Chin. Ann. of math.
4B (2) 1983

MEROMORPHIC FUNCTIONS WITH GIVEN BOREL DIRECTIONS

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Abstract

In this paper, the auther defines "orders" and corresponding Borel directions for meromorphic functions. Refering to the method in[1], construct the meromorphic functions with given Borel directions. Therefore, the author extends Yang Le and Zhang Guang-hou's theorem[1], and affirmatively answer a question posed by them.

Definition 1.

$$e_0^r = \ln_0 r = r, \quad e_n^r = e^{e_{n-1}^r}, \quad \ln_n r = \ln(\ln_{n-1} r).$$

Definition 2. *Suppose that $M(r)$ and $h(r)$ are positive functions on $[a, \infty)$ and that $\lim\limits_{r\to\infty} M(r) = \lim\limits_{r\to\infty} h(r) = \infty$; If*

$$\overline{\lim_{r\to\infty}} \frac{\ln M(r) - \ln_2 r}{\ln h(r)} = \rho,$$

where $0 \leqslant \rho \leqslant \infty$, then $M(r)$ is called a functuon of order ρ with respect to $h(r)$.

Definition 3. *If the characteristic function $T(r, f)$ of meromorphic function $f(z)$ is of order ρ with respect to $h(r)$, then $f(z)$ is called a meromorphic function of order ρ with respect to $h(r)$.*

Obviously, when $h(r) = r$, the order in the above definition is the same as the ordinary one.

Definition 4. *Suppose that $f(z)$ is a meromorchic function in the open plane. A half line $\arg z = \theta_0$, is called a Borel direction of order $\lambda (0 < \lambda \leqslant \infty)$ with respect to $h(r)$, if the equality*

$$\rho(\theta_0, \alpha) = \lim_{\delta\to 0} \left\{ \overline{\lim_{r\to\infty}} \frac{\ln n(r, \theta_0, \delta, f=\alpha)}{\ln h(r)} \right\} = \lambda$$

holds for all complex number α except at most two complex numbers, where $n(r, \theta_0, \delta, f=\alpha)$ denots the numbers of zeros of $f(z) - \alpha$ in the region $(|z| \leqslant r) \cap (|\arg z - \theta_0| \leqslant \alpha)$, counted according to their multiplicities.

A half line $\arg z = \theta_0$ is called a Borel direction of order zero with reapect to $h(r)$, if $\rho(\theta_0, \alpha) = 0$ and

$$\lim_{\delta\to 0} \left\{ \overline{\lim_{r\to\infty}} \frac{n(r, \theta_0, \delta, f=\alpha)}{(\ln h(r))^2} \right\} = +\infty$$

Manuscript received July 11, 1981.

hold for all complex numbers α except at most two complex numlers.

Definition 5. *Suppose that E is a non-empty bounded closed set of real numbers, and that $\rho(\theta)$ is a function on E, if the inequality*

$$\overline{\lim_{\substack{\theta\in E\\ \theta\to\theta_0}}}\rho(\theta)\leqslant\rho(\theta_0)$$

holds for any $\theta_0\in E$, then we call $\rho(\theta)$ an upper semi-continuous function on E.

In the following, C and K denote constants and they do not always denote the same numbers when they occur.

Lemma 1. *Suppose that $h(r)$ is a non-decreasing positive function on $[a, \infty)$ and $h(r)\leqslant\ln^k r\,(0<k<\infty)$, then there exists a sequence $\{r_n\}$, $\lim\limits_{n\to\infty} r_n=\infty$, such that*

$$h(r_n^3)<9^k h(r_n)\,(n=1,\ 2,\ \cdots).$$

Proof If the conclusion of lemma were not true, then there would exist $b\geqslant a$ such that for $r\geqslant b$ we would have $h(r^3)\geqslant 9^k h(r)$.

Put
$$r_0=\max\{b,\ c,\ 2\},$$
where $c=\inf\{r\,|\,h(r)\geqslant 2\}$, then we would have

$$\ln^k r_0^{3^n}\geqslant h(r_0^{3^n})\geqslant 9^k h(r_0^{3^{n-1}})\geqslant 9^{2k}h(r_0^{3^{n-2}})\geqslant\cdots\geqslant 9^{nk}h(r_0).$$

Therefore

$$h(r_0)\geqslant 9^{-nk}\ln^k r_0^{3^n}=9^{-nk}\cdot 3^{nk}\ln^k r_0=3^{-nk}\ln^k r_0.$$

But

$$h(r_0)\geqslant 2 \text{ and } \lim_{n\to\infty}3^{-nk}\ln^k r_0=0.$$

Lemma 2. *Suppose that $h(r)$ is a non-decreasiong positive function on $[a, \infty)$, If there exist two sequences $\{x_n\}$, $\{y_n\}$, and $\lim\limits_{n\to\infty} x_n=\lim\limits_{n\to\infty} y_n=\infty$ such that*

$$h(x_n)>\ln^{k+1}x_n \quad (0<K<\infty),$$
$$h(y_n)<\ln^k y_n \quad (n=1,\ 2,\ \cdots),$$

then there exists a sequence $\{r_m\}$, $\lim\limits_{m\to\infty} r_m=\infty$ such that

$$h(r_m^3)<Ch(r_m)\quad (m=1,\ 2,\ \cdots;\ 1<c<\infty).$$

Proof For and fixed M, there exist n_0 and n' such that

$$x_{n_0}>\max\{m,\ a,\ e^{3^k}\},\quad y_{n'}>x_{n_0}.$$

Let

$$E=\{x\,|\,h(x)\geqslant\ln^{k+1}x;\ x<y_{n'}\},$$
$$r_m=\sup\{x\,|\,x\in E\}\geqslant x_{n_0}>e^{3^k}.$$

Since $\ln^{k+1}x$ is continuous and $h(r)$ is non-decreasing, we have $r_m\in E$. Therefore

$$\ln^k y_{n'}>h(y_{n'})\geqslant h(r_m)\geqslant\ln^{k+1}r_m.$$

Consequently

$$y_{n'}>e^{\ln^{1+\frac{1}{k}}r_m}=r^{\ln^{\frac{1}{k}}r_m}>r_m.$$

Since $r_m>1$, we have $r_m^3\overline{\in}E$.

Therefore

$$h(r_m^3)<\ln^{k+1} r_m^3<3^{k+1}\ln^{k+1} r_m<Ch(r_m).$$

Lemma 3. *Suppose that $h(r)$ is a non-decreasing positive function on $[a, \infty)$ and $h(a)>1, \lim\limits_{r\to\infty} h(r)=\infty$, then there exists a sequence $\{x_n\}$, $\lim\limits_{n\to\infty} x_n=\infty$ such that*

1) *for any $r<x_n$*

$$r+\frac{r}{4\ln h(r)}\leqslant x_n+\frac{x_n}{2\ln h(x_n)},$$

2)

$$h\left(x_n+\frac{2x_n}{\ln h(x_n)}\right)<h^{1+\frac{1}{n}}(x_n),$$

where

$$n=1,\ 2,\ \cdots.$$

Proof For every fixed n, we take sufficiently large r such that

$$\ln h(r_0)>36n.$$

Let

$$\underline{r}_1=\inf\left\{r\,|\,r_0\leqslant r<\infty,\ h\left(r+\frac{2r}{\ln h(r)}\right)\geqslant h^{1+\frac{1}{n}}(r)\right\}.$$

(When $\underline{r}_1=\infty$, Lemma 3 holds evidently).

We take $r_1(\underline{r}_1\leqslant r_1<\underline{r}_1+1)$ such that

$$h\left(r_1+\frac{2r_1}{\ln h(r_1)}\right)\geqslant h^{1+\frac{1}{n}}(r_1)$$

and

$$r_1<\underline{r}_1+2\underline{r}_1/\ln h(\underline{r}_1+1)<\underline{r}_1\cdot\left(1+\frac{1}{18n}\right).$$

Let $r_1'=r_1+\Delta_1$, where $\Delta_1=2r_1/\ln h(r_1)$.

Obviously

$$r_1-\underline{r}_1<\frac{2r_1}{\ln h(\underline{r}_1+1)}<\Delta_1<\frac{r_1}{18n}.$$

We can choose sequence $\{r_m\}$, $\{\underline{r}_m\}$, $\{r_m'\}$ and $\{\Delta_m\}$ such that they have the following properties

$$r_{m-1}'\leqslant\underline{r}_m\leqslant r_m<r_m',$$

$$r_m'-\underline{r}_m<2\Delta_m,\quad \Delta_m=r_m'-r_m,$$

$$r_m'<r_m\cdot\left(1+\frac{1}{18n}\right)<\underline{r}_m\cdot\left(1+\frac{1}{18n}\right)^2,$$

$$h(r_m')=h\left(r_m+\frac{2r_m}{\ln h(r_m)}\right)\geqslant h^{1+\frac{1}{n}}(r_m),$$

where

$$m=1,\ 2,\ \cdots,$$

and

$$h\left(r+\frac{2r}{\ln h(r)}\right)<h^{1+\frac{1}{n}}(r),\quad\text{for } r\in\bigcup_{i=1}^{\infty}(r_i',\ \underline{r}_{i+1}).$$

If there exists a fixed positive number M for $m>M$ such that

$$\bar{r}_m\geqslant\underline{r}_{m+1},$$

where

$$\bar{r}_m = \sup\left\{r + \frac{r}{4\ln h(r)} \,\middle|\, r_M \leqslant r \leqslant r'_m\right\},$$

then

$$r'_m < r_m \cdot \left(1 + \frac{1}{18n}\right) < \underline{r}_m\left(1 + \frac{1}{18n}\right)^2 \leqslant \bar{r}_{m-1}\left(1 + \frac{1}{18n}\right)^2$$
$$\leqslant r'_{m-1} \cdot \left(1 + \frac{1}{4\ln h(r_M)}\right) \cdot \left(1 + \frac{1}{18n}\right)^2$$
$$< r'_{m-1}\left(1 + \frac{1}{18n}\right)^3 < \cdots < r_M\left(1 + \frac{1}{18n}\right)^{3(m-M)+1}.$$

Moreover, since

$$h(r'_m) \geqslant h^{1+\frac{1}{n}}(r_m) \geqslant h^{1+\frac{1}{n}}(r'_{m-1}) \geqslant \cdots \geqslant h^{\left(1+\frac{1}{n}\right)^{m-M}}(r_M),$$

we deduce that

$$\lim_{m\to\infty} h(r'_m) = \infty, \quad \lim_{m\to\infty} r'_m = \infty.$$

On the other hand, we have for any $m > M$

$$r'_{m+1} - r'_{M+1} = m\left(\bigcup_{i=M+1}^{m} [r'_i,\ r'_{i+1}]\right) = m\left(\bigcup_{i=M+1}^{m} [r'_i,\ \underline{r}_{i+1}]\right) + m\left(\bigcup_{i=M+1}^{m} [\underline{r}_{i+1},\ r'_{i+1}]\right)$$
$$\leqslant m\left(\bigcup_{i=M+1}^{m} [r'_i,\ \bar{r}_i]\right) + 2\sum_{i=M+1}^{m} \Delta_{i+1}.$$

According to the definition of $\bar{r}_i$, for every i in the above first term, there exists a sequence $\{r_j\}$ such that

$$r'_{t(i)-1} \leqslant r_j \leqslant r'_{t(i)} \quad (j = 1,\ 2,\ \cdots),$$
$$\lim_{j\to\infty} r_j + r_j/4\ln h(r_j) = \bar{r}_i \leqslant r'_{t(i)} + r'_{t(i)}/4\ln h(r'_{t(i)-1}),$$

where $t(i)$ is a positive integer whidh depends only on $i (M \leqslant t(i) \leqslant i)$.

Therefore

$$r'_{m+1} - r'_{M+1} \leqslant m\left(\bigcup_{i=M+1}^{m}\left[r'_{t(i)},\ r'_{t(i)} + \frac{r'_{t(i)}}{4\ln h(r'_{t(i)-1})}\right]\right) + 2\sum_{i=M+1}^{m} \Delta_{i+1}$$
$$\leqslant m\left(\bigcup_{t=M}^{m}\left[r'_t,\ r'_t + \frac{r'_t}{4\ln h(r'_{t-1})}\right]\right) + 2\sum_{t=M+2}^{m} \Delta_t < 5\sum_{t=M}^{m+1} r'_t/4\ln h(r'_{t-1})$$
$$< \frac{5r_M(1+1/18n)}{\ln h(r_M)} \sum_{t=M}^{\infty}\left\{\frac{(1+1/18n)^3}{1+1/n}\right\}^t < K < \infty.$$

The contradiction shows that there exists a sufficiently large m_0 such that

$$r_{m_0} > \bar{r}_{m_0-1} > r'_{m_0-1} > r_M \cdot (1 + 1/h(a)).$$

Let

$$\underline{r} = \sup\left\{r + \frac{r}{4\ln h(r)} \,\middle|\, r'_{m_0-1} < r < r_{m_0}\right\}.$$

Obviously $y > \bar{r}_{m_0-1}$.

Choose $x_n (r'_{m_0-1} < x_n < \underline{r}_{m_0}$, now (2) holds for x_n.) such that

$$y - \frac{r'_{m_0-1}}{4\ln h(\underline{r}_{m_0})} < x_n + \frac{x_n}{4\ln h(x_n)} < y.$$

Thus, we have for any $r < x_n$

$$r + \frac{r}{4\ln h(r)} < y < x_n + \frac{x_n}{4\ln h(x_n)} + \frac{r'_{m_0-1}}{4\ln h(\underline{r}_{m_0})} < x_n + \frac{x_n}{2\ln h(x_n)}.$$

Lemma 4. *Suppose that* (1) $h(r)$ *is a non-decreasing positive function on* $[a, \infty)$; (2) *the product*

$$p_m=\prod_{n=1}^{m}\prod_{k=1}^{K_n}(1+\alpha_{nk})^{m_{nk}};\quad \lim_{m\to\infty}p_m=p,$$

($0<p<\infty$; $0<\alpha_{nk}\leqslant 1$; $n=1, 2, 3, \cdots$; $k=1, 2, \cdots, K_n$; $1\leqslant K_n\leqslant n$; m_{nk} *are positive integers.*); (3) *the complex sequences* $\{a_{nk}\}$, $\{b_{nk}\}$ satisfy $\lim\limits_{n\to\infty}|a_{nk}|=\infty$, $|a_{nk}|\geqslant 1$, $h(|a_{nk}|)\geqslant 1$, $|a_{nk}-a_{n'k'}|\geqslant 4(n\neq n'\vee k\neq k')$, $\arg b_{nk}=\arg a_{nk}=\theta_{nk}$, $|b_{nk}|=|a_{nk}|+\alpha_{nk}/h(|a_{nk}|)$.

Under these conditions, (*a*) *the infinite product*

$$f(z)=\prod_{n=1}^{\infty}\prod_{k=1}^{K_n}\left(1+\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}} \tag{1}$$

defines a meromorphic function on the open plane P; (b) $\forall\varepsilon>0$, $\exists\Delta>0$, *when* $|z|>\Delta$ *and* $|z-b_{nk}|\geqslant\dfrac{1}{h(|a_{nk}|)}$, *we have* $|f(z)-1|<\varepsilon$.

Proof (a) Let C_{nk} be the domain $|z-b_{nk}|<\dfrac{1}{h(|a_{nk}|)}$ and $\bar{C}_{nk}$ be the closed domain $|z-b_{nk}|\leqslant\dfrac{1}{h(|a_{nk}|)}$. On the closed domain

$$D=P-\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}C_{nk},$$

we have

$$\left|\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right|\leqslant\frac{\alpha_{nk}}{h(|a_{nk}|)}\cdot h(|a_{nk}|)=\alpha_{nk}.$$

Since $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{K_n}\alpha_{nk}m_{nk}$ is covergent, the product (1) is uniformly convergent on D. Hence $f(z)$ is a meromorphic function on D. If $z\in\bar{C}_{n'k'}$, the product $f(z)$ is meromorphic on $\bar{C}_{n'k'}$,since $H(z)=\left(1+\dfrac{b_{n'k'}-a_{n'k'}}{z-a_{n'k'}}\right)^{m_{n'k'}}$ is meromorphic on $\bar{C}_{n'k'}$ and $f(z)/H(z)$ is holomorphic on $\bar{C}_{n'k'}$. (b) when $|z-b_{nk}|\geqslant 1/h(|a_{nk}|)$, we have

$$\begin{aligned}&\left|\prod_{n=n_0}^{n_0+T}\prod_{k=1}^{K_n}\left(1+\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}-1\right|\\&=\left|\sum_{n=n_0}^{n_0+T}\sum_{k=1}^{K_n}m_{nk}\frac{b_{nk}-a_{nk}}{z-b_{nk}}+\cdots+\prod_{n=n_0}^{n_0+T}\prod_{k=1}^{K_n}\left(\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}\right|\\&\leqslant\left|\sum_{n=n_0}^{n_0+T}\sum_{k=1}^{K_n}m_{nk}\alpha_{nk}+\cdots+\prod_{n=n_0}^{n_0+T}\prod_{k=1}^{K_n}\alpha_{nk}^{m_{nk}}\right|=|p_{n_0+T}/p_{n_0-1}-1|.\end{aligned}$$

Sicne p_m is convergent, there exists $\Delta_1>0$ such that however large T is, we always have $|p_{n_0+T}/p_{n_0-1}-1|<\varepsilon/2p$ for $n_0>\Delta_1$.

Therefore, we have, for a fixed $N>\Delta_1$.

$$A=\left|\prod_{n=N}^{\infty}\prod_{k=1}^{K_n}\left(1+\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}-1\right|<\frac{\varepsilon}{2p}.$$

Let $$b=\max\{|b_{nk}|\,|1\leqslant n\leqslant N,\ 1\leqslant k\leqslant K_n\}.$$

We have, for $|z|>\Delta_2=2(p_{N-1}-1)/\varepsilon+b$,

$$B=\left|\prod_{n=1}^{N-1}\prod_{k=1}^{K_n}\left(1+\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}-1\right|$$
$$\leqslant\left(\sum_{n=1}^{N-1}\sum_{k=1}^{K_n}m_{nk}\frac{\alpha_{nk}}{|z-b_{nk}|}+\cdots+\prod_{n=1}^{N-1}\prod_{k=1}^{K_n}\left|\frac{\alpha_{nk}}{z-b_{nk}}\right|^{m_{nk}}\right)\leqslant|p_{N-1}-1|/(|z|-b)<\frac{\varepsilon}{2}.$$

For any $\varepsilon>0$, we choose successively $\varDelta_1$, N, b, $\varDelta_2$, such that we have, for $|z|>\varDelta_2$ and $|z-b_{nk}|\geqslant 1/h(|a_{nk}|)$

$$|f(z)-1|\leqslant\left|\prod_{n=1}^{N-1}\prod_{k=1}^{K_n}\left(1+\frac{b_{nk}-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}\right|\cdot A+B\leqslant\left(\prod_{n=1}^{N-1}\prod_{k=1}^{K_n}(1+\alpha_{nk})^{m_{nk}}\right)\cdot A+B$$
$$\leqslant p\cdot A+B<\varepsilon.$$

Theorem 1. *Suppose that $h(r)$ is a non-dereasing positive function on $[a, \infty)$ satisfying $\lim_{r\to\infty} h(r)=\infty$, that E is an arbitrary non-empty closed set of real numbers* (mod 2π) *and that $\rho(\theta)$ is an upper semi-continuous function on E, $0\leqslant\rho(\theta)\leqslant\infty$. Denote $\rho_0=\max\{\rho(\theta)\,|\,\theta\in E\}$. Then there exists a meromorphic function $f(z)$ of order ρ_0 with respect to $h(r)$ on the open plane such that all the half streight lines $\arg z=\theta\ (\theta\in E)$ are Borel directions of order $\rho(\theta)$ of $f(z)$ with respect to $h(r)$ and $f(z)$ has no other singular direction.*

Proof We divided successively the open plane into equal parts by the rays whose vertices are at the origin. In the nth step, we divide the open plane into n angular domains by n rays $\arg z=\frac{2j\pi}{n}$ $(j=0, 1, 2, \cdots, n-1)$. If for some ray $\arg z=\frac{2j_0\pi}{n}$, both of the closed domains $\frac{2(j_0-1)\pi}{n}\leqslant\arg z\leqslant\frac{2j_0\pi}{n}$ and $\frac{2j_0\pi}{n}\leqslant\arg z\leqslant\frac{2(j_0+1)\pi}{n}$ do not contain any point $e^{i\theta}\ (\theta\in E)$, then we reject this ray. Suppose that the remained rays are denoted by L_{nk} $(n=1, 2, \cdots;\ k=1, 2, K_n;\ 1\leqslant K_n\leqslant n)$, and whose arguments are denoted by θ_{nk}.

(1) Suppose that $h(r)$ satisfies the condition in Lemma 1 or Lemma 2, then there exists a sequence $\{x_n\}$, $\lim_{n\to\infty}x_n=\infty$, such that $h(x_n^3)<Kh(x_n)$. We select a subsequence of $\{x_n\}$. for convenience, let $\{x_n\}$ still denote the subsequence for which $x_n>\max\{x_{n-1}^3, n+1\}$ and

$$h(x_n^2)>\{2[h(x_{n-1}^2)]^{\ln(n-1)}\}^{\ln(n+1)}>2^{n\ln(n+1)}. \tag{A}$$

Put

$$\rho'_{nk}=\max\left\{\rho(\theta)\,|\,\theta\in E,\ |\theta-\theta_{nk}|\leqslant\frac{2\pi}{n}\right\},$$
$$\rho_{nk}=\begin{cases}\ln n, & \ln n\leqslant\rho'_{nk}\leqslant\infty,\\ \rho'_{nk}, & 1/\ln(n+1)\leqslant\rho'_{nk}\leqslant\ln n,\\ 1/\ln(n+1), & 0\leqslant\rho'_{nk}\leqslant 1/\ln(n+1),\end{cases}$$
$$m_{nk}=\left[\frac{h(x_n^2)^{\rho_{nk}}}{n}\right],$$

where $[x]$ denotes the integral part of x.

Let $$a_{nk}=x_n^2e^{i\theta_{nk}},$$

$$b_{nk}=\left(x_n^2+\frac{\alpha_{nk}}{4h^2(x_n^2)}\right)e^{i\theta_{nk}},$$

where α_{nk} is the same as in Lemma 4.

Therefore

$$f(z)=\prod_{n=1}^{\infty}\prod_{k=1}^{K_n}\left(\frac{z-a_{nk}}{z-b_{nk}}\right)^{m_{nk}}$$

is a meromorphic function on the open plane, and when $|z|>\Delta$, $|z-b_{nk}|\geqslant 1/2h(x_n^2)$ we have $|f(z)-1|<\varepsilon$ so that if $a=1+C(C\neq 0)$, we have

$$\begin{aligned}\left|\frac{f'(z)}{f(z)-a}\right|&\leqslant\left|\frac{f'(z)}{f(z)}\right|\cdot\frac{2}{|f(z)-1-C|}\\&\leqslant\left|\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}m_{nk}\cdot\frac{a_{nk}-b_{nk}}{(z-a_{nk})(z-b_{nk})}\right|\cdot\frac{2}{|C|-|f(z)-1|}\\&\leqslant\left(\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}m_{nk}\frac{|b_{nk}-a_{nk}|}{|z-b_{nk}|\cdot(|z-b_{nk}|-|b_{nk}-a_{nk}|)}\right)\cdot\frac{2}{|C|-\varepsilon}\\&\leqslant K_a\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}m_{nk}\frac{\alpha_{nk}}{1-\frac{\alpha_{nk}}{2h(x_n^2)}}\leqslant K_a\sum_{n=1}^{\infty}\sum_{k=1}^{K_n}m_{nk}\cdot 2\alpha_{nk}<K_a<\infty.\end{aligned}$$

When $a\neq 1$ for sufficiently large n, by Rouche's theorem, the difference between the number of the zeros of $f(z)-a$ and that of the poles of $f(z)$ is at most a constant N_a.

Put $\quad n_r=\max\{n\,|\,x_n^2\leqslant r\},\ \rho_n=\max\{\rho_{nk}\,|\,1\leqslant k\leqslant K_n\}.$

From $\quad h(x_{n-1}^2)^{\rho_{n-1}}\leqslant h(x_{n-1}^2)^{\ln(n-1)}\leqslant\frac{1}{2}h(x_n^2)^{\frac{1}{\ln(n+1)}}\leqslant\frac{1}{2}h(x_n^2)^{\rho_n},$

we have

$$\frac{h(x_{n_r}^2)^{\rho_{n_r}}}{n_r}\leqslant n(r,\ f=0)\leqslant\sum_{n=1}^{n_r}h(x_n^2)^{\rho_n}\leqslant 2h(x_{n_r}^2)^{\rho_{n_r}}\leqslant 2h(r)^{\rho_{n_r}}. \tag{B}$$

Hence $2f(0)\neq 0,\ 1,\ \infty$, choosing some b_{nk} suitably, we get $2f'(0)\neq 0$. By [2] we obtain

$$\begin{aligned}T(r,\ f)<T(r,\ 2f)<C\Big\{&N(R,\ 2f)+N\Big(R,\ \frac{1}{2f}\Big)+N\Big(R,\ \frac{1}{2f-1}\Big)\\&+\ln\frac{1}{R-r}+1+K(|f(0)|,\ |f'(0)|)\Big\}\\<C\,\Big\{&n\Big(r+\frac{1}{4\ln h(r)},\ 2f=0\Big)\ln r+2n_r^2N\ln r+\ln_2 r+K_f\Big\},\end{aligned}$$

where K_f is a constant dependent on f.

Therefore, when

$$x_{n-1}^2+\frac{1}{h(x_{n-1}^2)}<r<r+\frac{r}{4\ln h(r)}<x_n^2-\frac{1}{h(x_n)},$$

we have

$$\frac{T(r,\ f)}{\ln r}<C\{n(r,\ f=0)+o(1)h(r)^{\rho_{nr}}\}\leqslant(C+o(1))h(r)^{\rho_{nr}};$$

when

$$x_n<r<r+\frac{1}{4\ln h(r)}\leqslant x_n^3,$$

we have

$$\frac{T(r,f)}{\ln r}<(C+o(1))h\left(r+\frac{r}{4\ln h(r)}\right)^{\rho_{nr}}<Ch(x_n^3)^{\rho_{nr}}\leqslant Ch(x_n)^{\rho_{nr}}<Ch(r)^{\rho_{nr}}.$$

In the above two cases, all sufficiently lange r are considered. Moreover, since

$$\frac{h(x_n^3)^{\rho_n}}{n}\ln x_n^3\leqslant C\frac{h(x_n)^{\rho_n}}{n}\ln x_n\leqslant C\frac{h(x_n^2)^{\rho_n}}{n}\ln x_n\leqslant Cn(x_n^2,f=0)\ln x_n$$

$$\leqslant C\int_{x_n^2}^{x_n^3}\frac{n(r,f=0)}{r}dr\leqslant CN(x_n^3,f=0)\leqslant CT(x_n^3,f),$$

we have

$$\overline{\lim_{r\to\infty}}\frac{\ln T(r,f)-\ln_2 r}{\ln h(r)}=\rho_0.$$

(2) Let $\theta_0\in E$, Suppose that k_0 is an integer such that $|\theta_{nk_0}-\theta_0|\leqslant\frac{2\pi}{n}$ and when $\frac{2\pi}{n}<\delta$, we have for sufficiently large n

$$n(x_n^3,\theta_0,\delta,f=a)\geqslant(h(x_n^2)^{\rho_{nk_0}}-N)/n\geqslant(h(x_n)^{\rho_{nk_0}}-N)/n\geqslant(Ch(x_n^3)^{\rho_{nk_0}}-N)/n.$$

By the definition of ρ'_{nk_0}, we have $\rho'_{nk_0}\geqslant\rho(\theta_0)$. Combining this inequality with (A), we see that $\rho\geqslant\rho(\theta_0)$, where ρ is the order of direction $\arg z=\theta_0$ with respect to $h(r)$.

For any $\varepsilon>0$, there exists $\delta>0$ such that

$$\bar{\rho}_n=\max\left\{\rho_{nk}\,\middle|\,|\theta_{nk}-\theta_0|<\frac{\delta}{3},\ \frac{2\pi}{n}<\delta\right\}<\rho(\theta_0)+\varepsilon.$$

When n is sufficiently large, we have, for $x_n<r<r+\frac{1}{h(r)}<x_n^3$,

$$n\left(r,\theta_0,\frac{\delta}{3},f=a\right)\leqslant n\left(r+\frac{1}{h(r)},\theta_0,\frac{\delta}{3},f=0\right)+n_r^2N$$

$$\leqslant 2h\left(r+\frac{1}{h(r)}\right)^{\bar{\rho}_n}+n_r^2N$$

$$\leqslant 2h(x_n^3)^{\bar{\rho}_n}+n_r^2N<Ch(x_n)^{\bar{\rho}_n}+n_r^2N\leqslant Ch(r)^{\bar{\rho}_n}+n_r^2N,$$

and we have, for $x_{n-1}^2+\frac{1}{h(x_{n-1}^2)}<r<r+\frac{1}{h(r)}<x_n^2-\frac{1}{h(x_n^2)}$,

$$n\left(r,\theta_0,\frac{\delta}{3},f=a\right)\leqslant n\left(r,\theta_0,\frac{\delta}{3},f=0\right)+n_r^2N\leqslant 2h(r)^{\bar{\rho}_n}+n_r^2N.$$

Combining this with (A), we obtain that the order ρ of the direction $\arg z=\theta_0$ with respect to $h(r)$ is less than $\rho(\theta_0)+\varepsilon$.

(3) Let $\theta_0\bar{\in}E$. By the properties of θ_{nk}, there exist $\delta>0$ and sufficiently large η such that there does not exist any a_{nk} in the domain$(|z-\theta_0|<\delta)\cap(|z|>\eta)$. By Lemma4, for sufficiently large $\Delta(>\eta)$, we have uniformly $|f(z)-1|<\varepsilon$ in the domain $\left(|z-\theta_0|<\frac{\delta}{2}\right)\cap(|z|>\Delta)$. Hence $\arg z=\theta_0$ isn't a singular direction (we see that 1 is unique asymptotic value of $f(z)$.).

(4) If $h(r)$ does not satisfy the conditions of Lemma 1 or Lemma 2, i. e. for any K and for sufficiently large r, we have $h(r)\geqslant\ln^k r\Rightarrow\lim_{r\to\infty}\frac{\ln_2 r}{\ln h(r)}=0$. Then by Lemma3,

we can choose a sequence $\{x_n\}$, $\lim_{n\to\infty} x_n=\infty$ such that

$$h\left(x_n+\frac{2x_n}{\ln h(x_n)}\right)<Ch^{1+\frac{1}{n}}(x_n).$$

Taking $x_n+x_n/\ln h(x_n)$ instead of x_n^2 and $x_n+2x_n/\ln h(x_n)$ instead of x_n^3, we can obtain desired results with a similar method.

It is easy to see that when $h(r)=r$, Theorem 1 is Theorem 2 in [1].

At the end of [1] the authors posed a question: "For an arbitrary, non-empty and closed set of the rays whose verteces are at the origin, is there a meromorpic function $f(z)$ whose set of Julia directions is exactly $\{\arg z=\theta \mid \theta\in E\}$ and for which $T(r, f)<\varphi(r)\ln r$ (for sufficiently large r), where $\varphi(r)$ is a given, non-decreasing real function such that $\lim_{r\to\infty}\varphi(r)=+\infty$?" Clearly Theorem 1 gives an affiirmative answer when we take $h(r)=\varphi(r)$ and $\rho_0=\max\{\rho(\theta)\mid\theta\in E\}<1$.

Taking account of the definitions in [3], we can also obtain analogous theorems.

Finally, the present author wishes to express his hearty thanks to Prof. Yu Jia-rong for his prtient guidance rnd to Prof. Yang Lo for his important advices.

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