function $g_-(x, s)$such that for $x\in\bar{\Omega}$, $u<s_-$ we have

$$f(x, u)\geqslant g_-(x, u),\int_\Omega g^-(x, u)dx\geqslant 0. \tag{2.18}$$

Theorem 3. *If $f(x, u)$ satisfies the conditions (A_+) and (A_-), then the problem (P) admits at least one solution.*

Proof By Theorem 2 it suffices to verify that there exist the supersolution u_+ and the subsolution u_- with $u_-(x)\leqslant u_+(x)$ in $\bar{\Omega}$.

Set

$$G(x, u)=g_+(x, u)-\frac{\int_\Omega g_+(x, u)dx}{\operatorname{mes}(\Omega)}. \tag{2.19}$$

By the assumption, we have constant k such that $|G|\leqslant k$ and $\int_\Omega Gdx=0$. By the Lemma 3 in the previous section, for any $w(x)\in C^2$ there exists a unique $v=Tw$ s uch that

$$\begin{cases} Lv=G(x, w), \text{ in } \Omega, \\ v|_\Gamma=\text{const}, \quad -\int_\Gamma\sum_{i,j=1}^n a_{ij}\dfrac{\partial v}{\partial x_j}\cos(x, x_i)ds=0, \end{cases} \tag{2.20}$$

and $\int_\Omega v dx=0$. Moreover

$$\alpha\int_\Omega |\nabla v|^2 dx\leqslant a(v,\ v)=\int_\Omega Gv dx\leqslant\frac{1}{2\varepsilon}\int_\Omega G^2 dx+\frac{\varepsilon}{2}\int_\Omega v^2 dx. \tag{2.21}$$

On the other hand, applying the Poincaré inequality as in The theorem 1 we have

$$\|v\|_{L^2}^2\leqslant\frac{nd^2}{2}\int_\Omega |\nabla v|^2 dx\leqslant\frac{nd^2}{2\alpha}a(v,\ v)\leqslant\frac{nd^2}{4\alpha\varepsilon}\int_\Omega G^2 dx+\frac{nd^2\varepsilon}{4\alpha}\|v\|_{L^2}^2. \tag{2.22}$$

Set $\varepsilon=\frac{2\alpha}{nd^2}$ and from the boundness of G we obtain

$$\|v\|_{L^2}^2\leqslant C_1,\int_\Omega |\nabla v|^2 dx\leqslant C_2. \tag{2.23}$$

By imbedding theorem and (2.23), we have

$$(v|_\Gamma)^2\mathrm{mes}(\Gamma)\leqslant\mathrm{const}(c_1+c_2),$$

$$|v|_\Gamma|\leqslant c_3. \tag{2.24}$$

As $v-v|_\Gamma$ is the weak solution of

$$\begin{cases} Lu=G(x,\ w), \\ w|_\Gamma=0, \end{cases} \tag{2.25}$$

owing to L^p theory for the Dirichlet problem it follows $v\in H_p^2$ and

$$\|v-v|_\Gamma\|_{H_p^2}\leqslant\mathrm{const}\|G\|_{L^p}\leqslant c_4. \tag{2.26}$$

Choosing $p>n$, by imbedding theorem and (2.24), we have

$$\|v\|_{1,\infty}\leqslant c_5. \tag{2.27}$$

We emphasize that the above $c_i(i=1,\ \cdots,\ 5)$ only depends on k. Therefore, there exists a number β such that $v(x)>\beta$, $x\in\bar{\Omega}$, $\forall w$. Choose α large enough so that $\alpha+\beta>s_+$.

Now we solve

$$\begin{cases} Lv=G(x,\ \alpha+v), \\ a|_\Gamma=\mathrm{const},\quad -\int_\Gamma\sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_j}\cos(n,\ x_i)ds=0. \end{cases} \tag{2.28}$$

Let

$$S=\left\{v\,|\,v\in C^1,\ \|v\|_{1,\infty}\leqslant c_5,\ \int_\Omega v dx=0\right\}.$$

It is easy to see that S is a closed convex set in C^1 and T maps S int o the compact susbset of S. By the Schauder fixed point theorem, (2.28) has a solution $v_0\in C^{2+\mu}$, $\int_\Omega v_0 dx=0$.

Set

$$u_+=\alpha+v_0>\alpha+\beta>s_+, \tag{2.29}$$

and owing to the condition (A_+), we have

$$\int_\Omega g_+(x,\ u_+)dx\leqslant 0,\quad f(x,\ u_+)\leqslant g_+(x,\ u_+), \tag{2.30}$$

$$G(x, u_+) \geqslant g_+(x, u_+) \geqslant f(x, u_+), \tag{2.31}$$

$$\begin{cases} Lu_+ = G(x, u_+) \geqslant f(x, u_+), \\ u_+|_\Gamma = \text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial u_+}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.32}$$

This means that u_+ is the supersolution. Using the same argument as above, from (A_-) we can fi nd the subsolution with the form $u_- = -\tilde{\alpha} + v_1$, Choose the positive numbers α and $\tilde{\alpha}$ large enough sothat $u_- < u_+$. The proof is completed by Theorem 2.

Corollary. *If $f_s(x, s) \leqslant 0$, then the necessary and sufficient conditions for existence of the problem (P) is that there exists a smooth function u_0 such that $u_0|_\Gamma = \text{const}$, $\int_\Omega f(x, u_0) dx = 0$.*

Proof The necessity is obvious. In order to prove the sufficiency we choose numbers s_+ and s_- such that

$$s_- < u_0 < s_+.$$

By $f_s \leqslant 0$, we obtain

$$f(x, u) \leqslant f(x, u_0), \quad \text{as } u > s_+ > u_0,$$

and

$$f(x, u) \geqslant f(x, u_0), \quad \text{as } u < s_- < u_0.$$

Let v_0 be the solution for

$$\begin{cases} Lv = f(x, u_0), \\ v|_\Gamma = \text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial v}{\partial x_j} \cos(n, x_i) ds = 0 \end{cases}$$

such that $\int_\Omega v\, dx = 0$. Choose $\alpha > 0$, $\beta < 0$ absolute value large enough such that $\beta + v_0 < s_-$ and $\alpha + v_0 > s_+$. Evidently $u_+ = \alpha + v_0$ and $u_- = \beta + v_0$ are the supersolution and the subsolution respectively. Moreover $u_- < u_+$. Thus, by Theorem 2, the proof is complete.

Theorem 4. *If f satisfies one of the following conditions, then the problem (P) admits at least one solution.*

(1) $\lim\limits_{|s|\to\infty} \dfrac{sf(x, s)}{|s|} < h(x)$, *where* $h(x) \in C^\mu$, $\int_\Omega h(x) dx < 0$,

(2) $\limsup\limits_{|s|\to\infty} \dfrac{f(x, s)}{s} < 0$,

(3) $\limsup\limits_{|s|\to\infty} f_s(x, s) < 0$,

(4) $f(x, s) = F(x, s) + g(x, s)$, *where F satisfies* (2) *or* (3) *and* $\lim\limits_{|s|\to\infty} \dfrac{g(x, s)}{s} = 0$,

(5) $f(x, s) = F(x, s)s + g(x, s)$, *where* $\limsup\limits_{|s|\to\infty} F(x, s) < 0$,

$$\lim_{|s|\to\infty} \frac{g(x, s)}{s} = 0.$$

The above limits are assumed to be uniform with respect to x.

Proof It is easy to see that (1) implies that f satisfies (A_+) and (A_-) with

$g_+=g_-=0$. (5)⇒(2) and (3)⇒(2), (4) either ⇒ (3)⇒(2) or ⇒(2). By Theorem 3 the proof is complete. In what follows we consider some cases in which f does not satisfy the conditions (A_+) and (A_-). Assume that f can be written in the form

$$f(x, s)=F(x)+g(x, s), \tag{2.33}$$

where

$$\lim_{|s|\to\infty} g(x, s)=0. \tag{2.34}$$

Set

$$\int_\Omega F(x)dx=k, \tag{2.35}$$

then F can be written as

$$F(x)=\tilde{F}(x)+t, \tag{2.36}$$

where

$$\begin{cases} \tilde{F}(x)=F(x)-\dfrac{k}{\operatorname{mes}(\Omega)} \text{ with } \displaystyle\int_\Omega \tilde{F}(x)dx=0, \\ t=\dfrac{k}{\operatorname{mes}(\Omega)}. \end{cases} \tag{2.37}$$

Let w be the solution for

$$\begin{cases} Lw=\tilde{F}, \\ w|_\Gamma=\text{const}, \ -\displaystyle\int_\Gamma \sum_{i,j=1}^n a_{ij}\frac{\partial w}{\partial x_j}\cos(n, x_i)ds=0 \end{cases} \tag{2.38}$$

and $\int_\Omega w\,dx=0$. As seen in the proof of Theorem 3, we have a uniqus $w\in C^{2+\mu}$. In addition we assume that there exists numbers $\alpha_+<0$, $\alpha_->0$, $k_+>0$ and $k_-<0$ such that

$$g(x, w(x)+k_+)<\alpha_+<0, \quad g(x, w(x)+k_-)>\alpha_->0, \quad x\in\bar{\Omega}.$$

Theorem 5. *If g satisfies the above conditions, then there exist numbers t_+ and t_-, $t_-<0<t_+$ such that when $t_-\leqslant t\leqslant t_+$ the problem*

$$(P_t)\begin{cases} Lu=\tilde{F}(x)+t+g(x, u), \ \text{in } \Omega, \\ u|_\Gamma=\text{const}, \ -\displaystyle\int_\Gamma \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)ds=0 \end{cases} \tag{2.39}$$

admits at least one solution. On the other hand, when $t>t_+$ or $t<t_-$ the problem (P_t) has no solution.

The proof is similar to those in [6], We omit the details.

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