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CHEBYSHEV APPROXIMATION BY LINEAR ALTERNATING FAMILIES WITH FIXED VALUES AT NODES

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Abstract

In this paper the author discusses a problem of Chebyshev approximation by linear alternating families with fixed values at nodes and gives the analogues of all results for the classical Chebyshev approximation, which include the theorems on existence, alternation, uniqueness, strong uniqueness and the continuty of the best approximation operator, etc.

1. Introduction

Dunham[1,2] considered an approximation problem by alternating families with a fixed value at an endpoint. In this paper we discuss this problem by linear alternating families with fixed values at some points (called nodes) and give the analogues of all results for the classical Chebyshev approximation.

Let X be a compact subset of $[a, b]$ containing at least $n+1$ points, where n is a fixed natural number. By $C(X)$ we mean the usual Banach space of all continuous real-valued functions defined on X with a uniform norm $\|f\|=\max_{x\in X}|f(x)|$.

Given $X_0=\{x_1, \cdots, x_m\}\subset X\,(m<n)$, let K be a subset in $C[a, b]$ having the property: For each element $p\in K$

$$p(x_i)=\alpha_i,\ i=1,\ 2,\ \cdots,\ m,$$

where the α' s are constants. We need the following two assumptions for K:

(a) For each element $p\in K$, $K-p$ is an $(n-m)$-dimensional subspace;

(b) Every non-zero element in $K-p$ has at most $n-1$ zeros in $[a, b]$.

The Chebyshev approximation problem is to find an element $p\in K$ for given $f\in C(X)$, such that

$$\|f-p\|=\inf_{q\in K}\|f-q\|.$$

Such an element p is called a best approximation to f from K.

We follow the approach in [3].

Definition. *For fixed $f\in C(X)$ and an arbitrary element $p\in K$ define*

$$X_{+1}=\{x\in X: f(x)-p(x)=\|f-p\|\},$$

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$$X_{-1}=\{x\in X: f(x)-p(x)=-\|f-p\|\},$$
$$X_p=X_{+1}\cup X_{-1};$$
$$\sigma(x)=\begin{cases} 1, & x\in X_{+1}\cup X_0,\\ -1, & x\in X_{-1}\cup X_0.\end{cases}$$

By the definition, in the case $x\in X_0$, $\sigma(x)$ may take the value either 1 or -1.

2. Main Results

The existence of best approximations easily follows from the usual compacteness arguments.

Theorem 1 (Existence). *For each $f\in C(X)$ there exists a best approximation $p\in K$ to f from K.*

First we establish the following simple result.

Theorem 2. *If $p\in K$ and*

$$X_p\cap X_0\neq\emptyset, \tag{1}$$

then p is a best approximation to f from K.

Proof Since for any $q\in K$

$$\|f-q\|\geqslant\max_{x\in X_0}|f(x)-q(x)|=\max_{1\leqslant i\leqslant m}|f(x_i)-\alpha_i|$$

and the condition (1) implies

$$\|f-p\|=\max_{x\in X_p}|f(x)-p(x)|=\max_{x\in X_0}|f(x)-p(x)|=\max_{1\leqslant i\leqslant m}|f(x_i)-\alpha_i|,$$

we have

$$\|f-p\|\leqslant\|f-q\|,\ \forall q\in K.$$

Next if we compare the setting in the present paper with the one in [3], we can see that the former may be considered as a special case of the latter in which $l=-\infty$ and $u=+\infty$. Thus in the setting of the present paper K is the same as M_0 in [3]. Furthermore, from the proof of Theorem 3.2 in [3], we can see that Theorem 3.2 still holds if every non-zero element $q\in K-p$ has at most $n-1$ zeros in $[a, b]$. Fortunately, it is just assumption (b). So we have the theorem, which is similar to Theorem 3.2 in [3].

Theorem 3 (Characterization). *Let $f\in C(X)$ and $p\in K$. If*

$$X_p\cap X_0=\emptyset, \tag{2}$$

then the following three statements are equivalent to each other:

(a) *p is a best approximation to f from K;*

(b) *The origin of the $(n-m)$-dimensional Euclidean space belongs to the convex hull of the set of $(n-m)$-tuples $\{\sigma(x)\hat{x}: x\in X_p\}$, where $\hat{x}=(\phi_1(x), \cdots, \phi_{n-m}(x))$ with a basis $\phi_1, \cdots, \phi_{n-m}$ for the subspace $K-p$;*

(c) *There exists an alternating system, i. e., a system of $n+1$ points in $X_p\cup X_0$*

$$\xi_1<\xi_2<\cdots<\xi_{n+1} \tag{3}$$

satisfying

$$\sigma(\xi_j)=(-1)^{j+1}\sigma(\xi_1),\ j=1,2,\cdots,n+1,$$

if the values at $\xi_j\in X_0$ *are appropriately chosen.*

Proof (a)⇒(b). Suppose $0\overline{\in}$ the hull of the set $\{\sigma(x)\hat{x}: x\in X_p\}$. Since X_p is compact, by Theorem of Linear Inequalities [4, p. 19] there exists an element $q\in K-p$ such that

$$\sigma(x)q(x)>0,\ \forall x\in X_p. \tag{4}$$

We will show that there exists $t\in(0,1]$ such that $\|f-r_t\|<e\equiv\|f-p\|$, where $r_t=p+td$.

Let $y\in X_p$. From (4) it follows that

$$|f(x)-r_t(x)|=|f(x)-p(x)-tq(x)|=|f(x)-p(x)|-t|q(x)|,$$

if both $t>0$ and $|x-y|$ are small enough. So there exists a number $t_y\in(0,1]$ and a neighborhood N_y of the point y such that

$$|f(x)-r_t(x)|<e,\ \forall t\in(0,t_y],\ \forall x\in N_y. \tag{5}$$

For $y\in X\setminus X_p$ we see that $|f(y)-p(y)|<e$. Since $r_t\to p$ as $t\to 0^+$, there also exists a number $t_y>0$ and a neighborhood N_y of y such that (5) is valid.

Now from the open cover $\{N_y\}$ of the compact set X we may select a finite subcover $\{N_{y_1},\cdots,N_{y_k}\}$. Taking the minimum of the corresponding positive numbers $t_{y_1},\cdots,t_{y_k}$, denoted by t, then $0<t\leqslant 1$ and

$$|f(x)-r_t(x)|<e,\ \forall x\in X.$$

Hence $\|f-r_t\|<e$. This is a contradiction, because $q\in K-p$ means $q=q^*-p$ for some $q^*\in K$ and $r_t=p+tq=(1-t)p+tq^*\in K$.

(b)⇒(c) and (c)⇒(a). Substituting $K-p$ for M and using assumption (b) for $K-p$ instead of the assumption of Haar subspace for M, we can repeat word by word the corresponding parts of the proof of Theorem 3.2 in [3].

Theorem 4 (Characterization). *Let* $f\in C(X)$ *and* $p\in K$. *Then* p *is a best approximation to* f *from* K *if and only if one of the following two statements is valid:*

(a) $X_p\cap X_0\neq\emptyset$;

(b) *There exists an alternating system.*

From the proof of (c)⇒(a) we can conclude the stronger result: If $p\in K$ has an alternating system, then it is the unique best approximation to f. But Theorem 3 shows that under condition (2) for any f its best approximation always possesses an alternating system. Thus for any $f\in C(X)$ it possesses the unique best approximation under condition (2).

Theorem 5 (Uniqueness). *Let* $f\in C(X)$ *and* p *be a best approximation to* f *from* K. *If condition* (2) *is valid, then* p *is unique.*

Theorem 6 (*Strong Uniqueness*). *Let* $f\in C(X)$ *and* p *be a best approximation to* f *from* K. *If condition* (2) *is valid, then tnere exists a constant* $\gamma>0$ *depending only on* f *such that for any* $q\in K$

$$\|f-q\| \geqslant \|f-p\|+\gamma\|p-q\|. \tag{6}$$

Proof If $\|f-p\|=0$, $f=p$. In this case we only need to put $\gamma=1$ to satisfy (6).

If $\|f-p\|>0$, by Theorem 3 (b) there exist r positive numbers θ_i and r points $y_i \in X_p$ such that

$$\sum_{i=1}^{r} \theta_i=1$$

and

$$\sum_{i=1}^{r} \theta_i \sigma(y_i) \phi_j(y_i)=0, \quad j=1, 2, \cdots, n-m.$$

By Carathéodory Theorem [4, p. 17] and assumption (b) we have $r=n-m+1$. Now for any $h \in K-p$ and $\|h\|=1$

$$\sum_{i=1}^{n-m+1} \theta_i \sigma(y_i) h(y_i)=0.$$

By assumption (b) the numbers $\sigma(y_i) h(y_i)$ are not all zeros. And from $\theta_i>0$ it follows that at least one of the numbers $\sigma(y_i) h(y_i)$ is positive. Hence $\max_i \sigma(y_i) h(y_i)>0$. Noting that the set $H=\{h \in K-p:\ \|h\|=1\}$ is compact, we obtain

$$\gamma=\min_{h \in H} \max_i \sigma(y_i) h(y_i)>0.$$

Now let $q \in K$ be arbitrary and $q \neq p$. Then $(p-q)/\|p-q\| \in H$ by assumption (a) and there exists an index i such that

$$\sigma(y_i)(p(y_i)-q(y_i)) \geqslant \gamma\|p-q\|.$$

Since $\sigma(y_i)(f(y_i)-p(y_i))=\|f-p\|$,

$$\begin{aligned}\|f-q\| &\geqslant \sigma(y_i)(f(y_i)-q(y_i))=\sigma(y_i)(f(y_i)-p(y_i))+\sigma(y_i)(p(y_i)-q(y_i)) \\ &\geqslant \|f-p\|+\gamma\|p-q\|.\end{aligned}$$

The continuity of the best approximation operator is a consequence of the strong uniqueness by the proof of Theorem 4.4 in [3]. So we have

Theorem 7 (Continuity of the Best Approximation Operator). *For each* $f \in W=\{f \in C(X):\ X_p \cap X_0=\emptyset\}$ *let* $\tau f \in K$ *be the best approximation to* f. *Then the best approximation operator* τ *is continuous in* W *and for each* $f \in W$ *there exists a constant* $\lambda>0$ *such that* $\|\tau f-\tau g\| \leqslant \lambda\|f-g\|$ *is valid for all* $g \in W$.

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