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ON THE TOPOLOGICAL DEGREE FOR THE SUM OF MAXIMAL MONOTONE OPERATOR AND GENERALIZED PSEUDOMONOTONE OPERATOR

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Abstract

Let X be a real separable reflexive Banach space, X^* its dual space, and let $T: X\to X^*$ be a maximal monotone operator, $P: X\to X^*$ a quasi-bounded generalized pseudomonotone operator or T-pseudomonotone operator. In this paper, We have constructed a topological degree for the operator $(T+P)$. As a by product a surjectvity result is obtained. In particular, we have given a partially affirmative answer to a Browder's question by using a topological method (cf., Mathematical Developments arising from Hilbert Problems, Vol. 1 (1976), 68—73 AMS)

Let X be a reflexive Banach space, X^* its conjugate space. Browder[1] posed the following open question: Let $T: X\to 2^{X^*}$ be a maximal monotone mapping and P a bounded finitely continuous mapping from X to X^*. Suppose that P is T-pseudomonotone and that $(T+P)$ is coercive. Is it then true that $(T+P)$ is surjective? An affirmative result was established by Brezis[2] if T is linear. Browder and Hess[3] introduced a class of generalized pseudomonotone operators which is a wider class than ones of maximal monotone and pseudomonotone operators. When we investigate the solvabilities for nonlinear elliptic boundary value problems, nonlinear parabolic problems and nonlinear integral equations of Hammerstein type, we often reduce the solvability to the surjectivity of perturbed maximal monotone operators. Therefore, the constructions of topological degree for the sum of maximal monotone operators and generalized pseudomonotone or T-pseudomonotone operatore provide useful aid for the study of the solvabilities of some nonlinear functional equations by using topological methods.

In the first section of this paper, we construct the topological degree for the sum $T+P$, where T is a maximal monotone operator and P a generalized pseudomonotone or T-pseudomonotone operator of a real reflexive Banach space X into X^* by using the topological degree for A-proper defined by Browder and Petryshyn[4,5]. We shall assume that there is an injective approximation scheme for (X, X^*). We don't assume

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that T and P are continuous. Since null operator is both maximal monotone and generalized pseudomonotone, and since pseudomonotone operator is generalized pseudomonotone, our results unify and extend the definitions of the topological degree for some operators of monotone type[6, 7]. In the second section of this paper, we consider some properties of the degree given by us. Moreover, we will proved the surjectivity for the sum $T+P$, where T is a maximal monotone operator and P is a generalized pseudomonotone or T-pseudomonotone operator. Our result of surjectivity unifies some basic results of the theory of monotone operators[3, 8, 9]. In particular, we answered Browder's question affirmatively if there exists an injective approximation scheme for (X, X^*) and T and P are singlevalued. In the third section of this paper, we consider the range of the sum of maximal monoteone oprators and generalized pseudomonotone or T-pseudomonotone operators by using our result of the surjectivity. In this paper, we assume throughout that all operators considered by us are singlevalued. For multivalued case, our all results are also true (see [10]).

§ 1. The Construction of Generalized Topological Degree

We give first the concepts of an injective approximation scheme and an A-proper mapping[4, 5]. Let X be a Banach space, X^* its conjugate space. We shall use the notations "$\longrightarrow$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively. We denote the collection of all natural numbers by $\mathcal{N}$.

I. $\Gamma=(\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ is said to be an injective approximation scheme for (X, X^*) provided that

(1) $\{X_n\}$ is a monotonically increasing sequence of finite-dimensional subspaces of X such that $\rho(x, X_n)\to 0$ as $n\to\infty$ for each x in X, where

$$\rho(x, V)=\inf_{y\in V}\|x-y\| \text{ for } V\subset X;$$

(2) for each n, X^* is the conjugate space of X, where X_n^* is taken as a Banach space with respect to the induced norm of X;

(3) for each $n\in\mathcal{N}$, P_n: $X_n\to X$ is the linear injection;

(4) for each $n\in\mathcal{N}$, $Q_n=P_n^*$: $X^*\to X_n^*$, where P_n^* is the adjoint operator of P_n.

Obviously, $\|P_n\|=\|P_n^*\|=\|Q_n\|=1$ and P_n and P_n^{-1} are bounded linear operators, $\forall n \in\mathcal{N}$.

If X is a separable and reflexive Banach space, it can be given that an injective approximation scheme $\Gamma=(\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ for (X, X^*), where $\{X_n\}$ is an increasing sequence of finite-dimensional subspaces of X with $\overline{\bigcup_{n=1}^{\infty} X_n}=X$ and P_n: $X_n\to X$ is the injection mapping, i. e., $P_n x=x$ $(\forall x\in X_n)$ and $Q_n=P_n^*$.

II. Let X be a Banach space, Ω a given set in X. Let $T: \Omega \to X^*$, $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ be an injective approximation scheme for (X, X^*). We write $\Omega_n = \Omega \cap X_n$ and $T_n = Q_n T P_n = Q_n T|_{\Omega_n}$. T is said to be an A-proper mapping on Ω with respect to Γ provided that

(i) for each $n \in \mathcal{N}$, $T_n: \Omega_n \to X_n^*$ is continuous;

(ii) for any sequence $\{n_j\}_{j=1}^{\infty} \subset \mathcal{N}$ and a corresponding sequence $\{x_{n_j}\}_{j=1}^{\infty}$ with $x_{n_j} \in \Omega_{n_j}$ such that

$$\|Q_{n_j} T P_{n_j} x_{n_j} - Q_{n_j} f\| \to 0 \quad (j \to \infty)$$

for some $f \in X^*$, there exists an infinite subsequence $\{x_{n_{j(k)}}\}_{k=1}^{\infty}$ and an element $x_0 \in \Omega$ such that $x_{n_{j(k)}} \to x_0$ $(k \to \infty)$ and $Tx_0 = f$.

III. Let $\Omega \subset X$ be a bounded and open set, $T: \bar{\Omega} \to X^*$ an A-proper with respect to the injective approximation scheme Γ and $f \in T(\partial\Omega)$. $\mathrm{Deg}_A(T, \Omega, f)$ is said to be the topological degree of T on $\bar{\Omega}$ over f with respect to Γ, as follows: Write $\hat{Z} = Z \cup \{-\infty, +\infty\}$, where Z is the set of all integers. Then $\mathrm{Deg}_A(T, \Omega, f)$ is the subset of $\hat{Z}$ given by

$\mathrm{Deg}_A(T, \Omega, f) = \{\gamma \mid \gamma \in \hat{Z}$, there exists an infinite sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ such that $\deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f) \to \gamma\}$, where $T_{n_j} = Q_{n_j} T P_{n_j}$ and $\Omega_{n_j} = \Omega \cap X_{n_j}$ and $\deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f)$ is Brouwer degree for mappings of finite-dimensional Euclidean spaces of the same dimension.

Although the scheme employed by us differs from one in [5], it has proved that Theorem 4.1B and Remark 4.1-1 in [5] remain valid for the topological degree given above by us (see [11]).

In this papaer, we always made following assumptions:

(p_1) X isareal reflexive Banach space with an injective approximation scheme. The scheme Γ mentioned above is assumed. Further, suppose that X has the property (h): if $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$ with $x_n, x_0 \in X$ implies $x_n \to x_0$,

(p_2) X^* is strictly convex.

We observe that a locally uniformly convex space has the property (h). The map J of X into X^* given by

$$Jx = \{f \in X^* \mid (f, x) = \|x\|^2 = \|f\|^2\}$$

is called the normalized duality map of X. Under the hypothesis (p_2), J is a singlevalued, bounded and demicontinuous operator. As for the concepts of monotonicity, maximal monotonicity and demicontinuous employed in this paper see e. g. [12]. An operator $P: X \rightharpoonup X^*$ is said to be generalized pseudomonotone if for any $x_n, x_0 \in D(P)$ with $x_n \rightharpoonup x_0$ in X and $Px_n \rightharpoonup g$ in X^* such that

$$\overline{\lim}\,(Px_n, x_n - x_0) \leqslant 0,$$

we have $Px_0 = g$ and $(Px_n, x_n) \to (Px_0, x_0)$. Suppose that $T: X \to X^*$ is an operator. An operator $P: X \to X^*$ is said to be T-pseudomonotone if for any $x_n, x_0 \in D(P)$ with

$x_n \rightarrow x_0$ and $\{Tx_n\}$ being bounded such that

$$\overline{\lim}(Px_n, x_n - x_0) \leqslant 0,$$

we have $Px_n \rightarrow Px_0$ and $(Px_n, x_n) \rightarrow (Px_0, x_0)$. An operator $P: X \rightarrow X^*$ is quasi-bounded if for each $M>0$ there is a constant $C(M)>0$ such that whenever $x \in D(P)$, $(Px, x) \leqslant M\|x\|$ and $\|x\| \leqslant M$, it follows that $\|Px\| \leqslant C$. It is known easily that the sum of finite number of quasi-bounded operators is also quasi-bounded. If $T: X \rightarrow X^*$ is monotone and $0 \in \operatorname{Int} D(T)$, then T is quasi-bounded[3]. An operator $P: X \rightarrow X^*$ is said to be finitely continuous on $D(P)$ if for each finite-dimensional space $X_n \subset X$, $T|_{D(P) \cap X_n}$ is continuous from the strong topology of X into the weak topology of X^*. This concept has been involved in the definitions of multivalued pseudomonotone operator and the mapping of type (M). Of course, we need naturally it when we consider singlevalued operators. We observe that a demicontinuous operator must be finitely continuous.

Remark 1. Our definition of the finite continuity slightly differs from one given in [6].

Lemma 1. *Let $T: X \rightarrow X^*$ be a monotone operator and $P: X \rightarrow X^*$ a quasi-bounded operator. Suppose that $\{x_n\}$ and $\{(T+P)x_n\}$ are both bounded with $\{x_n\} \subset D(T) \cap D(P) \neq \phi$. Then $\{Px_n\}$ is bounded.*

Proof First we assume $T0=0$. By the hypothesis, there exists $M>0$ such that $\|x_n\| \leqslant M$ and $\|(T+P)x_n\| \leqslant M$ for any $n \in \mathcal{N}$. From the monotonicity of T, we have

$$\begin{aligned}(Px_n, x_n) &= ((T+P)x_n, x_n) - (Tx_n, x_n) \\ &\leqslant ((T+P)x_n, x_n) \leqslant M\|x_n\|.\end{aligned}$$

Since P is quasi-bounded, $\|Px_n\| \leqslant M$, $\forall n \in \mathcal{N}$.

Next, set $T_1x = Tx - g$ when $T0 = g \neq 0$. So $T_10 = 0$. The boundedness of $\{(T+P)x_n\}$ implies that $\{(T_1+P)x_n\}$ is also bounded. Hence, $\{Px_n\}$ is bounded.

Lemma 2. *Let $T: X \rightarrow X^*$ be a maximal monotone operator and $P: X \rightarrow X^*$ finitely continuous operator on $D(T) \cap D(P)\ (\neq \phi)$. Suppose that X_n is a finite-dimensional subspace of X and that $Q_n: X^* \rightarrow X_n^*$ is a continuous operator. Then $Q_n(T+P): X_n \rightarrow X_n^*$ is a continuous operator on $D(T) \cap D(P)$.*

Proof T is demicontinuous since it is maximal monotone. Hence, T is finitely continuous. By the hypothesis of P, we know that $(T+P)$ is also finitely continuous. Since Q_n is continuous and since weak convergence implies strong convergence in a finite-dimensional space, $Q_n(T+P): X_n \rightarrow X_n^*$ is continuous on $D(T) \cap D(P)$.

Theorem 1. *Let $T: X \rightarrow X^*$ be a maximal monotone operator, $P: X \rightarrow X^*$ a quasi-bounded and finitely continuous operator which satisfies one of the following:*

(i) *P is generalized pseudomonotone,*

(ii) *P is T-pseudomonotone.*

Suppose that $\Omega\subset X$ is a bounded open set, $0\in \operatorname{Int}\Omega$, $\overline{\Omega}\subset D(T)\cap D(P)$ and $\overline{\Omega}$ is weakly closed. Then $T+P+\varepsilon J$ is an A-proper mapping on $\overline{\Omega}$ with respect to $\Gamma=(\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ for each $\varepsilon>0$, where J is the normalized duality map of X.

Proof We write $\Omega_n=\Omega\cap X_n$. By Lemma 2, $Q_n(T+P)P_n=Q_n(T+P)|_{X_n}: \Omega_n\to X_n^*$ is continuous ($\forall n\in\mathcal{N}$).

Step 1. Let $\{x_{n_j}\}_{j=1}^{\infty}\subset X$ and $x_{n_j}\in\Omega_{n_j}$ such that

$$\|Q_{n_j}(T+P+\varepsilon J)P_{n_j}x_{n_j}-Q_{n_j}f\|\to 0 \quad (j\to\infty) \tag{1}$$

for some $f\in X^*$. Since $\|Q_{n_j}f\|\leqslant\|f\|$ ($\forall j\in\mathcal{N}$), it follows from the equation (1) that $\{Q_{n_j}(T+P+\varepsilon J)P_{n_j}x_{n_j}\}$ is bounded. We write

$$\sup_j\{\|Q_{n_j}(T+P+\varepsilon J)P_{n_j}x_{n_j}\|\}=M_1, \ \sup_{x\in\overline{\Omega}}\|x\|=M_2$$

and

$$M=\max\{M_1, M_2\}.$$

Hence

$$\begin{aligned}((T+P+\varepsilon J)x_{n_j}, x_{n_j}) &= ((T+P+\varepsilon J)P_{n_j}x_{n_j}, P_{n_j}x_{n_j})\\ &=(Q_{n_j}(T+P+\varepsilon J)P_{n_j}x_{n_j}, x_{n_j})\\ &\leqslant\|Q_{n_j}(T+P+\varepsilon J)P_{n_j}x_{n_j}\|\|x_{n_j}\|\leqslant M\|x_{n_j}\| \quad (\forall j\in\mathcal{N}).\end{aligned}$$

Since T is monotone and $0\in\operatorname{Int}\Omega\subset D(T)$, T is quasi-bounded. Furthermore, since J is a bounded map, then $T+P+\varepsilon J$ is also quasi-bounded. It follows from the last inequality that $\|(T+P+\varepsilon J)x_{n_j}\|\leqslant M$ ($\forall j\in\mathcal{N}$). By the boundedness of J, we see that $\{(T+P)x_{n_j}\}$ is bounded. By Lemma 1, $\{Px_{n_j}\}$ is bounded. Therefore, $\{Tx_{n_j}\}$ is also bounded.

Step 2. For each X_n and each x in X_n, we have $x_{n_j}-x\in X_{n_j}$ as $n_j>n$. Hence, the equation (1) implies that

$$\begin{aligned}|((T+P+\varepsilon J)x_{n_j}-f, x_{n_j}-x)| &= |((T+P+\varepsilon J)x_{n_j}-f, P_{n_j}(x_{n_j}-x)|\\ &\leqslant\|Q_{n_j}(T+P+\varepsilon J)x_{n_j}-Q_{n_j}f\|\|x_{n_j}-x\|\\ &\geqslant(M+\|x\|)\|Q_{n_j}(T+P+\varepsilon J)x_{n_j}-Q_{n_j}f\|\to 0 \quad (j\to\infty).\end{aligned} \tag{2}$$

Since$X$ is reflexive and $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j(k)}}\rightharpoonup x_0\in\overline{\Omega}$ ($k\to\infty$). Hence, it follows from the equation (2) that

$$((T+P+\varepsilon J)x_{n_{j(k)}}, x_{n_{j(k)}}-x)\to(f, x_0-x) \quad (k\to\infty) \tag{3}$$

for each x in X_n. Indeed, it is known easily that the equation (3) is valid for any x in X because of $\rho(x, X_n)\to 0$ ($n\to\infty$) for each x in X and the boundedness of $\{(T+P+\varepsilon J)x_{n_{j(k)}}\}$. In particular, set $x=x_0$ in the equation (3), we obtain $((T+P+\varepsilon J)x_{n_{j(k)}}, x_{n_{j(k)}}-x_0)\to 0$ ($k\to\infty$). It follows that

$$((T+P+\varepsilon J)x_{n_{j(k)}}-(T+P+\varepsilon J)x_0, x_{n_{j(k)}}-x_0)\to 0 \quad (k\to\infty). \tag{4}$$

By the monotonicity of T, we have

$$\begin{aligned}&((T+P+\varepsilon J)x_{n_{j(k)}}-(T+P+\varepsilon J)x_0, x_{n_{j(k)}}-x_0)\\ &=(Tx_{n_{j(k)}}-Tx_0, x_{n_{j(k)}}-x_0)+(Px_{n_{j(k)}}-Px_0, x_{n_{j(k)}}-x_0)\\ &\quad+\varepsilon(Jx_{n_{j(k)}}-Jx_0, x_{n_{j(k)}}-x_0)\\ &\geqslant(Px_{n_{j(k)}}-Px_0, x_{n_{j(k)}}-x_0)+\varepsilon(\|x_{n_{j(k)}}\|-\|x_0\|)^2.\end{aligned} \tag{5}$$

We take the limit superior on both sides of the inequality (5) and take notice of the equation (4), then we obtain

$$0 \geqslant \overline{\lim_k} (Px_{n_{j(k)}} - Px_0,\ x_{n_{j(k)}} - x_0).$$

Since $x_{n_{j(k)}} \rightharpoonup x_0$, it follows that

$$\overline{\lim_k} (Px_{n_{j(k)}},\ x_{n_{j(k)}} - x_0) \leqslant 0.$$

We have known in Step 1 that $\{Tx_{n_{j(k)}}\}$ and $\{Px_{n_{j(k)}}\}$ are bounded. We may assume without loss of generality that $Px_{n_{j(k)}} \rightharpoonup g$ in X^*. We claim that $g = Px_0$ and $(Px_{n_{j(k)}}, x_{n_{j(k)}}) \to (Px_0, x_0)$ $(k \to \infty)$, because P is generalized pseudomonotone or T-pseudomonotone. Indeed, we have

$$\lim_k (Px_{n_{j(k)}} - Px_0,\ x_{n_{j(k)}} - x_0) = 0.$$

We take again the limit superior on both sides of (5), then we get

$$0 \geqslant \varepsilon\, \overline{\lim_k} (\|x_{n_{j(k)}}\| - \|x_0\|)^2,$$

it follows that $\|x_{n_{j(k)}}\| \to \|x_0\|$. We assert $x_{n_{j(k)}} \to x_0 (k \to \infty)$ since $x_{n_{j(k)}} \rightharpoonup x_0$ and the space X has the property (h).

Step 3. Since T and J are maximal monotone, they are demicontinuous. Hence, $(T+P+\varepsilon J)x_{n_{j(k)}} \rightharpoonup (T+P+\varepsilon J)x_0$. On the other hand, we have from the equation (3) and the boundedness of $\{(T+P+\varepsilon J)x_{n_{j(k)}}\}$ that

$$((T+P+\varepsilon J)x_{n_{j(k)}},\ x_0 - x) = ((T+P+\varepsilon J)x_{n_{j(k)}},\ x_0 - x_{n_{j(k)}}) + ((T+P+\varepsilon J)x_{n_{j(k)}},\ x_{n_{j(k)}} - x) \to (f,\ x_0 - x) \quad (k \to \infty)$$

for any x in X, i. e., $(T+P+\varepsilon J)x_{n_{j(k)}} \rightharpoonup f$. Thus $(T+P+\varepsilon J)x_0 = f$. This completes the proof of Theorem 1.

Lemma 3. *Let Ω be a bounded subset of X with $\Omega \neq \{0\}$, $T: \overline{\Omega} \to X^*$ an arbitrary operator, $f \in X^*$ and $\rho(f, T(\partial\Omega)) = \alpha > 0$. Then $\rho(f, (T+\varepsilon J)(\partial\Omega)) \geqslant \alpha/2$ as $0 < \varepsilon < \alpha/2d$, where $d = \sup_{x \in \overline{\Omega}} \|x\|$ and J is the normalized duality map of X.*

Proof It suffices to note $\|Jx\| = \|x\|$ and the others are similar to the proof of Lemma 2.2 in [7].

Theorem 2. *Suppose that T, P and Ω satisfy the assmptions in Theorem 1. Further, suppose that $\rho(f, (T+P)(\partial\Omega)) = \alpha > 0$ for $f \in X^*$. Then* $\mathrm{Deg}_A(T+P+\varepsilon J, \Omega, f)$ *is meaningful as $0 < \varepsilon < \frac{\alpha}{2d}$ and it is independent of ε, where $d = \sup_{x \in \overline{\Omega}} \|x\|$.*

Proof By virtue of Theorem 1 and Lemma 3, $T+P+\varepsilon J$ is A-proper on $\overline{\Omega}$ with respect to $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$. so $\mathrm{Deg}_A(T+P+\varepsilon J, \Omega, f)$ is meaningful as $0 < \varepsilon < \frac{\alpha}{2d}$.

For any ε_1 and ε_2 with $0 < \varepsilon_1 < \varepsilon_2 < \frac{\alpha}{2d}$, we will prove that the degrees corresponding to ε_1 and ε_2 respectively are the same by using the homotopy invariance of the topological degree for A-proper mappings. Set

$$T(x,\ t)=(T+P)x+(t\varepsilon_1+(1-t)\varepsilon_2)Jx,\ [x,\ t]\in\overline{\Omega}\times[0,\ 1].$$

By Theorem 1, $T(x,\ t)$ is A-proper for each t in $[0,\ 1]$. For each $n\in\mathcal{N}$, we have

$$Q_nT(P_nx,\ t)=Q_n(T+P)P_nx+(t\varepsilon_1+(1-t)\varepsilon_2)Q_nJP_nx. \tag{6}$$

By Lemma 2, $Q_n(T+P)P_n$ and Q_nJP_n(J is maximal monotone) in the right side of (6) are continuous from X_n into X_n^*. Since $\overline{\Omega}$ is a bounded set and $\|P_n\|=\|Q_n\|=1$, we have $\sup\limits_{x\in\overline{\Omega}}\|Q_nJP_nx\|=M<\infty$. For any $[x',\ t']$ and $[x'',\ t'']$ in $\overline{\Omega}_n\times[0,\ 1]$, we have

$$\begin{aligned}&\|(t'\varepsilon_1+(1-t')\varepsilon_2)Q_nJP_nx'-(t''\varepsilon_1+(1-t'')\varepsilon_2)Q_nJP_nx''\|\\&\quad\leqslant(t'(\varepsilon_1-\varepsilon_2)+\varepsilon_2)\|Q_nJP_nx'-Q_nJP_nx''\|+|(\varepsilon_1-\varepsilon_2)(t'-t'')|\,\|Q_nJP_nx''\|\\&\quad\leqslant\varepsilon_2\|Q_nJP_nx'-Q_nJP_nx''\|+M|(\varepsilon_1-\varepsilon_2)(t'-t'')|.\end{aligned} \tag{7}$$

Then, the second term in the right side of (6) is uniformly continuous from $\overline{\Omega}_n\times[0,\ 1]$ to X_n^*. Hence, $Q_nT(P_nx,\ t)$ is uniformly continuous from $\overline{\Omega}_n\times[0,\ 1]$ to X_n^*, too.

Let $x_{n_j}\in\partial\Omega_{n_j}(\subset\overline{\Omega})$ and $t_{n_j}\in[0,\ 1]$ such that

$$\|Q_{n_j}T(P_{n_j}x_{n_j},\ t_{n_j})-Q_{n_j}g\|\to 0\quad(j\to\infty) \tag{8}$$

for some $g\in X^*$. We may assume that $t_{n_j}\to t\in[0,\ 1]$. As the estimate of the inequality (7), we know easily that

$$\begin{aligned}&\|Q_{n_j}T(P_{n_j}x_{n_j},\ t)-Q_{n_j}T(P_{n_j}x_{n_j},\ t_{n_j})\|\\&\quad\leqslant M|(\varepsilon_1-\varepsilon_2)(t-t_{n_j})|\to 0\quad(j\to\infty)\end{aligned}$$

Combining the equality (8), we obtain

$$\|Q_{n_j}T(P_{n_j}x_{n_j},\ t)-Q_{n_j}g\|\to 0\quad(j\to\infty). \tag{9}$$

For fixed t, since $T(x,\ t)$ is A-proper, there exists $\{x_{n_{j(k)}}\}\subset\{x_{n_j}\}$ by (9) such that $x_{n_{j(k)}}\to x\in\overline{\Omega}$ and $T(x,\ t)=g$.

Furthermore, by the hypothesis and Lemma 3, $f\overline{\in}(T(x,\ t))(\partial\Omega\times[0,\ 1])$.

As was proved above that $T(x,\ t)$ satisfies all conditions of Theorem 4.1 (B_3') in [5]. Thus

$$\begin{aligned}\mathrm{Deg}_A(T+P+\varepsilon_1J,\ \Omega,\ f)&=\mathrm{Deg}_A(T(x,\ 0),\ \Omega,\ f)\\&=\mathrm{Deg}_A(T(x,\ 1),\ \Omega,\ f)=\mathrm{Deg}_A(T+P+\varepsilon_2J,\ \Omega,\ f).\end{aligned}$$

Theorem 2 is proved.

After getting above preliminary results, we can define the generalized topological degree as follows:

Definition 1. *Let T: $X\to X^*$ be a maximal monotone operator, P: $X\to X^*$ a quasi-bounded and finitely continuous generalized pseudomonotone or T-pseudomonotone operator. Suppose that $\Omega\subset X$ is a bounded open set, $0\in\mathrm{Int}\,\Omega$, $\overline{\Omega}\subset D(T)\cap D(P)$ and $\overline{\Omega}$ is weakly closed. Further, Let f be an element in X^* such that $\rho(f,\ (T+P)(\partial\Omega))=\alpha>0$. Then, we define that the topological degree of $(T+P)$ on Ω over f is*

$$\mathrm{Deg}((T+P),\ \Omega,\ f)=\mathrm{Deg}_A(T+P+\varepsilon J,\ \Omega,\ f),$$

where $0<\varepsilon<\dfrac{\alpha}{2d}$, $d=\sup\limits_{x\in\overline{\Omega}}\|x\|$ and J is the normalized duality map of X.

In what follows we will explain simply the topological degree. Obviously, null operator is maximal monotone, quasi-bounded and finitely continuous generalized pseudomonotone. A pseudomonotone must be generalized pseudomonotone. Moreover, a continuously and strongly monotone operator on X to X^* must be maximal monotone. Thus we have unified in Definition 1 the treatment of the topological degrees for some operators of monotone type[6, 7].

§ 2. The Properties for the Topological Degree and the Results of Surjectity

It is possible to prove that the degree given by us has the same as all properties of the topological degree for A-proper mapping. But, we will prove only some properties with relation to the surjectivity.

Lemma 4. *Let T: $X \to X^*$ be a maximal monotone operator, P: $X \to X^*$ a quasi-bounded and generalized pseudomonotone or T-pseudomonotone operator. Suppose that $x_n \to x_0$ with x_n, $x_0 \in D(T) \cap D(P)$ and $Tx_n + Px_n \to f$ for some $f \in X^*$. Then $Tx_0 + Px_0 = f$.*

Proof By Lemma 1, $\{Px_n\}$ is bounded since $\{x_n\}$ and $\{Tx_n + Px_n\}$ are bounded. Hence, $\{Tx_n\}$ is bounded, too. we may assume that $Px_n \rightharpoonup g \in X^*$ (if necessary, pass to an infinite subsequence). By the monotonicity of T, we have

$$\begin{aligned}(Px_n, x_n - x_0) &= ((T+P)x_n, x_n - x_0) - (Tx_n - Tx_0, x_n - x_0) - (Tx_0, x_n - x_0)\\ &\leqslant ((T+P)x_n, x_n - x_0) - (Tx_0, x_n - x_0).\end{aligned}$$

Hence

$$\overline{\lim}(Px_n, x_n - x_0) \leqslant 0.$$

Because P is generalized pseudomonotone or T-pseudomonotone, we claim that $g = Px_0$ and $(Px_n, x_n) \to (Px_0, x_0)$. Hence, $(Px_n - Px_0, x_n - x_0) \to 0 (n \to \infty)$. $\forall x \in D(T)$, since T is monotone, we have

$$(Tx_n + Px_n - Px_0 - Tx, x_n - x) \geqslant (Px_n - Px_0, x_n - x_0) + (Px_n - Px_0, x_n - x).$$

Hence, $(f - Px_0 - Tx, x_0 - x) \geqslant 0$. Since T is maximal monotone, we get $Tx_0 + Px_0 = f$.

Corollory. *If $\overline{\Omega} \subset X$ is a bounded and weakly closed set, then $(T+P)(\overline{\Omega})$ is a closed set.*

Proof It is obvious.

Remark 2. Lemma 1—4 remain valid if X is not separable.

We denote by $T_p(\overline{\Omega})$ the collection of all sum $(T+P)$ from this place to end where T and P are any operators in the sense of Definition 1.

Theorem 3. *Let $(T+P) \in T_p(\overline{\Omega})$ and $f \overline{\in} (T+P)(\overline{\Omega})$. Then* Deg $(T+P, \Omega, f) = \{0\}$. *Hence, if* Deg $(T+P, \Omega, f) \neq \{0\}$, *then the equation $(T+P)x = f$ has a solution* in $\overline{\Omega}$.

Proof By the corollary of Lemma 4, $(T+P)(\overline{\Omega})$ is a closed set. Hence

$$\rho(f, (T+P)(\overline{\Omega}))>0.$$

By Lemma 3, $\rho(f, (T+P+\varepsilon J)(\overline{\Omega}))>0$ for sufficiently small $\varepsilon>0$. For such ε, because of the property for the topological degree of an A-proper mapping,

$$\mathrm{Deg}(T+P, \Omega, f)=\mathrm{Deg}_A(T+P+\varepsilon J, \Omega, f)=\{0\}.$$

And the second part of this theorem is obvious.

Theorem 4. *Let $(T+P)\in T_p(\overline{\Omega})$, $\rho(0, (T+P)(\partial\Omega))>0$. Further, suppose that Ω is symmetric about the origin of X and $(T+P)$ is odd on $\partial\Omega$. Then* $\mathrm{Deg}(T+P, \Omega, 0)$ *is odd. In particular,* $\mathrm{Deg}(T+P, \Omega, 0)\neq\{0\}$.

Proof Since the normalized duality map is odd, $T+P+\varepsilon J$ is odd for any $\varepsilon>0$, too. Since P_n and Q_n are linear operators, $Q_n(T+P+\varepsilon J)\ P_n$ is also odd on $\partial\Omega_n$ for each $n\in\mathcal{N}$. By Theorem 4.1 (B_5) in [5], $\mathrm{Deg}(T+P+\varepsilon J, \Omega, 0)$ is odd.

Corollary. *Suppose the conditions on Ω as in Theorem 4. Then we have*

$$\mathrm{Deg}(kJ, \Omega, 0)\neq\{0\} \text{ for each } k>0.$$

Proof We take $T=kJ$ and $P=0$ in Theorem 4.

Lemma 5. *Suppose that a family of operators S_t: $\overline{\Omega}\times[0, 1]\to X^*$ satisfy $S_t\in T_p(\overline{\Omega})$ for each t in $[0, 1]$. Suppose that Q_nS_t: $\overline{\Omega}_n\times[0, 1]\to X^*$ is continuous ($\forall n\in\mathcal{N}$). If for $f\in X^*$, there are constants α_1, $\alpha_2>0$ an ad positive integer n_0 such that for all $t\in[0, 1]$ it is satisfied that*

$$\rho(f, S_t(\partial\Omega))\geqslant\alpha_1, \tag{10}$$

$$\rho(Q_nf, Q_nS_t(\partial\Omega_n))\geqslant\alpha_2, \text{ as } n\geqslant n_0. \tag{11}$$

Then $\mathrm{Deg}(S_t, \Omega, f)$ *is independent of t in* $[0, 1]$.

Proof We take ε with $0<\varepsilon<\min\left\{\frac{\alpha_1}{2d}, \frac{\alpha_2}{2d}\right\}$, where $d=\sup_{x\in\overline{\Omega}}\|x\|$. By the definition, we have $\mathrm{Deg}(S_t, \Omega, f)=\mathrm{Deg}_A(S_t+\varepsilon J, \Omega, f)$. Thus it sufficies to prove that the right stide of the equality is independent of ε. Since J: $X_n\to X_n^*$ and Q_nS_t: $\overline{\Omega}_n\times[0, 1]\to X_n^*$ are continuous, $Q_n(S_t+\varepsilon J)$: $\overline{\Omega}_n\times[0, 1]\to X_n^*$ is continuous for each $n\in\mathcal{N}$.

Next, when $n\geqslant n_0$, if we attend to $\|Q_n\|=1$ ($\forall n\in\mathcal{N}$) and $\|Jx\|=\|x\|$, then the inequality (11) implies

$$\|Q_nS_tx+\varepsilon Q_nJx-Q_nf\|\geqslant\|Q_nS_tx-Q_nf\|-\varepsilon\|Q_n\|\,\|Jx\|$$

$$\geqslant\alpha_2-\varepsilon\|x\|\geqslant\alpha_2-\varepsilon d\geqslant\frac{\alpha_2}{2}.$$

Therefore, $Q_nf\notin Q_n(S_t+\varepsilon J)(\partial\Omega_n)$ as $n\geqslant n_0$.

Finally, by the equality (10) and the homotopy invariance of Brouwer degree, we know that $\mathrm{Deg}(Q_n(S_t+\varepsilon J)P_n, \Omega_n, Q_nf)$ is independent of t in $[0, 1]$ (as $n\geqslant n_0$). Then $\mathrm{Deg}_A(S_t+\varepsilon J, \Omega, f)$ is independent of t according to the definition of the degree for A-proper. Lemma 5 is proved.

Lemma 6. *Let $S=T+P\in T_p(\bar{\Omega})$. Suppose that there is a constant $C>0$ such that*

$$(Sx,\ x)\geqslant C,\quad \forall x\in\partial\Omega. \tag{12}$$

Then $\operatorname{Deg}(S,\ \Omega,\ 0)=\operatorname{Deg}(J,\ \Omega,\ 0)$, *where J is the normalized duality map of X.*

Proof Set

$$S_tx=tSx+(1-t)Jx=tTx+tPx+(1-t)Jx\quad 0\leqslant t\leqslant 1.$$

We will verify that S_t satisfies all conditions in Lemma 5. For each $t\in[0,\ 1]$, since T and J are maximal monotone and $D(J)=X$, $tT+(1-t)J$ is maximal monotone, too[14]; Obviously, tP is quasi-bounded and finitely continuous generalized pseudomonotone or T-pseudomonotone. Therefore, $S_t\in T_p(\bar{\Omega})$.

By Lemma 2, $Q_n(T+P)$: $\bar{\Omega}_n\to X_n^*$ is continuous for each $n\in\mathcal{N}$. Since $\bar{\Omega}_n$ is a bounded and closed set, we may write $\sup\limits_{x\in\bar{\Omega}_n}\|Q_n(T+P)x\|=M<\infty$ and write $d=\sup\limits_{x\in\bar{\Omega}}\|x\|$. Hence, by $\|Q_nJx\|\leqslant\|x\|$, $\forall[x_0,\ t_0]$, $[x,\ t]\in\bar{\Omega}_n\times[0,\ 1]$ we have

$$\begin{aligned}\|Q_nS_tx-Q_nS_{t_0}x\|&\leqslant|t-t_0|\,\|Q_n(T+P)x\|\\&\quad+t_0\|Q_n(T+P)x-Q_n(T+P)x_0\|+|t-t_0|\,\|Q_nJx\|\\&\leqslant M|t-t_0|+t_0\|Q_n(T+P)x-Q_n(T+P)x_0\|+d|t-t_0|.\end{aligned}$$

It follows that Q_nS_t: $\bar{\Omega}_n\times[0,\ 1]\to X_n^*$ is continuous for each $n\in\mathcal{N}$.

Finally, the inequality (12) means $0\bar{\in}\partial\Omega$. Hence, there are positive constants d_1 and d_2 such that $d_1\leqslant\|x\|\leqslant d_2$ for any x in $\partial\Omega$. For $x\in\partial\Omega$, we have from the inequality (12) and from $(Jx,\ x)=\|x\|^2$ that

$$\begin{aligned}\|tSx+(1-t)Jx\|&\geqslant\frac{1}{\|x\|}(tSx+(1-t)Jx,\ x)\\&\geqslant\frac{tC}{\|x\|}+(1-t)\|x\|\geqslant\frac{tC}{d_2}+(1-t)d_1\\&\geqslant\min\left\{\frac{C}{d_2},\ d_1\right\}=\alpha>0,\end{aligned}$$

i.e., $\rho(0,\ S_t(\partial\Omega))\geqslant\alpha$ for all t in $[0,\ 1]$. By $Q_n=P_n^*$ and $P_n^{**}=P_n$, similarly, we can find $\rho(0,\ Q_nS_t(\partial\Omega))\geqslant\alpha$, i. e., $\rho(Q_n0,\ Q_nS_t(\partial\Omega))\geqslant\alpha$ because $Q_n0=0$ for each $n\in\mathcal{N}$. We know from Lemma 5 that

$$\operatorname{Deg}(S,\ \Omega,\ 0)=\operatorname{Deg}(S_1,\ \Omega,\ 0)=\operatorname{Deg}(S_0,\ \Omega,\ 0)=\operatorname{Deg}(J,\ \Omega,\ 0).$$

Lemma 6 is proved.

Theorem 5. *Let $(T+P)\in T_p(\bar{\Omega})$ and $f\in X^*$. If there exist two constants r, $C>0$ with $C>\|f\|$ such that $B(0,\ r)\subset\bar{\Omega}$ and*

$$((T+P)x,\ x)\geqslant C\|x\|\ \text{as}\ x\in\partial B(0,\ r), \tag{13}$$

Then the equation $(T+P)x=f$ has at least a solution in $B(0,\ r)$, where $B(0,\ r)$ is the open ball of radius r about origin of X.

Proof We write $S=T+P$. Set $S_1x=Sx-f$ if $f\neq 0$. It is clear that $S_1\in T_p(\bar{\Omega})$. It implies easily that $(S_1x,\ x)\geqslant(C-\|f\|)\|x\|$, as $x\in\partial B(0,\ r)$. Hence, we may assume without loss of generality $f=0$. The inequality (13) implies

$$(Sx, x)\geqslant Cr,\ \|Sx\|\geqslant\frac{(Sx, x)}{\|x\|}\geqslant C$$

for $x\in\partial B(0, r)$. By Lemma 6 and Cololiary of Theorem 4, we know that

$$\mathrm{Deg}(S, B(0, r), 0)=\mathrm{Deg}(J, B(0, r), 0)\neq\{0\}.$$

If the equation $Sx=0$ hasn't a solution in $B(0, r)$, then $0\overline{\in}S(\overline{B(0, r)})$. Because a closed ball is a weakly closed set, thus $S(\overline{B(0, r)})$ is the closed set from Corollary of Lemma 4. Consequently, $\rho(0, S(\overline{B(0, r)}))>0$. By Theorem 3, the equation $(T+P)x=0$ has a solution in $B(0, r)$. This completes the proof of Theorem 5.

Corollary 1. *Let T: $X\to X^*$ be a maximal monotone operator and P: $X\to X^*$ a quasi-bounded and generalized pseudomonotone or T-pseudomonotone operator, $D(T)=D(P)=X$.* If $(T+P)$ *is coercive, i. e.*

$$\lim_{\|x\|\to\infty}\frac{((T+P)x, x)}{\|x\|}=+\infty, \tag{14}$$

then $R(T+P)=X^*$.

Proof For each $f\in X^*$, there are two constants r and C with $C>\|f\|$ and $r>0$ such that $((T+P)x, x)\geqslant C\|x\|$ as $x\in\partial B(0, r)$ by the coercive condition (14). By Theorem 5, there exists x in X which satisfies $(T+P)x=f$.

Corollary 2. *Let T be a maximal monotone operator on X to X^*. If T is coercive, then $R(T)=X^*$.*

Proof It suffices to take null operator as P in Corollory 1.

Corollary 3. *Let P be a quasi-bounded and finitely continuous generalized pseudomonotone operator on X to X^*. If P is coercive, then $R(P)=X^*$.*

Proof We take null operator as T in Corollary 1.

Theorem 5 is the basic result in this paper. Corollary 1 has answered affirmatively Browder's problem[1] if X is a real separable and reflexive Banach space. Moreover, we require only that T-pseudomonotone operator is quasi-bounded. The assumptions about P in Corollary 1 are simpler than ones of Theorem 7 in [3]. Corollary 2 and Corollary 3 are contained essentially in [3, 9], but our methods differ from theirs.

§ 3. Another Application

Browder considered the range of the sum for a maximal monotone mapping and a bounded pseudomonotone mapping[13]. Here, we apply Corollary 1 in § 2 to consider the range of the sum for a maximal monotone operator and a quasi-bounded and finitely continuous pseudomonotone or T-pseudomonotone operator. We cite first the following concept and result.

Definition (see [13]). Let T, S and R be mappings with domains in the Banach space X and values in the Banach space Y and with $D(R)=X$. Then the pair $[T, S]$ is said to be in good position with respect to R if there exist a mapping ζ of X into

Y, a continuous function β from the positive real to the positive reals such that $\beta(r) \to 0$ as $r \to +\infty$ and two constants C, $C_0 > 0$ such that the following conditions are satisfied:

(1) ζ is uniformly continuous on bounded set of X to the strong topology of Y^*, $\zeta(rx) = r\zeta(x)$ for $r > 0$ and x in X

$$\|\zeta(x)\| \geqslant C\|x\|, \ x \in X.$$

(2) For each u in $D(T)$, v in $D(S)$

$$\inf\{(Tx - Tu, \ \zeta(x-u)) + \beta(\|x\|)\|x\|\} > -\infty, \tag{15}$$

$$\inf\{(Sx - Sv, \ \zeta(x-u)) + \beta(\|x\|)\|x\|\} > -\infty, \tag{16}$$

$$\inf\{(Rx, \ \zeta(x-u)) + C_0\|x\|\} > -\infty. \tag{17}$$

Theorem (Theorem 1 in [13]). *Let X and Y be Banach spaces, T, S and R be three mappings with domains in X and with values in Y such that the pair $[T, S]$ is in good position with respect to R, where $D(R) = X$ and R maps bounded sets in X into bounded set in Y. Suppose that the following additional conditions hold:*

(1) *For each $\xi > 0$, the mapping $T + S + \xi R$. has all of Y as its range;*

(2) *For each closed ball B_R about O in X, $(T+S)(D(T) \cap D(S) \cap B_R)$ is closed in Y.*

Then $\mathrm{Int}(R(T+S)) = \mathrm{Int}(R(T) + R(S))$.

Theorem 6. *Let T: $X \to X^*$ be a maximal monotone operator, S: $X \to X^*$ a quasi-bounded and finitely continuous generalized pseudomonotone or T-pseudomonotone operator, $D(T) = D(S) = X$. Suppose that the operator S satisfies the condition (16), i. e., there exists a continuous function $\beta(r)$: $R_+ \to R_+$, $\beta(r) \to 0 (r \to +\infty)$ such that for any u, $v \in X$, there exists a constant $C_{u,v}$ such that*

$$(Sx - Su, \ x - v) \geqslant C_{u,v} - \beta(\|x\|)\|x\| \quad \forall x \in X. \tag{18}$$

Then $\mathrm{Int}(R(T+S)) = \mathrm{Int}(R(T) + R(S))$.

Proof Since T is maximal monotone, it satisfies (15). We take the identity operator of X as ζ, and take the normalized duality map J of X as R. It is verified easily that $[T, S]$ satisfies the definition of good position with respect to J from the ondition (18)[13]. By Corollary of Lemma 4 in § 2, the condition (2) in the Browder's theorem is satisfied. For each $\xi \geqslant 0$, ξJ is also generalized pseudomonotone or T-pseudomonotone. Further, since $(\xi Jx, x) = \xi\|x\|^2$, $P = S + \xi J$ is also generalized pseudomonotone[3], when S is generalized pseudomonotone. It is proved easily that P is T-pseudomonotone, too, if S is T-pseudomonotone. In order to verify the condition (1) in the Browder's theorem, we put $u = v = 0$ in the inequality (18). It follows that for each $\xi > 0$ we obtain

$$\frac{(Px, x)}{\|x\|} = \frac{(Sx, x)}{\|x\|} + \frac{\xi(Jx, x)}{\|x\|} \geqslant C_{0,0} - \beta(\|x\|)\|x\| - \|S0\| + \xi\|x\|$$

$$= C_{0,0} - \|S0\| + (\xi - \beta(\|x\|)\|x\| \to +\infty \quad \text{as } \|x\| \to \infty,$$

i. e., the operator P is coercive. By Corollary 2 of Theorem 5, we get $R(T+S+\xi J)=X^*$. This theorem holds according to the Browder's theorem.

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