Chin. Ann. of math.
4B (2) 1983

NUMBER OF LIMIT CYCLES OF THE BOUNDED QUADRATIC SYSTEM WITH TWO FINITE SINGULAR POINTS

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Abstract

In this paper, the author proves the bounded quadratic system with two singular points at finite which corresponds to figures 12(a), 12(b), and 13(b) in [1] has at most one limit cycle, and shows under what conditions the limit cycle exists.

The aim of this paper is to investigate the existence and the number of limit cycles for figures 12(a), 12(b) and 13(b) in [1] (as follows).

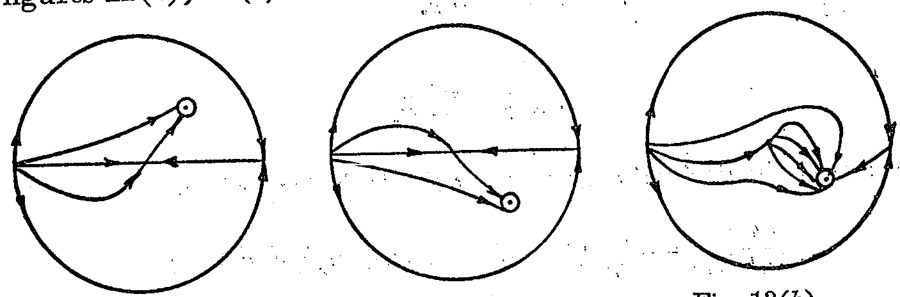

Fig. 12(a) Fig. 12(b) Fig. 13(b)

We shall show that in all these cases the system has at most one limit cycle.

For the sake of simplicity, let us consider the quadratic system

$$\frac{dx}{dt}=-y+\delta x+lx^2+mxy+ny^2,\quad \frac{dy}{dt}=x(1+ax+by). \tag{1}$$

By means of the method used in [1], we have:

Theorem 1. *All solutions of system* (1) *are bounded for* $t\geqslant 0$ *iff one of the following conditions holds*:

1) $n=0,\ (b-l)^2+4ma<0,\ mb<0;$

2) $n=0,\ (b-l)^2+4ma<0,\ b=m+a=0,\ m(m\delta+l)\leqslant 0,\ m\neq 0;$

3) $n=m=0,\ b=l,\ ab>0.$

(The proof is omitted here.)

Thus, for the case which has two finite singular points, it is sufficient to consider the system

$$\frac{dx}{dt}=-y+\delta x+lx^2+mxy,\quad \frac{dy}{dt}=x(1+ax+by), \tag{2}$$

Manuscript received March 22, 1981. revised october 25, 1982.

where

$$mb<0,\ (b-l)^2+4ma<0,\ (b\delta+a-m)^2=4(lb-ma), \tag{3}$$

or

$$m=0,\ b=l,\ ab>0,\ (a+b\delta)^2=4lb.$$

According to the behavior of the paths for system (2) in a neighbourhood of a higher order singular point, it is easy to see that the case $a+b\delta+m=0$ is corresponding to the figures 12(a), (b) in [1], and the case $B[a(a+b\delta+m)+2b^2]<0$ is corresponding to the figure 13(b) in [1], where $B=a+b\delta-m$.

Observe that if $|\delta|\geq 2$, then the unique elementary singular point of system (2) is a node and then that system has no limit cycle. Thus as the discussion confine to the existence and the number of the limit cycle for system (2), we may assume $|\delta|<2$. Under this condition, if $a(a+b\delta+m)+2b^2\neq 0$, we have $B>0$. Otherwise we translate the origin to the singular point (x_0, y_0), where $x_0=-\frac{2}{B}$, $y_0=\frac{2a-B}{bB}$, and system (2) becomes

$$\begin{aligned} \frac{d\bar{x}}{dt}&=-\frac{a}{bB}(a+b\delta+m)\bar{x}-\frac{1}{B}(a+b\delta+m)\bar{y}+l\bar{x}^2+m\bar{x}\bar{y},\\ \frac{d\bar{y}}{dt}&=-\frac{2a}{B}\bar{x}-\frac{2b}{B}\bar{y}+a\bar{x}^2+b\bar{x}\bar{y}. \end{aligned} \tag{3*}$$

Suppose

$$v=\bar{y}-\frac{y_0}{x_0}\bar{x},$$

it is easy to see that the derivative of v along the trajectory of system (3*) is

$$\left.\frac{dv}{dt}\right|_{v=0}=\frac{B}{8b^2}[(4-\delta^2)b^2+(a+m)^2]\bar{x}(\bar{x}+x_0)$$

and

$$\left.\frac{d\bar{x}}{dt}\right|_{\bar{x}=0}=-\frac{1}{B}(a+b\delta+m)\bar{y},\quad \left.\frac{d\bar{y}}{dt}\right|_{\bar{y}=0}=a\bar{x}(\bar{x}+x_0).$$

Without lost of generality, we may assume $a>0$. Here note that if $B<0$, then $x_0>0$, and there is a separatrix of the system (3*) as follows, which is impossible, and hence $B>0$.

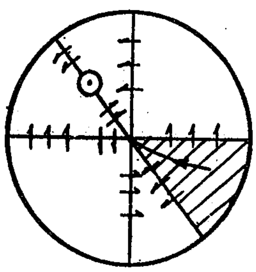

Fig.

Thus the inequality $B[a(a+b\delta+m)+2b^2]<0$ is equivalent to $B>0$, $a(a+b\delta+m)+2b^2<0$, if $|\delta|<2$.

So it is sufficient to consider the following two case:

case I: $a(a+b\delta+m)+2b^2<0$, $B>0$ ($|\delta|<2$),

case II: $a+b\delta+m=0$.

Now it is easy to see that in such two cases $m\neq 0$.

Without lost of generality, we may assume $b>0$, otherwies, we may put $-y$ and $-x$ instead of y and x respectively, in system (2). Hence $m<0$, and by the similarity transformation, we can reduce $m=-1$.

Thus system (2) can be transformed into the form

$$\frac{dx}{dt}=-y+\delta x+lx^2-xy,\quad \frac{dy}{dt}=x(1+ax+by), \tag{4}$$

and condition (3) into the form

$$b>0,\ (b-l)^2<4a,\quad (a+b\delta+1)^2=4(lb+a). \tag{5}$$

and the two cases which have been mentioned above are as follows:

case I: $\quad a(a+b\delta-1)+2b^2<0,\ B=a+b\delta+1>0,\ \ (|\delta|<2)$ (6)

case II: $\quad a+b\delta-1=0.$ (7)

§ 1. Limit cycles of case I

Theorem 2. *There is exactly one limit cycle for system (4) which satisfies conditions (5), (6) and $0<\delta<2$.*

Proof It is easy to see that $O(0,\ 0)$ is an unstable focus, and there exists at least one limit cycle surrounding it. We shall prove the uniqueness under conditions (5) and (6).

Let

$$x=x,\quad p=lx^2+\delta x-(1+x)y,$$

then system (4) becomes

$$\frac{dx}{dt}=p,$$

$$\frac{dp}{dt}=-x(1+ax)(1+x)-bx(lx^2+\delta x)+\left[\delta+(2l+b)x-\frac{x(lx+\delta)}{1+x}\right]p+\frac{p^2}{1+x},$$

let again

$$x=x,\quad u=\frac{p}{1+x}$$

we have

$$\frac{dx}{dt}=u(1+x),\quad \frac{du}{dt}=-x(1+ax)-\frac{bx^2(lx+\delta)}{1+x}+\left[\delta+(2l+b)x-\frac{x(lx+\delta)}{1+x}\right]u, \tag{8}$$

finally, we let

$$x=x,\ y=-u+\int_0^x\frac{\delta+(2l+b)s+(l+b)s^2}{(1+s)^2}ds,\quad \frac{dt}{d\tau}=\frac{1}{1+x},$$

then we have

$$\frac{dx}{d\tau}=-y-F(x),\quad \frac{dy}{d\tau}=g(x), \tag{9}$$

where

$$g(x)=\frac{x}{(1+x)^2}\left(1+\frac{a+b\delta+1}{2}x\right)^2,\quad F(x)=-\int_0^x\frac{\delta+(2l+b)s+(l+b)s^2}{(1+s)^2}ds.$$

It is evident that $xg(x)>0,\ \ x\neq0$ and if $f(x)=F'(x)=-\dfrac{\delta+(2l+b)x+(l+b)x^2}{(1+x)^2}$, then $f(0)=-\delta<0$. Furthermore

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{1}{g^2}[f'(x)g(x)-f(x)g'(x)]=\frac{1}{g^2}\frac{1+\frac{a+b\delta+1}{2}x}{(1+x)^4}w(x),$$

where

$$w(x)=\frac{1}{2}(l+b)(a+b\delta+1)x^3+[l(a+b\delta+1)+(l+b)(a+b\delta)]x^2$$

$$+\frac{3}{2}\delta(a+b\delta+1)x+\delta. \tag{10}$$

From (5) we have

$$a>0,\quad (a+b\delta-1)^2=4b(l-\delta), \tag{11}$$

hence

$$l>\delta>0,$$

and from (6), we have

$$a+b\delta-1<0$$

and since $a+b\delta+1>0$, we have

$$-\frac{2}{a+b\delta+1}<-1.$$

It is easy to verify that the straight line $x=-1$ is a line without contact, and limit cycles, if exist, must lie in the domain $x>-1$, and then could not have any separatrix which is terminating and beginning on non-elementary singular point.

Since

$$[\delta+(2l+b)x+(l+b)x^2]_{x=-\frac{2}{a+b\delta+1}}=\delta-\frac{2(2l+b)}{a+b\delta+1}+\frac{4(l+b)}{(a+b\delta+1)^2}$$

$$=-\frac{a+b\delta-1}{b(a+b\delta+1)}[a(a+b\delta-1)+2b^2]<0,$$

we have

$$4(l+b)-2(2l+b)(a+b\delta+1)+\delta(a+b\delta+1)^2<0,$$

or

$$4(l+b)-2(2l+b)(a+b\delta+1)+4\delta(lb+a)<0,$$

hence

$$-\delta(lb+a)>l+b-\frac{1}{2}(2l+b)(a+b\delta+1),$$

$$-2\delta(lb+a)>b-2l(a+b\delta)-b(a+b\delta),$$

$$b(a+b\delta-1)>-2a(l-\delta). \tag{12}$$

From (10), we have

$$w'(x)=\frac{3}{2}(l+b)(a+b\delta+1)x^2+2[l(a+b\delta+1)+(l+b)(a+b\delta)]x$$

$$+\frac{3}{2}\delta(a+b\delta+1),$$

and the roots of equation $w'(x)=0$ are

$$x_1=\frac{-2[l(a+b\delta+1)+(l+b)(a+b\delta)]-\sqrt{\Delta}}{3(l+b)(a+b\delta+1)}$$

and

$$x_2=\frac{-2[l(a+b\delta+1)+(l+b)(a+b\delta)]+\sqrt{\Delta}}{3(l+b)(a+b\delta+1)},$$

where

$$\Delta=4[l(a+b\delta+1)+(l+b)(a+b\delta)]^2-9\delta(l+b)(a+b\delta+1)^2.^{*)}$$

It is evident that

$$x_1<x_2<0$$

and $w(x)$ has a locally maximum value and a locally minimum value at $x=x_1$ and $x=x_2$, respectively, furthermore

$$w(x_1)>w(x_2),$$

$$w(x_2)=\frac{1}{3}[l(a+b\delta+1)+(l+b)(a+b\delta)]x_2^2+\delta(a+b\delta+1)x_2+\delta.$$

Let us consider

$$\psi(x)=\frac{1}{3}[l(a+b\delta+1)+(l+b)(a+b\delta)]x^2+\delta(a+b\delta+1)x+\delta,$$

$$\psi'(x)=\frac{2}{3}[l(a+b\delta+1)+(l+b)(a+b\delta)]x+\delta(a+b\delta+1).$$

The root of equation $\psi'(x)=0$ is

$$x_0=-\frac{3\delta(a+b\delta+1)}{2[l(a+b\delta+1)+(l+b)(a+b\delta)]},$$

and $\psi(x)$ has a locally minimum at $x=x_0$

$$\psi(x_0)=\frac{\delta}{l(a+b\delta+1)+(l+b)(a+b\delta)}[l(a+b\delta+1)+(l+b)(a+b\delta)-3\delta(lb+a)].$$

Since

$$a(l-\delta)>0,$$

we have

$$-\delta(lb+a)>-l(a+b\delta),$$

and therefore

$$l(a+b\delta+1)+(l+b)(a+b\delta)-3\delta(lb+a)>l+(a+b\delta)(2l+b-3l)$$
$$=b(a+b\delta)-l(a+b\delta-1)>0,$$

hence

$$w(x_2)=\psi(x_2)\geqslant\psi(x_0)>0.$$

Furthermore

$$w(-1)=\frac{3}{2}(l-\delta)(a+b\delta)+\frac{1}{2}(l-\delta)+\frac{b}{2}(a+b\delta-1),$$

on account of (12), we have

$$w(-1)>\frac{3}{2}(l-\delta)(a+b\delta)+\frac{1}{2}(l-\delta)-a(l-\delta)=\frac{1}{2}(l-\delta)(a+3b\delta+1)>0.$$

Thus

$$w(x)>0,\quad x>-1,$$

*) If $\Delta<0$, then $w'(x)>0$ and $w(x)$ is an increasing function, in order to prove $w(x)>0$, for $x>-1$, it is sufficient to show that $w(-1)\geqslant 0$ as below.

so that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]>0, \quad x>-1,$$

and by Theorem 6.6 of [2], our theorem has been proved.

Theorem 3. *There is no limit cycle for system* (4) *which satisfies conditions* (5), (6) *and* $\delta\leqslant 0$ *or* $\delta\geqslant 2$.

Proof If $|\delta|\geqslant 2$, then the elementary singular point $O(0, 0)$ is a node, and there is no limit cycle surround it. In the case $\delta=0$, from

$$\bar{v}_3=\frac{\pi}{4}[m(l+n)-a(b+2l)]=-\frac{\pi}{4}[l+a(b+2l)]<0,$$

implies $O(0, 0)$ is a stable focus, and if it exists some limit cycles, the number of them must be even.

But, in this case, (10) can be rewritten in the form

$$w(x)=\frac{1}{2}(l+b)(a+1)x^2\left[x+\frac{2(l(a+1)+a(l+b))}{(l+b)(a+1)}\right].$$

and (12) has the form

$$b(a-1)>-2al. \tag{13}$$

We observe

$$\frac{2[l(a+1)+a(l+b)]}{(l+b)(a+1)}-1=\frac{1}{(l+b)(a+1)}[l(a+1)+2al+(a-1)b]$$

and by (13), we obtain

$$\frac{2[l(a+1)+a(l+b)]}{(l+b)(a+1)}-1>\frac{l(a+1)}{(l+b)(a+1)}=\frac{l}{l+b}>0.$$

Hence

$$w(x)>\frac{1}{2}(l+b)(a+1)x^2(x+1)>0, \quad \text{for } x>-1.$$

In other words, there is at most one limit cycle, if it exists. This contradiction shows that there is no limit cycle for system (4) in this case.

Now, we divide our remaining proof into two parts, when $-2<\delta<0$——$l+b\leqslant 0$ and $l+b>0$.

(1) Case $l+b\leqslant 0$.

Let

$$-y-F(x)\equiv P(x, y), \quad g(x)\equiv Q(x, y)$$

and if Γ is an arbitrary limit cycle of system (9), then

$$D=\oint_\Gamma\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\tau=\oint_\Gamma\frac{(l+b)x^2+(2l+b)x+\delta}{(x+1)^2}d\tau=\oint_\Gamma\frac{v(x)}{(x+1)^2}d\tau, \tag{14}$$

where

$$v(x)=(l+b)x^2+(2l+b)x+\delta. \tag{15}$$

In the case $l+b=0$, $v(x)=lx+\delta=l(x+1)-(l-\delta)<0$, for $x>-1$. And in case $l+b<0$, the root of equation $v'(x)=0$ is

$$x_0=-\frac{2l+b}{2(l+b)}=-1+\frac{b}{2(l+b)}<-1, \tag{16}$$

and it is a locally maximum point of $v(x)$.

Since $a+b\delta+1>0$, if $-2<\delta<0$. Hence

$$\frac{-2}{a+b\delta+1}<-1,$$

furthermore

$$v(-1)=-(l-\delta)<0.$$

Thus $v(x)<0$, for $x>-1$, so that $D<0$, for $x>-1$ and then there exists at most one limit cycle, which leades again a contradiction as above. Thus system (9) and therefore system (4) has no limit cycle.

(2) Case $l+b>0$.

If $-2<\delta<0$, as we have showed above, $a+b\delta+1>0$, and again

$$\frac{-2}{a+b\delta+1}<-1$$

(a) If $(2l+b)(a+b\delta+1)-(l+b)\leqslant 0$ and if Γ is an arbitrary limit cycle of the system (4), Γ^* is a limit cycle of system (9) which corresponds to Γ, then

$$D=\oint_{\Gamma^*}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\tau=\oint_{\Gamma^*}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\tau-\frac{l+b}{a+b\delta+1}\oint_{\Gamma^*}g(x)d\tau$$
$$=\oint_{\Gamma^*}\frac{u(x)}{(1+x)^2}d\tau, \tag{17}$$

where

$$u(x)=-\frac{l+b}{4}(a+b\delta+1)x^3+\left(2l+b-\frac{l+b}{a+b\delta+1}\right)x+\delta, \tag{18}$$

$$u'(x)=-\frac{3(l+b)}{4}(a+b\delta+1)x^2+2l+b-\frac{l+b}{a+b\delta+1}<0 \tag{19}$$

and

$$\begin{aligned}
u(-1)&=\frac{(l+b)(a+b\delta+1)}{4}-\left(2l+b-\frac{l+b}{a+b\delta+1}\right)+\delta\\
&=\frac{1}{a+b\delta+1}[(l+b)(lb+a)-(2l+b)(a+b\delta+1)+l+b+\delta(a+b\delta+1)]\\
&=\frac{1}{a+b\delta+1}[-(a+1)(l-\delta)+b^2(l-\delta)+b(l-\delta)^2]\\
&=\frac{l-\delta}{a+b\delta+1}[-(a+b\delta+1)+b^2+lb]\\
&<\frac{l-\delta}{a+b\delta+1}\left[-(a+b\delta+1)-\frac{1}{2}a(a+b\delta-1)+lb\right]\\
&=\frac{l-\delta}{a+b\delta+1}\left[-(a+b\delta+1)-\frac{1}{2}a(a+b\delta+1)+lb+a\right]\\
&=\frac{l-\delta}{a+b\delta+1}\left[-(a+b\delta+1)-\frac{1}{2}a(a+b\delta+1)+\frac{1}{4}(a+b\delta+1)^2\right]\\
&=-\frac{l-\delta}{4}(a-b\delta+3)<0.
\end{aligned}$$

Hence $D<0$, for $x>-1$. It shows that system (4) has at most one limit cycle and it is impossible.

(b) If $(2l+b)(a+b\delta+1)-(l+b)>0$ and if Γ is an arbitrary limit cycle of the system (4), Γ^* is a limit cycle of system (9) which corresponds to Γ, then

$$D=\oint_{\Gamma^*}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\tau=\oint_{\Gamma^*}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\tau-(2l+b)\oint_{\Gamma^*}g(x)\,d\tau$$

$$=\oint_{\Gamma^*}\frac{z(x)}{(1+x)^2}\,d\tau, \tag{20}$$

where

$$z(x)=-(2l+b)(lb+a)x^3+[l+b-(2l+b)(a+b\delta+1)]x^2+\delta. \tag{21}$$

In this case $2l+b>0$, the roots of equation $z'(x)=0$ are $x_1=0$ and

$$x_2=\frac{2[(2l+b)(a+b\delta+1)-(l+b)]}{-3(2l+b)(lb+a)}<0,$$

they are the locally maximum point and locally minimum point of the function $z=z(x)$, respectively, and

$$z(x_2)<z(x_1), \tag{22}$$

$$z(x_1)=z(0)=\delta<0. \tag{23}$$

It is easy to verify that

$$z(-1)=(l-\delta)[b(2l+b)-1],$$

and on account of the fact that

$$0<\frac{1-b\delta}{2}(a+b\delta+1)-b\delta=1-2b\delta-\frac{1}{2}(a+b\delta-1)(b\delta-1)$$

$$=1-2b\delta-\frac{1}{2}(a+b\delta-1)^2+\frac{1}{2}a(a+b\delta-1)$$

$$=1-2b\delta-2b(l-\delta)+\frac{1}{2}a(a+b\delta-1)<1-2b\delta+2b\delta-2bl-b^2=1-b(2l+b),$$

we have

$$z(-1)<0.$$

Hence

$$D<0, \text{ for } x>-1$$

and as we have indicated above, there exists no limit cycle.

§ 2. Limit cycles of case II

Theorem 4. *There is exactly one limit cycle for system* (4) *which satisfies conditions* (5), (7) *and* $0<\delta<2$.

Proof The proof is similar to which we have stated in the proof of Theorem 2. We shall begin with system (9). We notice that $l=\delta>0$, $a+b\delta=1$ in this case, and hence $a+b\delta+1=2$. We also have

$$xg(x)>0,\ x\neq 0;\ f(0)=-\delta<0,$$