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A NOTE ON REDUCTIVITY OF OPERATORS

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Abstract

P. Rosenthal[1] introduced the following property (P):

If $\mathfrak{U}$ is any reductive algebra and $T\in\mathfrak{U}'$, then $T^*\in\mathfrak{U}'$.

In this note, the author proves

(1) A reductive spectral operator with polynomially compact quasinilpotent part has property (P);

(2) A reductive spectral operator with algebraic quasinilpotent part has property (P);

(3) The countable direct sum of scalar operators has property(P);

(4) If T is reductive and T is quasi-similar to a polynomially compact operator, then T is normal.

1. In [1], P. Rosenthal introduced the following property (P) which an operator T may have in connection with reductive algebras.

Property (P). If $\mathfrak{U}$ is any reductive algebra and $T\in\mathfrak{U}'$, then $T^*\in\mathfrak{U}'$.

In this note we present some sufficient conditions to guarantee that a reductive spectral operator has property (P), and that a reductive operator is normal.

2. when have reductive spectral operators have property (P)?

Theorem 1. *If T is a reductive spectral operator with polynomially compact quasinilpotent part, then T has property (P).*

Proof Since T is a reductive spectral operator, we can assume that the canonical decomposition of T is $T=N+Q$, where $N=\int\lambda E(d\lambda)$ is a normal operator and Q is a quasinilpotent operator commuting with N. Let $\mathfrak{U}$ be a reductive algebra such that $T\in\mathfrak{U}'$. Then for every Borel set $\sigma\in\mathbf{C}$, $E(\sigma)\in\mathfrak{U}'$, and consequently $N\in\mathfrak{U}'$ so that $Q\in\mathfrak{U}'$. Note that N is normal, therefore, by Fuglede's Theorem, $N\in\mathfrak{U}'$ implies $N^*\in\mathfrak{U}'$. Also, since the hypothesis that Q is polynomially compact, it follows from $Q\in\mathfrak{U}'$ that $Q^*\in\mathfrak{U}'$ by [2, Theorem 2]. Thus $T^*=N^*+Q^*\in\mathfrak{U}'$and hence T has pro perty (P).

Corollary 1. *If T is a reductive spectral operator with polynomially compact quasinilpotent part, then T is normal.*

Proof Let $\mathfrak{U}$ be the weakly closed algebra generated by T and I. Then $\mathfrak{U}$ is

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reductive and $T\in\mathfrak{U}'$. By Theorem 1, T^* commutes with T, that is, T is normal.

Theorem 2. *If T is a reductive spectral operator with algebraic quasinilpotent part, then T has property (P).*

Proof By the same argument as one we used in the proof of Theorem 1, we have $N^*\in\mathfrak{U}'$. Note that since Q is algebraic, therefore, by [3, Lemma 9.3], $Q\in\mathfrak{U}'$ implies $Q^*\in\mathfrak{U}'$. Thus $T^*\in\mathfrak{U}'$.

Corollary 2. *If T is a reductve spectral operator with algebraic quasinilpotent part, then T is normal.*

The above Corollaries 1 and 2 are generalizations of the corollary to Theorem 3.3 and Theorem 3.4 of [4], respectively.

Theorem 3. *The countable direct sum of scalar operators has property (P).*

Proof Let $S=\sum_{n=1}^{\infty}\oplus S_n$ denote a direct sum of scalar operators. Then there is a sequence of invertible self-adjoint operators $\{B_n\}$ such that $\{N_n\}=\{B_n^{-1}S_nB_n\}$ is a sequence of normal operators. Since we have

$$\|N_n\|=r(N_n)=r(S_n)\leqslant\|S_n\|,$$

so we may define the normal operator $N=\sum_{n=1}^{\infty}\oplus N_n$. To complete the proof, we note that since S_n is similar to N_n for each n, then [5, Theorem 2.5] implies that S is quasi-similar to N. Hence, by [2, Theorem 3], S has property (P).

3. A sufficient condition for reductive operators to be normal.

The argument used to prove [1, Theorem 5], with minor modifications will yield the following fact:

Theorem 4. *If T is reductive and T is quasi-similar to a polynomially compact operator S, then T is normal.*

Proof Suppose that $p(S)=K$ is compact where p is a nonconstant polynomial and there are quasi-affinities X and Y such that $TX=XS$, $YT=SY$. It is easy to check that T commutes with the compact operator $C=XKY$. Let $\mathfrak{M}$ be the reducing Kernel of C, that is $\mathfrak{M}=\ker C\cap\ker C^*$, then $\mathfrak{M}$ reduces T and $T|\mathfrak{M}^{\perp}$ is normal, by [6, Theorem 2]. If $x\in\mathfrak{M}$, then $x_X=0$ implies $XKYx=Xp(S)Yx=XYp(T)x=0$, and thus $p(T)x=0$. Hence $p(T|\mathfrak{M})=0$, that is, $T|\mathfrak{M}$ is algebraic. Now $T|\mathfrak{M}$ is also reductve. It follows that $T|\mathfrak{M}$ is normal, and thus so is $T=T|\mathfrak{M}\oplus T|\mathfrak{M}^{\perp}$.

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