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PROPERTIES OF CHARACTERISTIC FUNCTIONS AND EXISTENCE OF LIMIT CYCLES OF LIÉNARD'S EQUATION

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Abstract

In this paper, the author considers Liénard's equation, studies the properties of so-called characteristic functions and gives three theorems which ensure that the equation has at least one limit cycle. The theorems generlize Filippov's Theorem which is a repesentative result[1], Dragilev's Theorem[5], Theorem 1, 2 of [6], Theorem 9 of [8] and Theorems 1, 2 of [9] respectively.

Until now, Filippov's Theorem[4] is still the most general and representative result ensuring that Liènard's equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0$$

or its equivalent system

$$\dot{x}=y-F(x),\quad \dot{y}=-g(x) \tag{*}$$

$$\left(F(x)=\int_0^x f(\xi)d\xi\right)$$

has at least one limit cycle, this result is said to have the least conditions[1,2,3]. In this paper, we study the properties of so-called characteristic functions and give three existence theorems for limit cycles of (*). These theorems generalize Filippov's Theorem, Dragilev's Theorem[5], Theorems 1、2 of [6], Theorem 9 of [8] and Theorems 1、2 of [9].

We suppose that $f(x)$ and $g(x)$ are continuous on $(-\infty, +\infty)$, and $xg(x)>0$, for $x\neq 0$; $\int_0^{\pm\infty} g(x)dx=+\infty$.

We write $G(x)=\int_0^x g(\xi)d\xi$ and use Filippov's Transformation[4]:

when $x\geqslant 0$, let $z=z_1(x)=G(x)$, it has an inverse function $x=x_1(z)$, therefore $F(x)=F(x_1(z))=F_1(z)$;

when $x\leqslant 0$, let $z=z_2(x)=G(x)$, it has an inverse function $x=x_2(z)$, therefore $F(x)=F(x_2(z))=F_2(z)$.

Thus, the trajectories of (*) on righ-half (x, y) plane are transformed into integral curves of equation

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$$\frac{dz}{dy}=F_1(z)-y \tag{1}$$

on righ-half (z, y) plane; trajectories of $(*)$ on left-half (x, y) plane are transformed into integral curves of equation

$$\frac{dz}{dy}=F_2(z)-y \tag{2}$$

on righ-half (z, y) plane.

On righ-half (z, y) plane, except the rigion $(0, 0)$, both equations (1) and (2) satisfy the conditions of existence and uniqueness of solutions with initial condition[1,2,3].

On righ-half plane $z\geqslant 0$, $y=F(z)$ is called the characteristic function of equation

$$\frac{dz}{dy}=F(z)-y. \tag{3}$$

Obviously, we have[1,2,3,4]: if $F(z)$ is continuously differentiable and $F(0)=0$, then the trajectory of (3) passing through point $(z_0, F(z_0))$ on the characteristic curve must interect $y-$axis at two points A and B, where either $y_A<0$, $y_B\geqslant 0$, or $y_A\leqslant 0$, $y_B>0$; besides, when $z_0\to+\infty$, either y_A decreases monotonely and y_B does not decrease, or y_A does not increase and y_B increases monotonely.

For convenience, the trajectory passing through point $(z_0, F(z_0))$ is called z_0 characteristic trajectory of (3) or $F(z)$; and the two points at which the z_0 characteristic trajectory intersects $y-$axis are called characteristic points of the z_0 characteristic trajectory, where the upper (lower) point is called the z_0 upper (lower) characteristic point; it can be proved that the set of all characteristic points is an interval, and we call it the characteristic interval of $F(z)$.

We define:

1. If point $(0, F(0))$ is between the upper and lower characteristic points, we agree that this point is also a characteristic point; if characteristic points are at one-side of point $(0, F(0))$, we agree that this point is not a characteristic point. Then, it can be proved that characteristic inteaval is an open interval.

2. Unless special explanation, we always suppose that the functions in the next part are continuously differentiable with respect to z and are equal to zero at $z=0$.

By Poincaré-Bendixson's annular region Theorem and the proof of Filippov's Theorem, we have the following criterion.

Criterion:

If i) there exist numbers $z_{10}>0$, $z_{20}>0$ such that the z_{10} upper and lower characteristic points B_{10} and A_{10} of $F_1(z)$ are not above the z_{20} upper and lower characteristic points B_{20} and A_{20} of $F_2(z)$ respectively;

ii) there exist numbers $z_{11}>z_{10}$, $z_{21}>z_{20}$ such that the z_{11} upper and lower characteristic points B_{11} and A_{11} of $F_1(z)$ are not below the z_{21} upper and lower characteristic points B_{21} and A_{21} of $F_2(z)$ respectively, then (*) has at least one limit cycle.

§ 1. Properties of Characteristic Function

Let $m=\min\limits_{[0,z_0]} F(z)$, $M=\max\limits_{[0,z_0]} F(z)$. For equation (3), $\frac{dz}{dy}=m-y$ and $\frac{dz}{dy}=M-y$, using the comparison theorem and the method of [9], we obtain

Lemma 1. *Suppose that* $z_0>0$, $H_0\geqslant F(z_0)$ $(H_0\leqslant F(z_0))$ *and that the upper (lower) intersection point of y-axis with the trajectory passing through point* (z_0, H_0) *of* (3) *is* $B(A)$, *then we have*

$$\max\{M,\ m+\sqrt{(m-H_0)^2+2z_0}\}\leqslant y_B\leqslant M+\sqrt{2z_0} \tag{4}$$

for $m\leqslant H_0\leqslant M$;

$$m+\sqrt{(m-H_0)^2+2z_0}\leqslant y_B\leqslant M+\sqrt{(M-H_0)^2+2z_0} \tag{5}$$

for $H_0>M$

$$m-\sqrt{2z_0}\leqslant y_A\leqslant \min\{m,\ M-\sqrt{(M-H_0)^2+2z_0}\} \tag{6}$$

for $H_0\geqslant m$;

$$m-\sqrt{(m-H_0)^2+2z_0}\leqslant y_A\leqslant M-\sqrt{(M-H_0)^2+2z_0} \tag{7}$$

for $H_0<m$.

Lemma 2. *Suppose that* $F_1(z)\leqslant F_2(z)$ *and that the characteristic intervals of* $F_i(z)$ *are* (a_i, b_i) $(i=1, 2)$, *then* $a_1\leqslant a_2$, $b_1\leqslant b_2$.

Using the method in [6], we obtain

Lemma 3. *Let the characteristic interval of* $F(z)$ *be* (a, b). *If there exists number* $b_0>0$ $(a_0<0)$ *such that*

$$\inf_{[0,+\infty)}\left[b_0-F(z)+\int_0^z\frac{dz}{F(z)-b_0}\right]<0 \tag{8}$$

$$\left(\sup_{[0,+\infty)}\left[a_0-F(z)+\int_0^z\frac{dz}{F(z)-a_0}\right]>0\right), \tag{9}$$

then $b_0<b$ $(a<a_0)$.

A special case of Lemma 3 in [8] is the following lemma.

Lemma 4. *Let the characteristic interval of* $F(z)$ *be* (a, b). *If there exist constants* $C>0$, $L>0$ $(D>0,\ N>0)$ *such that*

$$F(z)+C>0 \quad (F(z)-D<0),$$

$$\lim_{z\to+\infty}\int_0^z\frac{dz}{F(z)+C}\leqslant L \tag{10}$$

$$\left(\lim_{z\to+\infty}\int_0^z\frac{dz}{F(z)-D}\geqslant -N\right), \tag{11}$$

then $-(C+L)>a$ $(b<D+N)$.

In Lemma 1, let $H_0=F(z_0)$, we can easily obtain

Lemma 5. *The characteristic interval of $F(z)$ can only have the following three types*

$$(a, +\infty);\ (-\infty, b);\ (-\infty, +\infty),$$

where $-\infty<a\leqslant 0,\ 0\leqslant b<+\infty$.

Example 1. The characteristic interval of $F(z)=C$ (constant) is $(-\infty, +\infty)$.

Example 2. (See [1]) The characteristic interval of $F(z)=a\sqrt{z}$ is as follows: $(-\infty, 0)$, for $a<-\sqrt{8}$; $(-\infty, +\infty)$, for $|a|<\sqrt{8}$; $(0, +\infty)$, for $a>\sqrt{8}$.

Example 3. (See [7, 9]) The characteristic interval of $F(z)=az$ is as follows: $\left(-\infty, -\frac{1}{a}\right)$, for $a<0$; $(-\infty, +\infty)$, for $a=0$; $\left(-\frac{1}{a}, +\infty\right)$, for $a>0$.

Example 4. The characteristic interval of $F(z)=Az^k\sin z$ ($|A|<+\infty$, $k>0$) is $(-\infty, +\infty)$.

Example 5. Let the characteristic interval of $F(z)=Az^k\sin^2 z$ ($A>0$, $k>2$) be (a, b), then $-\left(1+\pi\sum\limits_{n=0}^{+\infty}\frac{1}{\sqrt{1+A[n\pi]^k}}\right)<a<0$, $b=+\infty$.

In fact, in $[n\pi, (n+1)\pi]$, we have

$$\frac{1}{1+A[(n+1)\pi]^k\sin^2 z}\leqslant\frac{1}{1+Az^k\sin^2 z}\leqslant\frac{1}{1+A[n\pi]^k\sin^2 z}.$$

Using equality

$$\int_{n\pi}^{(n+1)\pi}\frac{dz}{1+B\sin^2 z}=\int_0^{\pi}\frac{dz}{1+B\sin^2 z}=\frac{\pi}{\sqrt{1+B}},$$

we can obtain

$$\sum_{n=0}^{+\infty}\frac{\pi}{\sqrt{1+A[(n+1)\pi]^k}}\leqslant\int_0^{+\infty}\frac{dz}{1+A^k\sin^2 z}\leqslant\sum_{n=0}^{+\infty}\frac{\pi}{\sqrt{1+A[n\pi]^k}}.$$

For $k>2$, series $\sum\limits_{n=0}^{+\infty}\frac{\pi}{\sqrt{1+A[n\pi]^k}}$ is convergent. Thus, from Lemma 3, we have $-\left(1+\pi\sum\limits_{n=0}^{+\infty}\frac{1}{\sqrt{1+A[n\pi]^k}}\right)<a<0$.

Example 6. Let the characteristic interval of $F(z)=Az^2$ ($A>0$) be (a, b), then $-\left[\left(\frac{1}{4}\right)^{\frac{2}{3}}+\frac{1}{2}\right]\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}\leqslant a\leqslant-\frac{1}{4}\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}$, $b=+\infty$.

In fact, from Lemma 3, if $0<\varepsilon<\frac{1}{2}\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{1}{3}}$, then we have

$$\sup_{[0,+\infty)}\left[-\varepsilon^3-Az^2+\int_0^z\frac{dz}{Az^2+\varepsilon^2}\right]\geqslant\left[-\varepsilon^2-Az^2+\frac{1}{\varepsilon\sqrt{A}}\operatorname{arc\,tg}\frac{\sqrt{A}z}{\varepsilon}\right]\Bigg|_{z=\frac{\varepsilon}{\sqrt{A}}}$$

$$\geqslant-2\varepsilon^2+\frac{\pi}{4\sqrt{A}\varepsilon}>0.$$

Thus, $a\leqslant\inf\{-\varepsilon^2\}=-\frac{1}{4}\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}$. Besides, from Lemma 4, we have

$$\lim_{z\to+\infty}\int_0^z\frac{dz}{Az^2+C^2}=\lim_{z\to+\infty}\frac{1}{C\sqrt{A}}\operatorname{arc\,tg}\frac{\sqrt{A}z}{C}\leqslant\frac{\pi}{2\sqrt{AC}},$$

where C is a positive number. Therefore $-\left(C^2+\frac{\pi}{2\sqrt{AC}}\right)<a$, it follows that $C^2+\frac{\pi}{2\sqrt{AC}}\geqslant\left[\left(\frac{1}{4}\right)^{\frac{2}{3}}+\frac{1}{2}\right]\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}$ and we obtain $-\left[\left(\frac{1}{4}\right)^{\frac{2}{3}}+\frac{1}{2}\right]\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}\leqslant a$.

Theorem 1. *Suppose $F(z)$ satisfies at least one of the following conditions:*

i) $\inf_{[0,+\infty)} F(z)=-\infty$ $\left(\sup_{[0,+\infty)} F(z)=+\infty\right)$;

ii) $\sup_{[0,+\infty)} F(z)<+\infty$ $\left(\inf_{[0,+\infty)} F(z)>-\infty\right)$;

iii) $\inf\{a_0|a_0\in(9)\}=-\infty$ $(\sup\{b_0|b_0\in(8)\}=+\infty)$;

iv) *the upper (lower) bound of characteristic interval of $F(z)$ is finite;*

v) *there exist $\Delta>0$, $F_0(z)$ such that the lower (upper) bound of characteristic interval of $F_0(z)$ is $-\infty(+\infty)$ and $F(z)\leqslant F_0(z)$ $(F(z)\geqslant F_0(z))$ for $z>\Delta$, then the lower (upper) bound of characteristic interval of $F(z)$* is $-\infty(+\infty)$.

Theorem 2. *Suppose $F(z)$ satisfies at least one of the following conditions:*

i) *there exist numbers $\Delta>0$, $K>0$ and H such that $F(z)\geqslant Kz+H$ $(F(z)\leqslant-Kz+H)$* for $z>\Delta$;

ii) *there exist numbers $\Delta>0$, $a\geqslant\sqrt{8}$ and H such that $F(z)\geqslant a\sqrt{z}+H$ $(F(z)\leqslant-a\sqrt{z}+H)$* for $z>\Delta$;

iii) *there exist numbers $\Delta>0$, $A>0$ and H such that $F(z)\geqslant Az^2+H$ $(F(z)\leqslant-Az^2+H)$* for $z>\Delta$;

iv) *$F(z)$ satisfies formula* (10) *in Lemma* 4 *(formula* (11) *in Lemma* 4);

v) *there exist $\Delta>0$ and $F_0(z)$ such that the lower (upper) bound of characteristic interval of $F_0(z)$ is finite and $F(z)\geqslant F_0(z)$ $(F(z)\leqslant F_0(z))$ for $z>\Delta$, then the lower (upper) bound of characteristic interval of $F(z)$ is finite.*

For $F_1(z)$ and $F_2(z)$ of (*), their characteristic intervals have nine groups of combinations, we list them in the following table:

Table 1. $-\infty<a_i\leqslant0,\ 0\leqslant b_i<+\infty,\ i=1, 2.$

group	the characteristic interval of $F_1(z)$	the characteristic interval of $F_2(z)$
1	$(-\infty, +\infty)$	$(-\infty, +\infty)$
2	$(-\infty, +\infty)$	$(-\infty, b_2)$
3	$(a_1, +\infty)$	$(-\infty, +\infty)$
4	$(a_1, +\infty)$	$(-\infty, b_2)$
5	$(a_1, +\infty)$	$(a_2, +\infty)$
6	$(-\infty, b_1)$	$(-\infty, b_2)$
7	$(-\infty, +\infty)$	$(a_2, +\infty)$
8	$(-\infty, b_1)$	$(-\infty, +\infty)$
9	$(-\infty, b_1)$	$(a_2, +\infty)$

By Theorems 1, 2 and with the lemmas and the examples mentioned above, we can determine when a certain group in Table 1 happens. When the 5^{th} and 6^{th} groups happen, we can give some substantial conditions to determine the bounds, for example, we shall give four corollaries of Theorem 5 later.

§ 2. Existence of Limit Cycles

Let $m_{ij}=\min\limits_{[0,z_{ij}]} F_i(z)$, $M_{ij}=\max\limits_{[0,z_{ij}]} F_i(z)$ $(i=1, 2; j=0, 1)$. For all the nine cases in Table 1, by the method of Filippov's Theorem, Lemma 1 and the preceding Criterion, we have

Theorem 3. *Suppose* i) $F_1(z)$ *and* $F_2(z)$ *satisfy one of the following conditions:*

1) *there exists a number* $\delta>0$ *such that*

$$F_1(z)\leqslant F_2(z),\ F_1(z)\not\equiv F_2(z),\ F_1(z)\leqslant a\sqrt{z},\ F_2(z)\geqslant -a\sqrt{z},\ 0<a<\sqrt{8},$$

for $0<z<\delta$;

2) *there exist numbers* $z_{10}>0$, $z_{20}>0$ *such that*

$$M_{10}+\sqrt{2z_{10}}\leqslant\max\{M_{20},\ m_{20}+\sqrt{(m_{20}-F_2(z_{20}))^2+2z_{20}}\},$$
$$m_{20}-\sqrt{2z_{20}}\geqslant\min\{m_{10},\ M_{10}-\sqrt{(M_{10}-F_1(z_{10}))^2+2z_{10}}\};$$

ii) *there exist numbers* $z_{11}>\max\{\delta, z_{10}\}$, $z_{21}>\max\{\delta, z_{20}\}$ *such that*

$$M_{21}+\sqrt{2z_{21}}\leqslant\max\{M_{11},\ m_{11}+\sqrt{(m_{11}-F_1(z_{11}))^2+2z_{11}}\},$$
$$m_{11}-\sqrt{2z_{11}}\geqslant\min\{m_{21},\ M_{21}-\sqrt{(M_{21}-F_2(z_{21}))^2+2z_{21}}\},$$

then (*) *has at least one limit cycle.*

Theorem 3 and Filippov's Theorem do not contain each other, the condition i) in Theorem 3 is more general than the condition for the interior boundary of Filippov's Theorem.

Corollary 3.1. *Suppose that there exist numbers* $z_{11}>z_{10}>0$ *and* $z_{21}>z_{20}>0$ *such that*

i) $M_{10}+\sqrt{2z_{10}}\leqslant M_{20},\ m_{20}-\sqrt{2z_{20}}\geqslant m_{10}$;

ii) $M_{21}+\sqrt{2z_{21}}\leqslant M_{11},\ m_{11}-\sqrt{2z_{11}}\geqslant m_{21}$,

then (*) *has at least one limit cycle.*

For the cases of the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} groups in Table 1, we obtain

Theorem 4. *Suppose* i) *condition* i) *in Theorem* 3 *is satisfied;*

ii) *there exists a number* $\Delta>\max\{\delta, z_{10}, z_{20}\}$ *such that* $\int_0^\Delta (F_1(z)-F_2(z))dz>0$ *and* $F_1(z)\geqslant F_2(z)$ *for* $z>\Delta$;

iii) *the upper bound of characteristic interval of* $F_1(z)$ *is* $+\infty$, *and the lower bound of characteristic interval of* $F_2(z)$ *is* $-\infty$,

then (*) *has at least one limit cycle.*

By Example 2, Lemma 2 and Theorem 2, we easily know that Filippov's Theorem is a special case of Theorem 4. By Lemma 3 and Theorem 3, applying the transformation: $x=-X$, $y=Y$, $t=-\tau$[3] and Filippov's Transformation upon (*), we can know that Theorems 1, 2 in [6] are special cases of this theorem. By Lemma 3, Theorem 9 in [8] is also one of its special cases.

Example **7.**

Let

$$F_1(z)=\begin{cases} A\sin^2 z, & 0\leqslant z<\frac{\pi}{2},\ A>1, \\ A+\left(z-\frac{\pi}{2}\right)\sin\left(z-\frac{\pi}{2}\right), & z>\frac{\pi}{2}, \end{cases}$$

$$F_2(z)=\begin{cases} \sin z, & 0\leqslant z<\frac{\pi}{2}, \\ 1+\left(z-\frac{\pi}{2}\right)\sin\left(z-\frac{\pi}{2}\right), & z>\frac{\pi}{2}, \end{cases}$$

then (*) has at least one limit cycle.

Example 7 satisfies the conditions in Theorem 4, but does not satisfy the conditions in Filippov's Theorem.

Corollary 4.1. *Suppose* i) *conditions* i) *and* ii) *in Theorem* 4 *are satisfied;*

ii) *for* $z>\Delta$, *there are*

$$F_1(z)\geqslant a\sqrt{z}+H\geqslant F_2(z),\quad |a|<\sqrt{8},\quad |H|<+\infty,$$

then (*) *has at least one limit cycle.*

For $a=0$, Corollary 4.1 generalizes Dragilev's Theorem[5].

Corollary 4.2. *Suppose* i) *conditions* i) *and* ii) *in Theorem* 4 *are satisfied;*

ii) $|F_1(z)|<+\infty$, $|F_2(z)|<+\infty$; *or* $\sup F_1(z)=+\infty$, $\inf F_2(z)=-\infty$, *for* $z>\Delta$,

then (*) *has at least one limit cycle.*

Corollary 4.3. *Suppose* i) *conditions* i) *and* ii) *in Theorem* 4 *are satisfied;*

ii) *for* $F_1(z)$ *and* $F_2(z)$, *a certain case of the* 2nd, 3rd *and* 4th *groups in Table* 1 *happens,*

then (*) *has at least one limit cycle.*

For the cases of the 5th and 6th groups in Table 1, we obtain

Theorem 5. *Suppose* i) *condition i) in Theorem* 4 *is satisfied;*

ii) *the lower (upper) bound* $a_i(b_i)$ *of characteristic interval of* $F_i(z)$ $(i=1, 2)$ *is finite; and* $a_1>a_2$ $(b_1>b_2)$, *then* (*) *has at least one limit cycle.*

Theorem 5 and Filippov's Theorem do not contain each other. Theorems 1, 2 in [9] are special cases of this theorem.

Let $m_1=\min\limits_{[0,\Delta]} F_1(z)$, $m_2^*=\min\limits_{[0,z^*]} F_2(z)$, and $M_2^*=\max\limits_{[0,z^*]} F_2(z)$. For the 5th group in Table 1, By Lemma 1, Theorems 1, 2 and the method in [9], we obtain

Corollary 5.1. *Suppose* i) *condition* i) *in Theorem* 3 *is satisfied;*

ii) *there exists number* $\Delta>\max\{\delta, z_{10}, z_{20}\}$ *such that* $F_1(z)\geqslant K(z-\Delta)+H$, $K>0$, $|H|<+\infty$, *for* $z>\Delta$;

iii) *there exists number* $z^*>z_{20}$ *such that*

$$m_1-\sqrt{2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},\ \text{for}-\frac{1}{K}+H\geqslant m_1;$$

$$m_1-\sqrt{\left(m_1+\frac{1}{K}-H\right)^2+2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},$$

for $-\frac{1}{K}+H<m_1$,

then (*) *has at least one limit cycle.*

Corollary 5.2. *Suppose* i) *condition* i) *in Theorem* 3 *is satisfied;*

ii) *there exists a number* $\Delta>\max\{\delta, z_{10}, z_{20}\}$ *such that*

$$F_1(z)\geqslant a\sqrt{z-\Delta}+H,\quad a\geqslant\sqrt{8},\quad |H|<+\infty,\quad \text{for } z>\Delta;$$

iii) *there exists a number* $z^*>z_{20}$ *such that*

$$m_1-\sqrt{2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},\ \text{for } H\geqslant m_1;$$

$$m_1-\sqrt{(m_1-H)^2+2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},$$

for $H<m_1$,

then (*) *has at least one limit cycle.*

Corollary 5.3. *Suppose* i) *condition* i) *in Theorem* 3 *is satisfied;*

ii) *there exists anumber* $\Delta>\max\{\delta, z_{10}, z_{20}\}$ *such that* $F_1(z)\geqslant A(z-\Delta)^2+H$, $A>0$, $|H|<+\infty$, *for* $z>\Delta$;

iii) *there exists a number* $z^*>z_{20}$ *such that*

$$m_1-\sqrt{2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},$$

for $-\left[\left(\frac{1}{4}\right)^{\frac{2}{3}}+\frac{1}{2}\right]\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}+H\geqslant m_1$;

$$m_1-\sqrt{\left(m_1+\frac{1}{4}\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}-H\right)^2+2\Delta}>\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\},$$

for $-\frac{1}{4}\left(\frac{\pi}{\sqrt{A}}\right)^{\frac{2}{3}}+H<m_1$,

tnen (*) *has at least one limit cycle.*

The three corollaries of Theorem 5 can be used in the cases of the 3rd and 4th groups in Table 1. In these corollaries, conditions ii) and iii) about the exterior boundary of the annular region never contain conditions "$\int_0^\Delta(F_1(z)-F_2(z))dz>0$" and "$F_1(z)\geqslant F_2(z)$ for $z>\Delta$". In conditions iii) of these corollaries, "$\min\{m_2^*,\ M_2^*-\sqrt{(M_2^*-F_2(z^*))^2+2z^*}\}$" can be substituted by "$\min\{m_2^*,\ F_2(z^*)\}$" or "$\min\{m_2^*,\ M_2^*-\sqrt{2z^*}\}$".

Example 8. Let

$$F_1(z)=z,\ F_2(z)=\begin{cases} A\sin z, & 0\leqslant z<\frac{3\pi}{2},\quad A>1, \\ -A+\left(z-\frac{3\pi}{2}\right)^3\sin^2\left(z-\frac{3\pi}{2}\right), & z\geqslant\frac{3\pi}{2}, \end{cases}$$

then (*) has at least one limit cycle.

Example 8 satisfies the conditions of Theorem 5, but does not satisfy the conditions of Filippov's Theorem.

Corollary 5.4. *Suppose* i) *condition* i) *in Theorem* 3 *is satisfied;*

ii) *there exist numbers* $C>0, L>0$ *such that* $F_1(z)$ *satisfies formula* (10) *of Lemma* 4, *there exists a number* $a_0<0$ *such that* $F_2(z)$ *satisfies formula* (9) *of Lemma* 3; *and* $a_0<-(C+L)$,

then (*) *has at least one limit cycle.*

Example 9. Let $F_1(z)=A_1z^4\sin^2 z$, $F_2(z)=A_2z^2$ and

$$-\left(1+\pi\sum_{n=0}^{+\infty}\frac{1}{\sqrt{1+A_1[n\pi]^4}}\right)>-\frac{1}{4}\left(\frac{\pi}{\sqrt{A_2}}\right)^{\frac{2}{3}},\ A_1>0,\ A_2>0,$$

then (*) has at least one limit cycle.

Example 9 satisfies the conditions of Corollary 5.4, but does not satisfy the conditions of Filippov's Theorem.

For the 6^{th} group in Table 1, we can obtain analogous corollaries.

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Reference

[1] 叶彦谦,极限环论,上海科技出版社, 1965.
[2] 张芷芬,常微分方程定性理论,北京大学, 1979.
[3] 张芷芬,关于 Liénard 方程的极限环问题,东北师大, 1980.
[4] Филиппов, А. Ф., Достаточное условие существования устойчивого предельного чикла для уравнения второго порядка, *Матем. СБ.*, **30** (70), 1952.
[5] Драгилёв, А. В., Периодические решения дифференциального уравнений нелинейных колебаний, *Прикладкая мат-ка и мех-ка*, **XVI**, B.1, 1952.
[6] Железнов, И. Й., Некоторые достаточиные условия существования пределеных циклов, *Известия высших учебных заведений, Мат-ка*, №1 (2), 1958.
[7] 佘澍祥,证明极限环存在唯一性的 Филиппов 方法,数学学报, **14**: 3 (1964).
[8] 丁大正, Liénard 方程极限环的存在性,全国常微分方程定性理论学术会北京大学论文集, 1981.
[9] 陈秀东、黄启昌,两个 Liénard 方程极限环存在性定理,科学通报, **26**: 13 (1981); (全文),东北师大学报(自然科学版), 1982, 第一期.