Chin. Ann. of math.
4B (2) 1983

THE ASYMPTOTIC BEHAVIOUR OF ANALYTIC FUNCTIONS

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Abstract

In this paper, the author obtains the following results:

(1) If Taylor coeffients of a function $w(z)=\sum_{n=1}^{\infty} A_n z^n$ satisfy the conditions:

(i) $\sum_{k=1}^{\infty} k|A_k|^2<\infty$, (ii) $\mathrm{Re}\sum_{k=1}^{u} A_k=O(1)(n\to\infty)$, (iii) $A_k=O\left(\frac{1}{k}\right)$, the for any $h>0$ the function $\varphi(z)=\exp\{w(z)\}=\sum_{k=0}^{\infty} D_k z^k$ satisfies the asym ptotic equality

$$\left|\frac{\{\varphi(z)(1-z)^{-h}\}_n}{d_n(h)}-\sum_{k=0}^{u} D_k\right|=o(1)\,(n\to\infty),$$

the case $h>\frac{1}{2}$ was proved by Milin[1].

(2) If $f(z)=z+a_2z^2+\cdots\in S^*$ and $\lim_{r\to1}\frac{(1-r^2)}{r}\max_{|z|=r}|f(z)|=\alpha$, then for $\lambda>\frac{1}{2}$

$$\lim_{n\to\infty}\frac{\left|\left|\left\{\left(\frac{f(z)}{z}\right)^{\lambda}\right\}_n\right|-\left|\left\{\left(\frac{f(z)}{z}\right)^{\lambda}\right\}_{n-1}\right|\right|}{d_n(2\lambda-1)}=\alpha^{\lambda}.$$

Let S denote all analytic and univalent functions f in unit disc U with $f(0)=0$, $f'(0)=1$ and let S^* denote the subclass of starlike functions. Let T denote all functions $w(z)=\sum_{n=1}^{\infty} A_n z^n$ which are analytic in U with $\sum_{k=1}^{\infty} k|A_k|^2<\infty$ and

$$\frac{1}{(1-x)^h}=\sum_{n=0}^{\infty} d_n(h)x^n.$$

In this paper, we prove the following theorems:

Theorem 1. *Let $w(z)\in T$ and satisfy the following conditions*

$$\text{(I)}\ \mathrm{Re}\sum_{k=1}^{n} A_k=O(1),\quad \text{(II)}\ A_k=O\left(\frac{1}{k}\right).$$

If $\varphi(z)=e^{w(z)}=\sum_{k=0}^{\infty} D_k z^k$, *then*

$$\frac{\{\varphi(z)(1-z)^{-h}\}_n}{d_n(h)}-\sum_{k=0}^{n} D_k=o(1),\quad h>0\,(n\to\infty), \tag{1}$$

where $d_1, d_2, \cdots$, are the Taylor coefficients of the binomial function

$$\frac{1}{(1-x)}=\sum_{n=0}^{\infty} d_n(h)x^n.$$

Manuscript received January 31, 1981.

Milin[1] *proved the case* $h>\frac{1}{2}$ *without condition (II).*

Theorem 2. *Let* $f(z)\in S^*$ *with* $\lim_{r\to 1}\frac{(1-r)^2}{r}\max_{|z|=r}|f(z)|=\alpha\,(\alpha\neq 0)$ *and* $\{f(z)/z\}^\lambda=\sum_{n=0}^{\infty}D_n(\lambda)z^n$, *then*

$$|D_n(\lambda)|-|D_{n-1}(\lambda)|\sim\alpha^\lambda d_n(2\lambda-1),\lambda>\frac{1}{2}(n\to\infty). \tag{2}$$

The case $\lambda>\frac{3}{4}$ was proved by Milin[1].

For the proof of the theorem we need the following lemma.

Lemma. *Let* $w(z)=\sum_{k=1}^{\infty}A_kz^k$, $\varphi(z)=\exp\{w(z)\}=\sum_{k=0}^{\infty}D_kz^k$. *Then*

$$\sum_{k=0}^{n}|D_k|^2/d_k(\lambda)\leqslant d_n(\lambda+1)\exp\left\{\frac{1}{d_n(\lambda+1)}\sum_{v=1}^{n}d_{n-v}(\lambda)\Delta_v(\lambda)\right\}, \tag{3}$$

where

$$\Delta_v=\frac{1}{\lambda^2}\sum_{k=1}^{v}k|A_k|^2-\sum_{k=1}^{n}1/k.$$

Now we come to the proof of Theorem 1. Since

$$S_n^{(h)}=\{\varphi(z)(1-z)^{-h}\}_n=\sum_{k=0}^{\infty}d_{n-k}(h)D_k$$

and

$$nD_n=\sum_{k=1}^{n}kA_kD_{n-k},$$

Hence we have

$$\begin{aligned}\frac{S_n(h)}{d_n(h)}-\frac{S_n(h+1)}{d_n(h+1)}&=\sum_{k=0}^{n-1}\left\{\frac{d_k(h)}{d_n(h)}-\frac{d_k(h+1)}{d_n(h+1)}\right\}D_{n-k}\\&=\sum_{k=1}^{n}\frac{kA_kS_{n-k}S_{n-k}(h)}{(n+h)d_n(h)}\\&=\frac{1}{(n+h)d_n(h)}\{\sum_{0<k<n-\varepsilon n}+\sum_{n-\varepsilon n\leqslant k\leqslant n}\}kA_kS_{n-k}(h)\\&=\frac{1}{(n+h)d_n(h)}\{\Sigma_1+\Sigma_2\},\end{aligned} \tag{5}$$

It is known that if $\sum_{k=1}^{\infty}k|A_k|^2<\infty$, given an arbitrary positive number $\varepsilon<\frac{1}{2}$, we can find N such that $\sum_{k=1}^{m}|kA|_k^2<\varepsilon m$ when $m>N$. Let us choos n such that $\varepsilon n>N+1$. Since $d_n(h)\leqslant d_{n-1}(h)$, $n=1,2,3\cdots$ for $0<h\leqslant 1$, Applying Cauchy inequality to Σ_1 we have

$$\begin{aligned}|\Sigma_1|^2&=|\sum_{k\leqslant n-\varepsilon n}kA_kd_{n-k}^{1/2}(h)S_{n-k}(h)/\{d_{n-k}(h)\}^{1/2}|^2\\&\leqslant[\sum_{0\leqslant k<n-\varepsilon n}|kA_k|^2d_{n-k}(h)]\,[\sum_{\varepsilon n<k<n}|S_n(h)|^2/d_k(h)]\\&\leqslant[d_{[\varepsilon n]}(d)\sum_{k=1}^{m}|kA_k|^2]\,[\sum_{k=0}^{n}|S_k(h)|^2/d_k(h)]\\&=o(d_n(h+1))\sum_{k=0}^{n}|S_k(h)|^2/d_k(h),\end{aligned}$$

$$(d_{[\varepsilon n]}(h) \sim [\varepsilon n]^{h-1}/(\Gamma(h)) \sim [\varepsilon n]^{h-1}/\Gamma(h)).$$

Similarly, applying Cauchy inequality to Σ_2, we have (Using condition (II))

$$\begin{aligned}(\Sigma_2)^2 &\leqslant \sum_{n-\varepsilon n<k<n} d_{n-k}(h)\,|kA_k|^2 \sum_{k=0}^{n} |S_k(h)|^2/d_k(h) \\ &= O\left(\Big[\sum_{k<\varepsilon n} d_k(h)\Big]\Big[\sum_{k=0}^{n} |S_k(h)|^2/d_k(h)\Big]\right) \\ &= O\left(d_{[\varepsilon n]}(h+1) \sum_{k=0}^{n} |S_k(h)|^2/d_k(h)\right) \\ &= o\left\{d_n(h+1) \sum_{k=0}^{n} |S_k(h)|^2/d_k(h)\right\}. \end{aligned} \tag{7}$$

Since $w(z)\in T$, Re $\sum_{k=1}^{n} A_k = O(1)$, it follows that

$$\sum_{k=1}^{n} k|A_k+h/k|^2 - h^2 \sum_{k=1}^{n} 1/k = \sum_{k=1}^{n} k|A_k|^2 - 2h\,\mathrm{Re}\sum_{k=1}^{n} A_k = O(1). \tag{8}$$

Let $\quad w_1(z) = \log\varphi(z)(1-z)^{-h} = w(z) - h\log(1-z)$

$$= \sum_{k=1}^{\infty} A_k z^k + h\sum_{h=1}^{\infty} z^k/k = \sum_{h=1}^{\infty} A_k^{(1)} z^k.$$

We apply the lemma to this function $w(z)$, obtaining

$$\sum_{k=0}^{n} |S_k(h)|^2/d_k(h) \leqslant d_n(h+1)\exp\left\{\frac{k}{d_n(h+1)} \sum_{\nu=1}^{n} d_{n-\nu}(h)\Delta_\nu^{(1)}(h)\right\},$$

where

$$\Delta_\nu^{(1)}(h) = \frac{1}{h^2}\sum_{k=1}^{\nu} k|A_k+h/k|^2 - \sum_{k=1}^{\nu} 1/k.$$

In virture of (8), $\Delta_\nu^{(1)}(h) = O(1)$. Therefore,

$$\sum_{k=0}^{n} |S_k(h)|^2/d_k(h) = O(d_n(h+1)). \tag{9}$$

From (5), (6), (7) and (9), we get

$$S_n(h)/d_n(h) - S_n(h+1)/d_n(h+1) = o(1),\ 0<h<1\ (n\to\infty). \tag{10}$$

The case $h\geqslant 1$ was proved by Milin. Hence (10) holds for every real number $h>0$. In [1]

$$\sum_{k=0}^{n} D_k - S_n(h)/d_n(h) = o(1), \quad h>2. \tag{11}$$

In virture of (10) and (11), the theorem follows at once.

Corollary. *If $w(z)\in T$ and the conditions of Theorem 1 are satisfied, then*

$$\{\varphi(z)(1-z)^{-h}/d_n(h)\}_n \sim \varphi(r) \sim \exp\sum_{k=1}^{n} A_k,\ h>0\ (n\to\infty),$$

where $r = 1-\theta/n,\ 0<m<\theta<M$.

From [1]

$$\sum_{k=1}^{n} D_k \sim \varphi(r) \sim \exp\sum_{k=1}^{n} A_k.$$

To prove Theorem 2 it is sufficient to verify the conditions of Theorem 1 for the functions $f\in S^*$. WS suppose $\lim_{r\to\infty}(1-r)^2/r|f(r)| = \alpha$, $\log f(z)/z = 2\sum_{k=1}^{\infty} \gamma_k z^k$, then

$|r_n|<2/n$ for $f(z)\in S^*$. Consequently, Let $\varphi(z)=(1-z)^{2\lambda}\{f(z)/z\}^{\lambda}, w(z)=\sum_{k=1}^{\infty}A_k z^k$, then $A_k=2\lambda(r_k-1/k)=O(1/k)$.

Bazilevich theorem[2] asserts that

$$\sum_{k=1}^{\infty}k|r_k-1/k|^2\leqslant\frac{1}{2}\log 1/\alpha \quad (\alpha\neq 0).$$

This shows $w_1(z)\in T$, and

$$\operatorname{Re}\sum_{k=1}^{n}A_k\sim\log|\varphi(r)|\sim\log\alpha^{\lambda},\ r=1-\theta/n\,(n\to+\infty)$$

by corollary.

Then all conditions of Theorem 1 are verified. Take $h=2\lambda-1>0$, Theorem 2 follows.

It is to be noticed that condition (II) may be replaced by

$$\text{(II)}' \qquad \sum_{k=1}^{n}d_{n-k}(h)\,|kA_k|^2=O(d_n(h+1)).$$

References

[1] Milin, I. M., Uinivalent Functions and Orthonormal Systems, (1977), 32—71.
[2] Базилевич, И. Е., *Матем. сб.*, **68** (110) (1965), 549—560.