Chin. Ann. of math.
4B (2) 1983

FINITE GROUPS IN WHICH EVERY NON-MAXIMAL PROPER SUBGROUP OF EVEN ORDER IS 2-CLOSED

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Abstract

In this paper the author classifies the finite groups satisfying the following condition:
(A) every non-maximal proper subgroup of even order is 2-closed.
Particularly, the following theorem is shown: A_5 and SL(2, 5) are the only non-solvable groups satisfying the condition (A).

A lot of results have been obtained in characterizing the alternating group A_5 by its subgroups. In 1957, M. Suzuki showed that a nonabelian simple group is isomorphic to A_5 if its maximal subgroups contatin only nilpotent proper subgroups[1]; In 1973, O. Pretzel studied a class of finite groups satisfying the condition (A_1)

Every proper subgroup is a p-group or the normalizer of a Sylow subgroup, and showed that A_5 is the only nonabelian simple group satisfying the condition (A_1)[2]; Furthermore, in 1977, A. Machi and A. Siconolfi showed that the condition (A_2)

Every proper subgroup is a p-group or the normalizer of a p-group is equivalent to (A_1)[3]. Thus another characterization of A_5 was obtained.

In this paper we will study a larger class of finite groups, this will lead to a new result characterizing A_5.

Let G be a finite group satisfying the following condition:

(A) every non-maximal proper subgroup of even order is 2-closed.

Our main purpose is to prove that A_5 is the unique nonabelian simple group satisfying the condition (A).

In this paper our notation is standard. A finite group is called inner 2-closed, if G is not 2-closed but every proper subgroup of G is 2-closed. A inner nilpotent group is called a p-elementary group, if it has a nontrivial normal Sylow p-subgroup[4]. the properties of p-elementary groups are well known.

§ 1. Proof of the main theorem

In [4], Π_Ω order solvable groups were definited, and the following theorem was

Manuscript received February 28, 1981.

proved: any inner Π_Ω order solvable group is a q-elementary group. Our proof begins with this theorem.

Lemma 1. *If a finite group G is inner 2-closed, then G is a q-elementary group, where q is an odd prime divisor of $|G|$.*

Proof First show that G is solvable. Suppose the contrary and let G be a counter-example of minimal order. It is clear that G is an inner solvable group, hence G is a simple group. It follows that there exists an element y of odd prime order in G for an involution u of G such that $y^u=y^{-1}$. Since the dihedral group $\langle y, u\rangle$ is not 2-closed, $\langle y, u\rangle=G$, contrary to the fact that G is simple.

Now suppose that 2, p and q are three distinct prime factors of $|G|$. By the solvability of G we know that G contains a p-complement N_p. Let S be a Sylow 2-subgroup of N_p, then $S\lhd N_p$. By the same reason G contains a q-complement N_q and $S\subseteq N_q$, hence $S\lhd N_q$. It follows that $S\lhd\langle N_p, N_q\rangle=G$, contrary to the fact that G is not 2-closed. Therefore G contains only two distinct prime factors, thus G is an inner Π_Ω order solvable group. By the theorem mentioned above in [4] it follows that G is a q-elementary group, then the proof is completed.

By the way we point out that if G is inner p-closed and p is an odd prime, then G is not always a q-elementary group. A_5 may be taken as a counterexample, since it is inner 5-closed but not solvable. However, it is clear from the proof of Lemma 1 that the following statement is true: if a solvable finite group is inner p-closed, then it is a q-elementary group.

Lemma 2. *Let G be a finite simple group, and M a maximal subgroup of G. If M is a p-elementary group, then $|M|=p^aq$, where p and q are distinct primes.*

Proof By the properties of p-elementary groups we have that $|M|=p^aq^b$ and a Sylow q-subgroup Q of M must be cyclic, and each proper subgroup of Q is contained in $Z(M)$. It is easy to prove $N_G(Q)\not\subseteq M$. In fact, if $N_G(Q)\subseteq M$, then Q must be a Sylow q-subgroup of G. Moreover, since Q is not normal in M, $N_G(Q)$ is a proper subgroup of M, hence it is nilpotent, this implies $Q\subseteq Z(N_G(Q))$. It follows by [6, theorem 9.9] that G has a normal p-complement, contradicting the fact that G is simple. In view of $N_G(Q)\not\subseteq M$ we now know that there exists an element t such that $t\in N_G(Q)\setminus M$. Obviously, $M^t\neq M$ and $Q\subseteq M\cap M^t$. Let Q_1 be a proper subgroup of Q, then $Q_1\subseteq Z(M)\cap Z(M^t)$, therefore $Q_1\lhd\langle M, M^t\rangle=G$, this implies $Q_1=1$. Thus Q is of prime order, i. e. $b=1$, the lemma is proved.

Lemma 3. *Let G be a nonabelian simple group satisfying the condition (A), then the following statements are true:*

(1) *every maximal subgroup of even order of G is either 2-closed or q-elementary, where q is an odd prime;*

(2) *the Sylow 2-subgroups of G have trivial intersections;*

(3) *involutions of G form a single conjugate class, and involutions of S are conjugate to each other in $N_G(S)$, where S is a Sylow 2-subgroup of G;*

(4) *let S_0 be the subgroup of S generated by involutions of S, then S_0 is an elementary abelian group and $S_0 \subseteq Z(S)$;*

(5) *let u and v be two involutions which lie in distinct Sylow 2-subgroups of G, then the dihedral group $\langle u, v\rangle$ is a p-elementary group of order $2p$, and a maximal subgroup of G. Hence p is the highest power of p dividing $|G|$, i. e., $p \| |G|$.*

Proof (1) is clear by Lemma 1.

Suppose (2) is false. So there exist two distinct Sylow 2-subgroups S and S_1 of G such that $D = S \cap S_1 \neq 1$ is a maximal intersection. Obviously, $N_G(D)$ is not 2-closed, hence $N_G(D)$ is a p-elementary group by (1). Thus D is a Sylow 2-subgroup of $N_G(D)$ by Lemma 2. It follows that $N_G(D)$ is a 2-closed group, this is a contradiction. (2) is proved.

(3) and (4) are clear by (2) and [7, Lemmas 6 and 7].

Now let us prove (5). Put $M = \langle u, v\rangle$ and $x = uv$. Clearly, $M \neq G$, and M is not 2-closed by (2), hence M is a p-elementary groug by (1). Thus we know that M is a maximal subgroup of G. It follows, by Lemma 2, that $|M| = 2p^a$, where p^a is the order of x. Let $y = x^{p^{a-1}}$, then y is of order p. Clearly, $\langle y, u\rangle$ is a p-elementary group of order $2p$. Since every proper subgroup of M is nilpotent, $M = \langle y, u\rangle$. Hence $|M| = 2p$. Finally, since the Sylow p-subgroup of M is normal in M, it is a Sylow subgroup of G. Therefore, $p \| |G|$. (5) is proved.

Lemma 4. *Let G be a nonabelian simple group satisfying the condition (A), and S a Sylow 2-subgroup of G. Let u be an involution of S. Put $H = N_G(S)$ and $U = C_G(u)$. Then*

(1) *$|H:U| = p$, where p is an odd prime;*

(2) *G contains a p-elementary group of order $2p$;*

(3) *$U \cap U^g = 1$, for all $g \in G \backslash H$.*

Proof It is clear that $U \subseteq H$ by Lemma 3(2). If $|H:U| = 1$, then $U = H$. Thus, by Lemma 3(3) we know that S contains only one involution, therefore S is either a cyclic group or a generalized quaternion group. But in both cases G is not a nonabelian simple group by [6, 5.7 and Brauer-Suzuki Theorem]. Hence $|H:U| \neq 1$.

Now let t be an involution which is not in S. Put $K = H \cap H^t$. Thus, by Lemma 3(2) and [7, Lemma 8], K is a group of odd order and K contains exactly $|H: U|$ elements which can be written as products of two involutions, one of which is t. Since $|H: U| \neq 1$, there exists an element $x \neq 1$ in K such that $x = tv$, where v is an involution. Thus $\langle t, v\rangle$ is a p-elementary group of order $2p$ and a maximal subgroup of G by Lemma 3(5). We may assert that $K = \langle x\rangle$. In fact, note that $t \in N_G(K)$, we have $\langle t, v\rangle = \langle x, t\rangle \subseteq N_G(K)$. Hence $N_G(K) = \langle x, t\rangle$, but $t \notin K$. Therefore $K = \langle x\rangle$.

This means that K is a group of order p. (1) is proved. Clearly, (2) is also proved.

It remains to prove the statement (3). Let $g\in G\backslash H$ and $x\in U\cap U^g$. We know that both u and u^g commute with x by the definition of U, hence $\langle u, u^g\rangle\subseteq C_G(x)$. On the other hand, from $g\notin H$ and Lemma 3(2) we know that u and u^g belong to distinct Sylow 2-subgroups of G, hence $\langle u, u^g\rangle$ is a q-elementary group of order $2q$ and a maximal subgroup of G. Thus $C_G(x)=\langle u, u^g\rangle$, this implies $x=1$. (3) is proved.

Lemma 5. *Let G be a nonabelian simple group satisfying the condition (A). Preserve the symobls of Lemma 4. Let V be the set of involutions which do not belong to S, and put $M_i=\langle u, v_i\rangle$ for each $v_i\in V$. Denote by T the set of all M_i, and set $|U|=2m$. Then the following statements hold.*

(1) $|T|=2m$ and T contains m elements which are p-elementary groups, and m elements which are q-elementary groups, where p and q are distinct odd primes, and $p=|H: U|$.

(2) $q\equiv 2 \mod (p)$.

Proof Set $g=|G|$. Firstly, by Lemma 3 we have

$$|V|=|G: U|-|H: U|=g/2m-p. \qquad (*)$$

Now prove (1). From Lemma 4 we know that T contains at least one element which is a p-elementary group. Suppose that T has l elements which are p-elementary groups, then the involution u must lie in l p-elementary groups. Note that u is arbitrary, we know that every involution of G must lie in l p-elementary groups. Moreover, it is easy to know that G involves $g/2p$ distinct p-elementary groups, each of which contains p involutions. Thus we get $g/2$ involutions when every involution of G is repeated l times. Clearly, among them only $g/2l$ involutions are distinct, hence the number of involutions of G, i. e., $g/2m$, must be $g/2l$. Thus we get $l=m$.

If each element of T is a p-elementary group, then $T=m$, hence $T=\{M_1, M_2, \cdots, M_{2m}\}$. Since every element of V is exactly in one M_i and all the involutions of M_i belong to V except u, by (*) we have the following equality

$$(p-1)m=g/2m-p.$$

Hence

$$2(p-1)m^2+2pm-g=0,$$

or

$$p^2+2(p-1)g=[p+2(p-1)m]^2. \qquad (**)$$

From this it is easy to know that p^2 divides $|G|$, contrary to Lemma 3(5). Thus we have proved that there exist in T q-elementary groups as well as p-elementary groups. Hence, as we prove that T contains exactly m p-elementary groups, we can prove that T has m elements which are q-elementary groups.

Now it is not difficult to complete the proof of (1). In fact, G contains $(p-1)g/2p\geqslant g/3$ elements of order p, and $(q-1)g/2q\geqslant g/3$ elements of order q. If there exists an M_i in T which is neither a p-elementary group nor a q-elementary group, then M_i must be an r-elementary group, thus G contains at least $g/3$ elements of order r. Clearly, the total number of the above three classes of elements$\geqslant g$, this is impossible, (1) is proved.

Next prove (2). By (1) we know that the total number of involutions contained in M_1, M_2, $\cdots$, and M_{2m} is $|V|$ exclusive of u, thus we have

$$(p-1)m+(q-1)m=g/2m-p.$$

Hence

$$2(p+q-2)m^2+2pm-g=0.$$

As we prove the equality (**) we can prove that there exists an integer n such that

$$p^2+2(p+q-2)g=n^2.$$

From this it is easy to know that p divides $g-2$. (2) is proved.

Now let us prove the main theorem of this paper.

Theorem 1. *The alternating group A_5 is the only nonabelian simple group satisfying the condition (A).*

Proof Preserve the symobles of Lemmas 3, 4 and 5. By Lemma 4 it is easy to know that there exist at least $|G:H|=g/2mp$ conjugates of U in G, which have trivial intersections. Hence they contain $(2m-1)g/2mp$ elements $(\neq 1)$. Moreover, G contains $(p-1)g/2p$ elements of order p and $(q-1)g/2q$ elements of order q. Thus we have the following inequality

$$(2m-1)g/2mp+(p-1)g/2p+(q-1)g/2q<g.$$

It follows that

$$1/p-1/mp-1/q<0. \qquad (*)$$

Since $q\geqslant p+2$ by Lemma 5(2), we have

$$2m<p+2.$$

On the other hand, $2m\geqslant|S_0|=2^a$ by Lemma 3(4), but Lemmas 3(3) and 4(1) imply $p=2^a-1$, hence

$$p+1\leqslant 2m.$$

From the above two inequalities we obtain

$$2m=p+1=2^a.$$

By Lemma 5(2) we have $q=kp+2$. If $k>1$, then $q>p+4$. Thus by (*) we get

$$2/p(p+4)<1/2mp.$$

Clearly, this is a contrary inequality. Hence $k=1$, i. e. $q=p+2$. Thus, from the following equality

$$pq=(2^a-1)(2^a+1)$$

we get $p=3$, $q=5$ and $a=2$.

Finaly, since $|S_0|=2^a=2m=|U|\geqslant|S|$, $S=S_0$. Hence the Sylow 2-subgroups of G are of order 4. Suppose that G contains more than three distinct prime factors. Let

r be the fourth prime factor, then by [7, Lemma 8] we know that there exist two involutions in G whose product is an element of order r. Thus the subgroup generated by them is an r-elementary group by Lemma 3, this is impossible. Theorem 1 is proved.

§ 2. Further results

In this section we prove the following

Theorem 2. *Let G be a finite group satisfying the condition (A) and suppose that G is not 2-closed, then G is a group of even order whose order contains at most three distinct prime factors and one of the following statements holds:*

(1) *G is isomorphic to A_5 or $SL(2, 5)$;*

(2) *$|G|$ contains three distinct prime factors, i. e., $|G|=2^a p^b q^c$, and $G=SO(G)$, where S is a Sylow 2-subgroup of G which has the following properties: S is cyclic and every proper subgroup of S is contained in $Z(G)$;*

(3) *$|G|$ contains two distinct prime factors, i. e. $G=2^a p^b$, and $G=SO(G)$, or $G/O_2(G)$ is a p-elementary group of order $2p^b$.*

Proof First prove (1). Suppose that G is nonsolvable, then G is an inner solvable group. Thus we have $G/\Phi(G)\cong A_5$ by Theorem 1 and [5, Theorem 4.1]. Let $M/\Phi(G)$ be a maximal subgroup of $G/\Phi(G)$, then $M/\Phi(G)$ is inner nilpotent, hence M is also inner nilpotent. It follows that $\Phi(G)\sqsubseteq Z(M)$, thus we have $\Phi(G)\sqsubseteq Z(G)$. Conversely, it is clear that $\Phi(G)\sqsupseteq Z(G)$. Hence we have $\Phi(G)=Z(G)$. It follows by [6, chap 11, exercise 3] that $G\cong SL(2, 5)$ if $Z(G)\neq 1$. (1) is proved.

Now suppose that G contains more than three distinct prime factors, then G is solvable by (1). Since the index of a maximal subgroup of a solvable group is a power of a prime, the order of each maximal subgroup of G contains at least three distinct factors. It follows by Lemma 1 that all the maximal subgroups of G are 2-closed. Again, applying Lemma 1, we know that G is 2-closed, a contradiction. This proves that $|G|$ contains at most three distinct prime factors.

Now suppose that G is solvable. First, we consider the case that $|G|$ contatins three distinct prime factors. Since G is solvable, we know that G has a Sylow base S, P and Q. Since G is not 2-closed, one of SP and SQ is not 2-closed, for example SP. Thus we know that SP is a p-elementary group by Lemma 1. Hence S is cyclic. It follows by [6, 5. 7] that $G=SO(G)$. Let S_1 be a proper subgroup of S, then $S_1O(G)$ is 2-closed, hence $S_1O(G)=S_1\times O(G)$. Since S is cyclic, we get $S_1\sqsubseteq Z(G)$. Thus (2) holds.

Next we consider the case that $|G|$ contains only two distinct prime factors, i. e. $|G|=2^a p^b$. Suppose that M is a maximal subgroup of G, and $M\lhd G$, then $|G: M|$ is a prime. If $|G: M|=p$, then the fact that G is not 2-closed implies that M is not 2-closed, hence M must be inner 2-closed, it follows that S is cyclic. Thus we have $G=SO(G)$ by [6, 5.7]. Hence (3) holds. Therefore we may assume $|G: M|=2$. If M is inner 2-closed, then the Sylow p-subgroup of M is norma lin G. Hence (3) holds also. If M is 2-closed, then $|G/O_2(G)|=2p^b$. Let $H/O_2(G)$ is a proper subgroup of $G/O_2(G)$. If $H/O_2(G)$ is not 2-closed, then H is also not 2-closed. Hence H is a p-elementary group. It follows that a Sylow 2-subgroup S of H is cyclic. Moreover, it is clear that S is also a Sylow 2-subgroup of G. This leads to $G=SO(G)$. Hence $G/O_2(G)$ is inner 2-closed when $G\neq SO(G)$. Thus it is a p-elementary group. This completes the proof of Theorem 2.

From Theorem 2 we immediately obtain the following

Corollary. *Let G be a non-solvable finite group. Suppose that all the non-maximal proper subgroups of even order are abelian, then $G=A_5$.*

It follows that. since the Sylow 2-subgroups of $SL(2, 5)$are quaternion groups by [6, Theorem 11.8], they are nonabelian.

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