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SMOOTH POINT MEASURES AND DIFFEOMORPHISM GROUPS

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Abstract

Let $X=R^d(d>1)$. Consider the unitary representations of Diff(X) given by quasi-invariant measures under the action of Diff(X). The author proposes smooth point measures as generalization of Poisson point measures and proves that every smooth point measure is quasi-invariant under the action of Diff(X) and if $\{U_g^i\}$, $i=1, 2$, are the unitary representations of Diff(X) given by the smooth point measures μ_i, $i=1, 2$, respectively, then $\{U_g^1\}$ is unitarily equivalent to $\{U_g^2\}$ iff μ_1 is equivalent to μ_2 as measure.

§ 1. Introduction

Let $X=R^d(d>1)$ and Diff(X) be the group of all C^∞ diffeomorphsm from X onto X having compact support, i. e., for every element ψ of Diff(X) there exists a compact subset C_ψ of X such that $\psi(x)=x$ for all $x\bar{\in} C_\psi$. Assume

$$\mathrm{Diff}(X, K)=\{\psi:\psi\in \mathrm{Diff}(X),\ C_\psi\subset K\},$$

where K is a compact subset of X. Endow Diff(X, K) with the topology defined by the countable family of metrics

$$\|\varphi-\psi\|_n=\max_{0\leq|m|\leq n}\sup_{x\in X}\|D^{(m)}\varphi(x)-D^{(m)}\psi(x)\|,$$

for $\varphi, \psi\in$ Diff(X, K) and $n=0, 1, 2, \cdots$, where $(m)=(m_1, m_2, \cdots, m_d)$ is a d-tuple of non-negative integer, $|m|=m_1+m_2+\cdots+m_d$, and $D^{(m)}=\partial^{|m|}/\partial x_1^{m_1}\cdots\partial x_d^{m_d}$. We say a sequence $\{\psi_n\}$ of Diff(X) converge to ψ in Diff(X) if there exists a compact subset K of X such that $\{\psi_n\}\subset$ Diff(X, K) and $\{\psi_n\}$ converges to ψ in Diff(X, K). In this way, Diff(X) becomes a non-locally compact topological group.

We write

$$\varDelta(A)=\{r:r=(x_n)\subset A,\ \text{for every compact subset } K,\ |K\cap r| \text{ is finite}\},$$

where A is a subset of X and $|B|$ is the cardinal number of the set B. For every compact set K, we define a function $K(\cdot)$ on $\varDelta(A)$ such that

$$K(r)=|r\cap K|,\quad \text{for } r=(x_n)\in\varDelta(A).$$

The σ-algebra generated by these functions is denoted by $\mathscr{F}_A$. We call the measurable

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space $(\Delta(X), \mathscr{F}_X)$ the configuration space with $\mathrm{Diff}(X)$. We define the action of $\mathrm{Diff}(X)$ on $(\Delta(X), \mathscr{F}_X)$ such that

$$\psi(r)=(\psi(a_n)), \quad \text{for } r=(a_n)\in\Delta(X) \text{ and } \psi\in\mathrm{Diff}(X).$$

This paper is devoted to the measures on $(\Delta(X), \mathscr{F}_X)$ which are quasi-invariant under the action of $\mathrm{Diff}(X)$ and the unitary representations of $\mathrm{Diff}(X)$ given by such measures. Our main result is as follows. First, we propose smooth point measures on $(\Delta(X), \mathscr{F}_X)$ as a generalization of Poisson measures and we prove that every smooth point measure is quasi-invariant under the action of $\mathrm{Diff}(X)$. If we define the unitary representation $\{U_\psi^\mu\}$ of $\mathrm{Diff}(X)$ given by μ such that

$$U_\psi^\mu(F)=F(\psi^{-1}r)\left(\frac{d\psi\mu}{d\mu}(r)\right)^{1/2}, \text{ for } F\in L_\mu^2(\Delta(X)),\ \psi\in\mathrm{Diff}(X),$$

then it must be continuous. It is obvious that $\{U_\psi^{\mu_1}\}$ and $\{U_\psi^{\mu_2}\}$ are unitarily equivalent if μ_1 and μ_2 are equivalent as measures. Conversely, it is difficult to find the necessary condition of the equivalence of $\{U_\psi^{\mu_1}\}$ and $\{U_\psi^{\mu_2}\}$. In this paper we prove that if μ_1 and μ_2 are smooth point measures, then $\{U_\psi^{\mu_1}\}$ is equivalent to $\{U_\psi^{\mu_2}\}$ iff μ_1 is equivalent to μ_2 as measure. In particular, if $\mathscr{P}_{m_i}$ is a Poission measure of smooth measure m_i, $i=1, 2$, then $\{U_\psi^{m_1}\}$ is unitary equivalent to $\{U_\psi^{m_2}\}$ iff its Kakutani's distance $\left(\int(\sqrt{dm_1}-\sqrt{dm_2})^2\right)^{1/2}$ is finite.

§ 2. Smooth point measures

Set

$$A=S_r=\{x: x\in X,\ \|x\|_d<r<\infty\},$$
$$\mathrm{Diff}(A)=\{\psi:\psi\in\mathrm{Diff}(X),\ C_\psi\subset A\}.$$

A measure m on X is called a smooth measure if $dm=f(x)dx$, where $f\in C^\infty$, $f(x)>0$ everywhere, and dx is the Lebesque measure on X. We write

$$\mathrm{Diff}(X, m)=\{\psi:\psi\in\mathrm{Diff}(X),\ \psi m=m\},$$
$$\mathrm{Diff}(A, m)=\mathrm{Diff}(X, m)\cap\mathrm{Diff}(A).$$

Definition 1. *A probability measure μ on $(\Delta(X), \mathscr{F}_X)$ is a smooth point measure if*

a. $\mu\{r:|r\cap A|=k\}>0$, *for $k\in Z$ and for every A.*

b. *there exists a smooth measure m such that μ is invariant under the action of the subgroup* $\mathrm{Diff}(X, m)$, i. e., $\psi\mu=\mu$ *for every* $\psi\in\mathrm{Diff}(X, m)$.

Remark. We write $Z=(0, 1, 2, \cdots)$ and $Z_+=(1, 2, \cdots)$.

If m is a smooth measure on X, we can construct a Poisson measure $\mathscr{P}_m$ on $(\Delta(X), \mathscr{F}_X)$ such that

$$\mathscr{P}_m\{r:|r\cap C|=k\}=e^{-m(C)}\frac{(m(C))^k}{k!}, \text{ for } k\in Z,$$

where C is a compact subset of X.

Obviously, every Poisson measure is smooth. Moreover, there exist a lot of smooth measures which are essentially different from Poisson measures. For example, let m be a smooth measure and $m(X)=\infty$, we define

$$\mathscr{P}=\frac{1}{2}(\mathscr{P}_{t_1m}+\mathscr{P}_{t_2m}),\text{ where } t_1>t_2>0.$$

Then $\mathscr{P}$ is smooth, but it is not equivalent to any Poisson measure. Now we consider

$$A^{(k)}=\{r:r \text{ is a finite subset of } A \text{ and } |r|=k\},\text{ for } k\in Z_+,$$
$$A^{(0)}=\{\phi\}.$$

By the definition, we have $\psi A^{(k)}=A^{(k)}$, for $k\in Z$ and $\psi\in \mathrm{Diff}(A)$.

Suppose

$$(A^k)=\{(x_1, x_2, \cdots, x_k):(x_1, \cdots, x_k)\in A^k,\ x_i\neq x_j,\ i,\ j=1, \cdots, k\}.$$

Thus (A^k) is an open subset of X^k. Let O be a connected component of (A^k), we define a map T from O onto $A^{(k)}$ such that

$$T(x_1, \cdots, x_k)=\{x_1, \cdots, x_k\}\in A^{(k)}.$$

Evidently, T is bijective. We endow $A^{(k)}$ with a topology such that T is a homeomorphism from O onto $A^{(k)}$. Hence $A^{(k)}$ becomes a C^∞ manifold of $k\cdot d$ dimension. Denote the Borel σ-algebra of $A^{(k)}$ by $B(A^{(k)})$, and write $Tm^k=m_k$, where m^k is the product measure of m on X^k, we conclude that m_k is invariant under the action of $\mathrm{Diff}(A, m)$.

Proposition 1. *Let* λ *be a finite measure on* $(A^{(k)}, B(A^{(k)}))$ *and* m *be a smooth measure on* X. *If* λ *is invariant under the action of* $\mathrm{Diff}(A, m)$, *then there exists non-negative constant* C *such that* $d\lambda=Cdm$.

Proof By [1], it is well known that λ is equivalent to m. Moreover, λ is invariant with respect to $\mathrm{Diff}(A, m)$, and this ends the proof.

Next we observe a smooth point measure μ with respect to m. Define a map $P_A(r)=r\cap A$, $P_{A^c}(r)=r\cap A^c$, for $r\in\Delta(X)$. Clearly, P_A and P_{A^c} are measurable maps from $(\Delta(X), \mathscr{F}_X)$ into $(\Delta(A), \mathscr{F}_A)$ and $(\Delta(A^c), \mathscr{F}_{A^c})$ respectively. Furthermore, we have $P_A(\psi r)=\psi(P_A r)$, $P_{A^c}(\psi r)=P_{A^c}(r)$, and $\psi\mathscr{F}_A=\mathscr{F}_A$, $\psi B=B$ for $r\in\Delta(X)$ $\psi\in\mathrm{Diff}(A)$, $B\in\mathscr{F}_{A^c}$. Let us write $\mu^A=P_A\mu$, $\mu^A_k(B)=\mu^A(B)$, for $B\in B(A^{(k)})$, then μ^A_k is invariant under $\mathrm{Diff}(A, m)$.

Proposition 2. *If* μ_1, μ_2 *are smooth point measure, then* μ^A_1 *is equivalent to* μ^A_2.

Proof It is immediate that $A=\bigcup_{k\geq 0}A^{(k)}$, and $A^{(k)}\in\mathscr{F}_A$ for $k\in Z$. So we have

$$\mu^A_1(B)=\sum_{k\geq 0}\mu^A_1(B\cap A^{(k)}),\quad \mu^A_2(B)=\sum_{k\geq 0}\mu^A_2(B\cap A^{(k)}),\text{ for } B\in\mathscr{F}_A.$$

According to Proposition 1 and Definition 1, there exist constants $C^{(i)}_k>0$, $i=1, 2$, $k\in Z_+$, such that

$$\mu^A_1(B\cap A^{(k)})=C^{(1)}_k m^{(1)}_k(B\cap A^{(k)}),\quad \mu^A_2(B\cap A^k)=C^{(2)}_k m^{(2)}_k(B\cap A^{(k)}),\text{ for } k\in Z_+,$$

where $m^{(1)}_k$, $m^{(2)}_k$ are smooth measures on $A^{(k)}$ which are defined as before. Because

$m_k^{(1)}$ is equivalent to $m_k^{(2)}$ for $k\in Z_+$, and $\mu_1^A(B\cap A^{(0)})=0$ iff $\mu_2^A(B\cap A^{(0)})=0$, thus the conclusion follows.

Corollary. *If K_1, K_2 are compact subsets of X and μ is a smooth point measure with respect to m. Suppose that $m(K_1\cap K_2)=0$, $\mu(B_1)>0$, $\mu(B_2)>0$, where $B_1\in P_{k_1}^{-1}\mathscr{F}_{k_1}$, $B_2\in P_{k_2}^{-1}\mathscr{F}_{k_2}$, then $\mu(B_1\cap B_2)>0$.*

Proposition 3. *Let (Ω, S) be a measurable space, μ_i be finite measure on (Ω, S), $i=1, 2$, and ψ be a measurable isomorphism on (Ω, S) such that*

$$\mu_1\approx\mu_2\approx\psi\mu_1\approx\psi\mu_2, \text{ then } \frac{d\psi\mu_1}{d\psi\mu_2}(\omega)=\frac{d\mu_1}{d\mu_2}(\psi^{-1}\omega), \text{ a. s., } \mu_2.$$

Proof Since $\psi\mu_1(B)=\mu_1(\psi^{-1}B)$, for $B\in S$, we have

$$\int_B \frac{d\psi\mu_1}{d\psi\mu_2}(\omega)\,d\psi\mu_2=\psi\mu_1(B)=\int \mathbf{1}_B(\psi\omega)\,d\mu_1=\int_B \frac{d\mu_1}{d\mu_2}(\psi^{-1}\omega)\,d\psi\mu_2,$$

and this ends the proof.

Proposition 4. *If (Ω, S) is a measurable space, G is a group of measurabl maps on (Ω, S). Let μ_1, μ_2 be finite measures on (Ω, S). Assume they both are quasi-invariant with G. We define G's unitary representations $\{U_g^1\}$ and $\{U_g^2\}$ on $L^2_{\mu_1}$ and $L^2_{\mu_2}$ respectively, $U_g^iF=F(g^{-1}\omega)\left(\frac{dg\mu_i}{d\mu_i}(\omega)\right)^{1/2}$, for $F\in L^2_{\mu_i}$, $i=1, 2$. If μ_1 is equivalent to μ_2 as measure, then $\{U_g^1\}$ is unitarily equivalent to $\{U_g^2\}$.*

Proof Set

$$T(F)=F(\omega)\left(\frac{d\mu_2}{d\mu_1}(\omega)\right)^{1/2}, \text{ for } F\in L^2\mu_1,$$

thus, T is a unitary map from $L^2_{\mu_1}$ onto $L^2_{\mu_2}$ and $T^{-1}U_g^2T=U_g^1$, for $g\in G$.

Proposition 5. *Let μ be a probability measure on $(\varDelta(X), \mathscr{F}_X)$, $\psi_n\in \mathrm{Diff}(X)$, $\psi\in\mathrm{Diff}(X)$. If $\psi_n\to\psi$ as $n\to\infty$ and $\psi_n\mu=\mu$ for every n, then $\psi\mu=\mu$.*

Proof Suppose that $A=S_r\supset\bigcup_n C_{\psi_n}$, and $\varphi_i\in C^\infty(X)$, $i=1, \cdots, m$, which support $C_i\subset A$. Define

$$P=\{r: r\in\varDelta(X),\ (\langle r, \varphi_1\rangle, \cdots, \langle r, \varphi_m\rangle)\in C\},$$

where $\langle r, \varphi_i\rangle=\sum_{x\in r}\varphi_i(x)$, C is a compact subset of R^m. By Fatou's lemma,

$$\overline{\lim_{n\to\infty}}\,\mu(\psi_n^{-1}P)\leqslant\mu(\overline{\lim_{n\to\infty}}\,\psi_n^{-1}P).$$

It is clear that $P=\{r: (\langle P_Ar, \varphi_1\rangle, \cdots, \langle P_Ar, \varphi_m\rangle)\in C\}$, and $\psi_nP_Ar=P_A(\psi_nr)$, for n. If $P_A(r)=(a_1, \cdots, a_k)$, $k\in Z_+$, then $\psi_nP_A(r)=(\psi_n(a_1), \cdots, \psi_n(a_k))$. Because $\psi_n(a_j)\to\psi(a_j)$, $j=1, \cdots, k$, it follows that $\overline{\lim_n}\,\psi_n^{-1}P\subset\psi_0^{-1}P$.

On the other hand, $\psi_n^{-1}\to\psi^{-1}$ as $n\to\infty$, by the fact that $\mathrm{Diff}(X)$ is a topological group. Thus, $\mu(P)=\mu(\psi_0P)$ for each P, this ends the proof.

Definition 2. *Let μ be a smooth point measure. for $A=S_r$, we define $\tilde{\mu}$ on $(\varDelta(X), \mathscr{F}_X)$ such that $\tilde{\mu}=\mu^A\times\mu^{A^c}$, where $\mu^A=P_A\mu$, $\mu^{A^c}=P_{A^c}\mu$.*

Remark. $(\varDelta(X), \mathscr{F}_X)=(\varDelta(A)\times\varDelta(A^c), \mathscr{F}_A\times\mathscr{F}_{A^c})$.

Proposition 6. *If μ is a smooth point measure, then $\mu \ll \tilde{\mu}$ and*

$$\frac{d\mu}{d\tilde{\mu}} = \sum_{k\geq 0} \rho_k(P_{A^c}r)\mathbf{1}_{A^{(k)}}(P_A r),$$

where $\rho_k(\cdot)$ is a non-negative measurable function on $(\Delta(A^c, \mathscr{F}_{A^c})$, and $\mathbf{1}_{A^{(k)}}$ is the indicator of $A^{(k)}$.

Proof If $\widehat{\mathscr{F}}_A = P_A^{-1}\mathscr{F}_A$, $\widehat{\mathscr{F}}_{A^c} = P_{A^c}\mathscr{F}_{A^c}$, then there exists a regular conditional probability of μ on $\mathscr{F}_X$ given by $\widehat{\mathscr{F}}_{A^c}$. It is denoted by $\mu(\cdot\,|\widehat{\mathscr{F}}_{A^c})$. Obviously

$$\mu(\hat{B}\cap\hat{C}) = \int_{\hat{C}} \mu^A(B|\widehat{\mathscr{F}}_{A^c})(r)\,d\mu,$$

where $\hat{B} = P_A^{-1}B$ and $\hat{C} = P_{A^c}^{-1}C$, $B\in\mathscr{F}_A$ and $C\in\mathscr{F}_{A^c}$, and

$$\mu^A(\cdot\,|\widehat{\mathscr{F}}_{A^c})(r) = P_A\mu(\cdot\,|\widehat{\mathscr{F}}_{A^c})(r).$$

Given $\psi\in\mathrm{Diff}(A, m)$, by assumption, μ is invariant under $\mathrm{Diff}(X, m)$. We can write

$$\int \mathbf{1}_{\hat{B}}(\psi r)\mathbf{1}_{\hat{C}}(\psi r)\,d\mu = \int \mathbf{1}_{\hat{B}}(\psi r)\mathbf{1}_{\hat{C}}(\psi r)\,d\psi^{-1}\mu = \mu(\hat{B}\cap\hat{C}).$$

Thus

$$\mu^A(\psi^{-1}B|\widehat{\mathscr{F}}_{A^c})(r) = \mu^A(B|\widehat{\mathscr{F}}_{A^c})(r), \quad \text{a. s., } \mu$$

By Proposition 5 and the fact that $\mathrm{Diff}(X)$ is separable topological group and the σ-algebra $\mathscr{F}_A$ is generated by some countable subalgebra, we can find a set $M\in\widehat{\mathscr{F}}_{A^c}$, $\mu(M)=0$ such that, for every $\psi\in\mathrm{Diff}(A, m)$,

$$\psi\mu^A(\cdot\,|\widehat{\mathscr{F}}_{A^c})(r) = \mu^A(\cdot\,|\widehat{\mathscr{F}}_{A^c})(r), \text{ for } r\overline{\in} M.$$

Accoding to Proposition 2, for every $k\in Z$, there exists non-negative number $C_k(r)$ such that

$$\mu^A(B|\widehat{\mathscr{F}}_{A^c})(r) = C_k(r)\mu^A(B), \text{ for } B\in B(A^{(k)}).$$

As $\mu^A(A^{(k)}) = \mu\{r: |r\cap A| = k\} > 0$, for $k\in Z$, so we have

$$C_k(r) = \mu^A(A^{(k)}|\widehat{\mathscr{F}}_{A^c})(r)/\mu^A(A^{(k)}),\ k\in Z,$$

which is tantamount to that $C_k(r)$ is $\widehat{\mathscr{F}}_{A^c}$-measurable. By Doob's lemma, we can write $C_k(r) = \rho_k(P_{A^c}r)$, where $\rho_k(\cdot)$ is a measurable function on $(\Delta(A^c), \mathscr{F}_{A^c})$. Therefore

$$\mu(\hat{B}\cap\hat{C}) = \sum_{k\geq 0}\int_{\hat{C}}\Big(\int_B \mathbf{1}_{A^{(k)}}(x)\,d\mu^A(r)\Big)\rho_k(P_{A^c}r)\,d\mu(r).$$

Consequently, $\mu\ll\tilde{\mu}$ and

$$\frac{d\mu}{d\tilde{\mu}} = \sum_{k\geq 0}\rho_k(P_{A^c}r)\mathbf{1}_{A^{(k)}}(P_A r), \text{ a. s., } \tilde{\mu}.$$

Proposition 7. *Let μ be a smooth point measure, then μ is quasi-invariant under the action of* $\mathrm{Diff}(X)$, *and*

$$\frac{d\psi\mu}{d\mu}(r) = \frac{d\psi\mu^A}{d\mu^A}(P_A r), \text{ a. s.,}$$

where $\psi\in\mathrm{Diff}(X)$ and $C_\psi\subset A = S_r$.

Proof Let

$$\rho(r)=\frac{d\mu}{d\tilde{\mu}}=\sum_{k\geq 0}\rho_k(P_{A^c}r)\mathbf{1}_{A^{(k)}}(P_Ar).$$

Hence, $\rho(\psi r)=\rho(r)$, $\mu(\psi^{-1}B)=\int_B \rho(r)d(\psi\tilde{\mu})$, for $B\in\mathscr{F}_X$.

Obviously, $\psi\tilde{\mu}\approx\tilde{\mu}$ and $\frac{d\psi\tilde{\mu}}{d\tilde{\mu}}(r)=\frac{d\psi\mu^A}{d\mu^A}(P_Ar)$, the conclusion holds.

Corollary. *If μ is a smooth point measure with respect to m, then*

$$\frac{d\psi\mu}{d\mu}(r)=\prod_{x\in r}\left(\frac{d\psi m}{dm}(x)\right), \textit{ for } \psi\in\mathrm{Diff}(X).$$

Proof Suppose that $C_\psi\subset A=S_r$. By Proposition 2, we have

$$\frac{d\mu^A}{d\mathscr{P}_m^A}(x)=\sum_{k\geq 0}C_k\mathbf{1}_{A^{(k)}}^{(x)},\ C>0.$$

Moreover

$$\frac{d\psi\mu^A}{d\mu^A}=\frac{d\psi\mu^A}{d\psi\mathscr{P}_m^A}\cdot\frac{d\psi\mathscr{P}_m^A}{d\mathscr{P}_m^A}\cdot\frac{d\mathscr{P}_m^A}{d\mu^A}.$$

According to Proposition 3

$$\left(\frac{d\psi\mu^A}{d\psi\mathscr{P}_m^A}\right)(x)=\frac{d\mu^A}{d\mathscr{P}_m^A}(\psi^{-1}x)=\frac{d\mu^A}{d\mathscr{P}_m^A}(x),$$

which is tantamount to that

$$\frac{d\psi\mu}{d\mu}(r)=\frac{d\psi\mathscr{P}_m^A}{d\mathscr{P}_m^A}(r)=\prod_{x\in r}\left(\frac{d\psi m}{dm}(x)\right).$$

Proposition 8. *Let m be a smooth measure on X. If $\{U_\psi^m\}$ is the unitary representation of* Diff(X) *given by $\mathscr{P}_m$, then it is continuous.*

Proof We first consider the unitary representation of Diff(A) on $L^2(A^{(k)}, \mathscr{P}_m^A)$ which is given by $\mathscr{P}_m^A$. Denote the Kakutani's inner product and distance by K_ρ and K_d respectively, then

$$M_1^2(\psi)=2\left(1-\exp\left(-\frac{1}{2}K_d(\psi m, m)\right)\right), \text{ for } \psi\in\mathrm{Diff}(A),$$

which ensures that $\lim_{n\to\infty}M_1^2(\psi_n)=0$ as ψ_n converges to I_d in Diff(A).

Following [2], the unitary representation of Diff(A) on $L^2(A^{(k)}, \mathscr{P}_m^A)$ given by $\mathscr{P}_m^A$ is continuous. Because $L^2(\Delta(A), \mathscr{P}_m^A)=\sum\oplus L^2(A^{(k)}, \mathscr{P}_m^A)$, it is easy to show that the unitary representation of Diff(A) on $L^2(\Delta(A), \mathscr{P}_m^A)$ given by $\mathscr{P}_m$ is continuous. Let $F\in L^\infty(\Delta(X), \hat{\mathscr{F}}_A)$, $J\in L^\infty(\Delta(X), \hat{\mathscr{F}}_{A^c})$, then

$$\|U_{\psi_n}^m(F\cdot J)-F\cdot J\|^2=\|J\|^2\|U_{\psi_n}^m(F)-F\|^2\to 0 \text{ as } n\to\infty.$$

Thus $\{U_\psi^m\}$ is contiouous.

Moreover, in a completely analogous fasion, we can prove that the conclusion holds in the case of smooth point measure.

Summing Propositions 1—8, we have the following

Theorem 1. *Let μ be a smooth point measure with respect to m, then μ is quasi-invariant under the action of* Diff(X), *and*

$$\frac{d\psi\mu}{d\mu}(r)=\prod_{x\in r}\left(\frac{d\psi m}{dm}(x)\right),\ \text{for}\ \psi\in \mathrm{Diff}(X).$$

Furthermore, the representation $\{U^{\mu}_{\psi}\}$ *of* Diff(X) *given by* μ *is continuous.*

§ 3. Unitary representation of Diff(X) given by smooth point measures

Let μ_1 and μ_2 be smooth point measures on $(\Delta(X), \mathscr{F}_X)$, and $\{U^1_\psi\}$, $\{U^2_\psi\}$ be unitary representations given by μ_1 and μ_2 respectively. In this section, we shall prove that $\{U^1_g\}$ and $\{U^2_g\}$ are unitarily equivalent iff μ_1 is equivalent to μ_2.

Proposition 9. *If* μ_i, $i=1, 2$, *is smooth point measure with respect to* m_i, $i=1, 2$, *and* $P_0\in L^2(\Delta(X), \mu_2)$ *such that, for every* $\psi\in$ Diff(A, m_1),

$$(*)\qquad P_0(\psi^{-1}r)\left(\frac{d\psi\mu_2}{d\mu_2}(r)\right)^{1/2}=P_0(r)\quad \text{a. s.}, \mu_2,$$

then

$$\rho_2^{1/2}(r)P_0(r)\ \mathbf{1}_{A^{(k)}}(P_A r)=C(A, k, P_{A^c}r)\left(\frac{d\mu_1^A}{d\mu_2^A}(P_A r)\right)^{1/2}\mathbf{1}_{A^{(k)}}(P_A r)\quad \text{a. s.}, \tilde{\mu}_2,$$

where $\tilde{\mu}_2=\mu_2^A\times\mu_2^{A^c}$, $\dfrac{d\mu_2}{d\tilde{\mu}_2}=\rho_2$, $C(A, k, \cdot)$ *is a measurable function on* $(\Delta(A^c), \mathscr{F}_{A^c})$, $k\in Z$.

Proof Define a map $T:\Delta(X)\to\Delta(A)\times\Delta(A^c)$ such that $T(r)=(P_A r, P_{A^c}r)$. Thus, T is a measurable isomorphsm from $(\Delta(X), \mathscr{F}_X)$ onto $(\Delta(A)\times\Delta(A^c), \mathscr{F}_A\times\mathscr{F}_{A^c})$. Obviously, we can write $F(r)=F(P_A r, P_{A^c}r)$, for $F\in L^0(\Delta(X), \mathscr{F}_X)$, where $F(\cdot, \cdot)$ is measurable on $(\Delta(A)\times\Delta(A^c), \mathscr{F}_A\times\mathscr{F}_{A^c})$, and $T\ \tilde{\mu}_2=\mu_2^A\times\mu_2^{A^c}$.

We set $P(r)=\rho_2^{1/2}(r)P(r)=P(P_A r, P_{A^c}r)$. Then

$$\iint|P(x, y)|^2d\mu_2^A(x)d\mu_2^{A^c}(y)=\int\rho_2(r)|P_0(r)|^2d\tilde{\mu}_2=\int|P_0(r)|^2d\mu_2<\infty,$$

so that $P(x, y)\in L^2(\Delta(A)\times\Delta(A^c), \mu_2^A\times\mu_2^{A^c})$.

By Fubinis theorem, there exists a set $M_1\in\mathscr{F}_{A^c}$ such that $\mu_2^{A^c}(M_1)=0$, and $P_y(\cdot)=P(\cdot, y)\in L^2(\mu_2^A)$ as $y\overline{\in} M$.

For $\psi\in$ Diff(A, m_1), there exists $M(\psi)\in\mathscr{F}_X$ such that $\mu_2(M(\psi))=0$, and $(*)$ holds as $r\overline{\in} M(\cdot)$. As

$$\int_{M(\psi)}\rho_2 d\tilde{\mu}_2=\iint\mathbf{1}_{M(\psi)}(x, y)\rho_2(x, y)d\mu_2^A(x)d\mu_2^{A^c}(y)=0,$$

we can find a set $M(\psi, A^c)\in\mathscr{F}_{A^c}$, $\mu_2^{A^c}(M(\psi, A^c))=0$ and, for $y\overline{\in} M(\psi, A^c)$,

$$\mathbf{1}_{M(\psi)}(x, y)\rho_2(x, y)=0,\ \text{a. s.}, \mu_2^A.$$

Clearly, as $y\overline{\in} M(\psi, A^c)$, we have

$$P(\psi^{-1}x, y)\left(\frac{d\psi\mu_2^A}{d\mu_2^A}(x)\right)^{1/2}=P(x, y),\quad \text{a. s.}, \mu_2^A.$$

Suppose that $\{\psi_n\}$ is a countable subset of Diff(A, m) which is dense in Diff(A, m_1).

Let $M=\bigcup_n M(\psi_n, A^c)\cup M_1$, then $M\in\mathscr{F}_{A^c}$, $\mu_2^{A^c}(M)=0$. As $y\bar{\in} M$, we have $P_y(\cdot)\in L^2(\mu_2^A)$ and

$$P_y(\psi_n^{-1}x)\Big(\frac{d\psi_n\mu_2^A}{d\mu_2^A}(x)\Big)^{1/2}=P_y(x),\quad \text{a. s., } \mu_2^A.$$

According to Theorem 1, if $\psi\in\mathrm{Diff}(A, m_1)$ and $y\bar{\in} M$, then

$$P_y(\psi^{-1}x)\Big(\frac{d\psi\mu_2^A}{d\mu_2^A}(x)\Big)^{1/2}=P_y(x),\quad \text{a. s., } \mu_2^A.$$

As $\mu_1^A=\psi\mu_1^A\approx\mu_2^A\approx\psi\mu_2^A$ for $\psi\in\mathrm{Diff}(A, m_1)$, we have

$$\frac{d\psi\mu_2^A}{d\mu_2^A}(x)=\frac{d\psi\mu_2^A}{d\psi\mu_1^A}(x)\cdot\frac{d\psi\mu_1^A}{d\mu_2^A}(x)=\frac{d\mu_2^A}{d\mu_1^A}(\psi^{-1}x)\cdot\frac{d\mu_1^A}{d\mu_2^A}(x).$$

Then

$$P_y(\psi^{-1}x)\Big(\frac{d\mu_2^A}{d\mu_1^A}(\psi^{-1}x)\Big)^{1/2}=P_y(x)\,\frac{d\mu_3^A}{d\mu_1^A}(x),\quad \text{a. s., } \mu_2^A.$$

for $y\bar{\in} M$.

Define

$$\Phi_y(x)=P_y(x)\Big(\frac{d\mu_2^A}{d\mu_1^A}(x)\Big)^{1/2},\quad \text{for } y\bar{\in} M,$$

thus, $\Phi_y(\cdot)$ belongs in $L^1(\Delta(A), \mu_1^A)$ and $\Phi_y(\psi^{-1}x)=\Phi_y(x)$ a. s., μ_1^A, for $\psi\in\mathrm{Diff}(A, m_1)$. By Proposition 1, we can find constant $C(A, r, y)$, $k\in Z$, such that

$$\Phi_y(x)\mathbf{1}_{A^{(k)}}(x)=C(A, r, y)\mathbf{1}_{A^{(k)}}(x),\quad \text{a. s., } \mu_1^A.$$

It is clear that

$$C(A, k, y)=\int_{A^{(k)}}P_0(x, y)\rho_2^{1/2}(x, y)\Big(\frac{d\mu_2^A}{d\mu_1^A}(x)\Big)^{1/2}d\mu_1^A\Big/\mu_1^A(A^{(k)}),$$

where $\mu_1^A(A^{(k)})=\mu_1\{r: |r\cap A|=k\}>0$ and

$$|C(A, k, y)|^2\leqslant\int_{A^{(k)}}|P_0(x, y)|^2\rho_2(x, y)d\mu_2^A(x)\Big/\mu_1^A(A^{(k)}).$$

It means that $C(A, k, y)$ must be in $L^2(\Delta(A^c), \mathscr{F}_{A^c})$. Then we can say that for $y\bar{\in} M$, $k\in Z$,

$$\rho_2^{1/2}(x, y)P_0(x, y)\mathbf{1}_{A^{(k)}}(x)=C(A, k, y)\Big(\frac{d\mu_1^A}{d\mu_2^A}(x)\Big)^{1/2}\mathbf{1}_{A^{(k)}}(x),\quad \text{a. s., } \mu_2^A.$$

By Fubinis theorem, we have

$$\rho_2^{1/2}(r)P_0(r)\mathbf{1}_{A^{(k)}}(r)=C(A, k, P_{A^c}r)\Big(\frac{d\mu_1^A}{d\mu_2^A}(P_Ar)\Big)^{1/2}\mathbf{1}_{A^{(k)}}(P_Ar),\quad \text{a. s., } \tilde{\mu}_2.$$

Theorem 2. *Let μ_i, $i=1, 2$, is a smooth point measure with respect to m_i, $i=1, 2$. If $\{U_\psi^1\}$ and $\{U_\psi^2\}$ are the unitary representations of* Diff(X) *given by μ_1 and μ_2 respectively, then $\{U_\psi^1\}$ is unitarily equivalent to $\{U_\psi^2\}$ iff μ_1 is equivalent to μ_2 as measure.*

Proof By Proposition 4, the sufficiency follows. Next we consider the necessity. Denote the inner product of $L^2_{\mu_i}$, $i=1, 2$, by $(\ ,\ )_i$, $i=1, 2$. If there exists a unitary map T from $L^2_{\mu_1}$ onto $L^2_{\mu_2}$ such that $TU_\psi^1T^{-1}=U_\psi^2$, for $\psi\in\mathrm{Diff}(X)$, setting $F=1\in L^2_{\mu_1}$, and $G=T(F)\in L^2_{\mu_2}$, then, for $\psi\in\mathrm{Diff}(X, m_1)$,

$$G(\psi^{-1}r)\left(\frac{d\psi\mu_2}{d\mu_2}(r)\right)^{1/2}=G(r), \quad \text{a. s., } \mu_2.$$

From Proposition 9, we claim that for each $A=S_r$,

$$\rho_2^{1/2}(x, y)G(x, y)=\sum_{k\geq 0}C(A, k, y)\left(\frac{d\mu_1^A}{d\mu_2^A}(x)\right)^{1/2}\mathbf{1}_{A^{(k)}}(x) \quad \text{a. s., } \tilde{\mu}_2,$$

where
$$\rho_2=\frac{d\mu_2}{d\tilde{\mu}_2}.$$

If $\psi\in\mathrm{Diff}(A)$, then

$$\rho_2^{1/2}(x, y)G(\psi^{-1}x, y)=\sum_{k\geq 0}C(A, k, y)\left(\frac{d\mu_1^A}{d\mu_2^A}(\psi^{-1}x)\right)^{1/2}\mathbf{1}_{A^{(k)}}(x) \quad \text{a. s., } \tilde{\mu}_2,$$

and
$$\frac{d\mu_1^A}{d\mu_2^A}(\psi^{-1}x)\,\frac{d\psi\mu_2^A}{d\mu_2^A}(x)=\frac{d\psi\mu_1^A}{d\mu_2^A}(x), \quad \text{a. s., } \mu_2^A.$$

$$\rho_2^{1/2}(x, y)G(\psi^{-1}x, y)\left(\frac{d\psi\mu_2^A}{d\mu_2^A}(x)\right)^{1/2}=\sum_{k\geq 0}C(A, k, y)\left(\frac{d\psi\mu_1^A}{d\mu_2^A}(x)\right)^{1/2}\mathbf{1}_{A^{(k)}}(x), \quad \text{a. s., } \mu_2^A.$$

$$\iint\rho_2^{1/2}(x, y)G(\psi^{-1}x, y)\left(\frac{d\psi\mu_2^A}{d\mu_2^A}(x)\right)^{1/2}\rho^{1/2}(x, y)\overline{G(x, y)}d\mu_2^A(x)d\mu_2^{A^c}(y)$$
$$=\sum_{k\geq 0}\iint|C(A, k, y)|^2\left(\frac{d\psi\mu_1^A}{d\mu_2^A}(x)\right)^{1/2}\left(\frac{d\mu_1^A}{d\mu_2^A}(x)\right)^{1/2}\mathbf{1}_{A^{(k)}}(x)d\mu_2^A(x)d\mu_2^{A^c}(y)$$
$$=\sum_{k\geq 0}\int\left(\int_{A^{(k)}}|C(A, k, y)|^2d\mu_2^{A^c}(y)\right)\left(\frac{d\psi\mu_1^A}{d\mu_1^A}(x)\right)^{1/2}d\mu_1^A(x).$$

Because

$$(U_\psi^2G, G)_2=\iint G(\psi^{-1}x, y)\left(\frac{d\psi\mu_2^A}{d\mu_2^A}(x)\right)^{1/2}\overline{G(x, y)}\rho_2d\mu_2^Ad\mu_2^{A^c},$$

$$(U_\psi^1F, F)=\sum_{k\geq 0}\int_{A^{(k)}}\left(\frac{d\psi\mu_1^A}{d\mu_1^A}(x)\right)^{1/2}d\mu_1^A$$
$$=\sum_{k\geq 0}\frac{e^{-m_1(A)}}{k!}D(k, A)\left(\int_A\left(\frac{d\psi m_1}{dm_1}(x)\right)^{1/2}dm_1\right)^k,$$

where $D(k, A)=\mu_1\{r: |r\cap A|=k\}/\mathscr{P}_{m_1}(A^{(k)})>0$.

Hence we have

$$\sum_{k\geq 0}\left(\int|C(A, k, y)|^2d\mu_2^{A^c}\right)\frac{D(k, A)}{k!}\left(\int_A\left(\frac{d\psi m_1}{dm_1}(x)\right)^{1/2}dm_1\right)^k$$
$$=\sum_{k\geq 0}\frac{D(k, A)}{k!}\left(\int_A\left(\frac{d\psi m_1}{dm_1}(x)\right)^{1/2}dm_1\right)^k.$$

It is easy to see that for every $A=S_r$, we always can find a sequence $\{\psi_n\}\subset\mathrm{Diff}(A)$ shoh that

$$\int_A\left(\frac{d\psi_n m_1}{dm_1}(x)\right)^{1/2}dm_1\to 0 \quad \text{as } n\to\infty.$$

Consequently

$$\int|C(A, k, y)|^2d\mu_2^{A^c}=1, \text{ for every } k\in Z.$$

Because

$$\rho_2(x, y)\,|G(x, y)|^2\mathbf{1}_{A^{(k)}}(x)=|C(A, k, y)|^2\left(\frac{d\mu_1^A}{d\mu_2^A}(x)\right)\mathbf{1}_{A^{(k)}}(x), \quad \text{a. s., } \tilde{\mu}_2,$$

according to Fubinis theorem, we have

$$\int \rho_2(x, y)\,|G(x, y)|^2 d\mu_2^{A^c}(y) = \left(\frac{d\mu_1^A}{d\mu_2^A}(x)\right), \quad \text{a. s., } \mu_2^A.$$

Set $A_n = S_n$, $n=1, 2, \cdots$ Clearly, $\{\mathscr{F}_{A_n}, n=1, 2, \cdots,\}$ is increasing and $\bigvee_n \widehat{\mathscr{F}}_{A_n} = \widehat{\mathscr{F}}_X$. Moreover, $\left\{\frac{d\mu_1^{A_n}}{d\mu_2^{A_n}}(P_{A_n}r),\ \widehat{\mathscr{F}}_{A_n},\ n \in Z_+\right\}$ is martingale. On the other hand

$$E^{\mu_2}(|G|^2|\widehat{\mathscr{F}}_{A_n}) = \int |G(P_{A_n}r, y)|^2 \rho_2(P_{A_n}r, y)\, d\mu^{A^c}(y), \quad \text{a. s., } \mu_2.$$

So that

$$\int \left| \left(\frac{d\mu_1^{A_n}}{d\mu_2^{A_n}}(P_{A_n}r)\right) - |G(r)|^2 \right| d\mu_2 \to 0 \quad \text{as } n\to\infty.$$

It is well known that $d\mu_1 = g\,d\mu_2 + d\mu_{12}$, where $g \in L^1_{\mu_2}$, μ_{12} is the singular part of μ_1 in its Lebesques decomposition. As $\mu_1(B)=0$ iff $\mu_2(B)=0$ for $B \in \widehat{\mathscr{F}}_{A_n}$, we have

$$\int_B d\mu_1 = \int_B g\,d\mu_2 \quad \text{for } B \in \widehat{\mathscr{F}}_{A_n}.$$

Thus

$$\int_B \frac{d\mu_1^{A_n}}{d\mu_2^{A_n}}(P_{A_n}r)\, d\mu_2 = \int_B g\,d\mu_2.$$

Setting $n\to\infty$, we see

$$\int_B |G|^2 d\mu_2 = \int_B g\,d\mu_2 \quad \text{for } B \in \mathscr{F}_X.$$

Consequently

$$1 = \int_{A(x)} |G|^2 d\mu_2 = \int_{A(x)} g\,d\mu_2,$$

which implies that $\mu_{12}=0$, i. e., $\mu_1 \ll \mu_2$. In completely analogous fasion, we can prove that $\mu_2 \ll \mu_1$, this ends the proof.

Corollary. *If* m, $i=1, 2$, *is a smooth measure on* X, *and* $\{U_\psi^{m_i}\}$, $i=1, 2$, *is the unitary representation of* Diff(X) *given by* $\mathscr{P}_{m_i}$, $i=1, 2$, *then* $\{U_\psi^{m_1}\}$ *is unitarily equivalent to* $\{U_\psi^{m_2}\}$ *iff the Kakutanis distance* $\left(\int (\sqrt{dm_1} - \sqrt{dm_2})^2\right)^{1/2}$ *is finite.*

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