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SOME APPLICATIONS OF BOREL TECHNIQUE IN PARTIAL DIFFERENTIAL EQUATIONS

CHEN SHUXING (陈恕行)
(*Institute of Mathematics, Fudan University*)

Abstract

In this work Borel's technique is applied to analyzing the solvability of partial differetial equations. It is proved that if P is analytic-hypoelliptic, f is C^∞, then the problem for $Pu=f$ with any number of conditions on a fixed point is ill-posed. Besides, some other results on the flat solution to Laplace equation and wave equation are obtained.

In this note we are going to apply the Borel technique to discuss some problems in partial differential equations, such as the solvability in the class of flat functions for differential operators, approximate solvability for differential operators, which may not be local solvable. At first, for reader's conveniency we recall Borel's lemma.

Lemma. *For any sequence* $\{h_\alpha\}$ *(*α *is multi-index), there exists a* C^∞ *function* $g(x)$ *in neighborhood of* 0 *in* R^n, *which satisfies conditions*

$$(D^\alpha g)(0)=h_\alpha. \tag{1}$$

Let us consider

$$Pu=f, \quad (x\in\Omega) \tag{2}$$

$$D^\alpha u=h_\alpha, \quad |\alpha|<N, \quad x\in M \tag{3}$$

where P is a differential operator of m^{-th} order defined in an open set Ω of $\mathbf{R}^n$, M is a submanifold of Ω, f and h_α are given functions defined on Ω and M respectively, h_α and f satisfy compatibility conditions, N is a finite integer or infinity. M. S. Baouendi and E. C. Zachmanoglou have proved (see [1]), if all coefficients of P are analytic, M is analytic and noncharacteristic, $f=h_\alpha=0$, then the solution u of (2), (3) will vanish in some neighborhood of M. This means the solution of (2), (3) is unique if it exists. However, does the solution of (2), (3) really exist? When $\dim(M)=n-1$, (2), (3) is a Cauchy problem. At that time, problem (2), (3) has a unique solution, so long as $N=m$. When codim $(M)\geqslant 2$, things will be quite different.

Theorem 1. *If* P *is Laplacian,* M *is a single point, then problem* (2), (3) *has more than one solution for any finite* N, *and this problem may have no solution in the case* $N=\infty$.

Proof Without loss of generality, we can choose M as origin. We also can assume $f=0$ by substracting an arbitrary solution of Poisson equation $\Delta u=f$. Notice

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that each h_α is a number because M is a point. We construct a series of polynomials Z_k of k degree as follows. First, choose $Z_0=h_0$, $Z_1=\sum_{i=1}^{n} h_i x_i$. For $k\geqslant 2$, the number of coefficients of the homogeneous polynomial Z_k of k degree with n variables is C^k_{n+k-1}. Meanwhile, the form of compatibility is

$$D^\beta \Delta u|_M=0. \qquad (4)$$

Since Z_k is a polynomial of k degree, this equality can be rewritten as

$$D^\beta \Delta Z_k|_M=0 \quad |\beta|=k-2. \qquad (5)$$

Equality (5) contains C^{k-2}_{n+k-3} conditions for every fixed $k(k\geqslant 2)$, so the linear algebraic system of coefficients of Z_k is solvable. Arbitrarily choosing Z_k according to (5), we make summation $\sum_{k=0}^{N-1} Z_k$, which satisfies (2), (3). Obviously, because of arbitrariness of Z_k, the solution of (2), (3) is not unique.

Now let us turn to the case of $N=\infty$. If there exists a solution of (2), (3); because of $P=\Delta$, the solution should be analytic. It is well known that the derivatives of analytic functions should satisfy an inequality

$$|D^\alpha u(x)|\leqslant C(C|\alpha|)^{|\alpha|} \qquad (6)$$

for some constant C (see [2]). So if we choose $\{h_\alpha\}$ in such a way, that $\{h_\alpha\}$ grows faster than $C(C|\alpha|)^{|\alpha|}$ with any constant C, for example, $h_\alpha=|\alpha|^{2|\alpha|}$, then problem (2), (3) can never have solution.

Remark 1. Similarly, if P is analytic-hypoelliptic, M is a point, then the second part of conclusion in Theorem 1 still holds.

Theorem 2. *If P is anylytic-hypoelliptic, M is single point, all h_α in (6) are equal to 0, then we can find a flat at M function f, such that $Pu=f$ doesn't have any flat solution.*

Proof The key point is to construct $\{h_\alpha\}$, which satisfies compatibility conditions, such that problem (2), (3) has no solution. We notice that for every integer k, the number of $h_\alpha(|\alpha|=k)$ is C^k_{n+k-1}, these h_α and only these h_α appear in compatibility conditions $D^\beta\Delta u|_M=0(|\beta|=k-2)$, so we can find non-trivial $h_\alpha(|\alpha|=k)$, for the number of conditions $D^\beta\Delta u|_M=0$ is C^{k-2}_{n+k-3}, which is less than C^k_{n+k-1}. Besides, for any constant γ, $\gamma h_\alpha(|\alpha|=k)$ still satisfy $D^\beta\Delta u|_M=0$. This fact indicates that there exists a desired set of h_α, which increases very quickly and does not allow a solution of (2), (3). Using Lemma 1 we get a C^∞ function g. Set $f=-\Delta g$, f is flat at M by compatibility conditions, and we can claim that the equation $\Delta u=f$ has no flat at M solution. In fact, if such u would exist, $u+g$ would satisfy $\Delta(u+g)=0$ and $D^\alpha(u+g)|_M=h_\alpha$ for any α, this contradicts to the construction of $\{h_\alpha\}$.

Using a different method, S. Alinhac and M. S. Baouendi have pointed out that the conclusion in Theorem 2 is valid in the case $n=2$, (see [4], p. 190).

Theorem 3. *If P in (2) is wave operator, M is t-axis, N in (3) is infinity, then there exists a sequence $\{h_\alpha\}$, such that $Pu=0$ with condition (3) has no solution which is temper distribution in direction t. Furthermore, if N in (3) is finite, then the solution is not unique.*

Proof We choose $\{h_\alpha\}$ as that one in the proof of Theorem 1, each h_α is independent of t. Suppose Theorem 3 is not true, that means we can find a solution u of $Pu=0$, which satisfies (3). It is well known that the singularity of solution u is contained in the characteristic set $\sum(P)$, which does not intersect with normal bundle of t-axis or any other lines parallel to t-axis, therefore, for every fixed x the restriction $u(t, x)$ is a well-defined distribution. Because u is a temper distribution, we can introduce an analytic function $g(t)=e^{-t^2}$, such that $\langle g(t)u(t, x)\rangle$ is well-defined. According to Hörmander's theorem ([2], theorem 4.1) we know $f(x)=\langle g(t), u(t, x)\rangle$ is analytic and

$$D^\alpha f(x)\,|_{x=0}=\langle g(t),\ D^\alpha u(t, x)\,|_{x=0}\rangle=h^\alpha\int g(t)\,dt.$$

Obviously, this is impossible because of growth property of h^α.

When N in (3) is a finite number, R. S. Strichartz proved (see [3]), there exists a solution of (2), which satisfies (3) and

$$D^\alpha u|_M=0 \quad |\alpha|\geqslant N. \tag{7}$$

Certainly, if we omit the condition (7), the solution will never be unique.

Remark 2. Similar to Theorem 2, we can prove, if P in (2) is wave operator, M is t-axis, N in (3) is infinity, then there exists a flat at M function f, such that $Pu=f$ has no flat at M solution.

Finally, we are going to apply Borel's lemma to the problem of solvability. It is well known that there exists partial differential operators P with C^∞ coefficients, defined in an open set Ω, for some C^∞ function f we cannot find any solution in any subdomain of Ω, but if we consider the solvability on some set of equivalent classes, the following fact will be valid.

Let $M_1, \cdots, M_k$ be arbitrary points in Ω, we call C^∞ functions f_1 and f_2 are equivalent and use the notation $f_1\sim f_2$, if $f_1\sim f_2$ is flat at $M_1, \cdots, M_k$. According to this equivalent relation we can divide $C^\infty(\Omega)$ to a set of equivalent classes $C^\infty/\sim$.

Theorem 4. *Suppose P is a partial differential operator with C^∞ coefficients defined in Ω, $M_1, \cdots, M_k$ are arbitrary points in Ω, the principal symbol P_m is not degenerate in $M_j(1\leqslant j\leqslant k)$, then the induced operator P_1 on quotient space is a surjective map from $C^\infty/\sim$ to $C^\infty/\sim$.*

Proof Let O_j be the neighborhood of point $M_j(j=1, \cdots, k)$, $O_j\subset\Omega$, $O_{j_1}\cap O_{j_2}=\emptyset$ for $1\leqslant j,\ j_1,\ j_2\leqslant k$. For any $f\in C^\infty(\Omega)$, we can calculate the possible values of u and its derivatives from the equation $Pu=f$. We denote the possible values $D^\alpha u$ at M_j by

$h_\alpha^j(1\leqslant j\leqslant k,\ |\alpha|>0)$. Making use of Borel's lemma we can construct C^∞ functions v_j, the derivatives of which are equal to h_α^j. Let ζ_j be a $C_c^\infty(O_j)$ function, which is equal to 1 in a smaller neighborhood $O'_j\subset O_j$ of M_j, then $u=\sum_{j=1}^{k}\zeta_j v_j$ satisfies the conditions

$$(D^\alpha Pu)(M_j)=D^\alpha f(M_j) \quad 1\leqslant j\leqslant k,\ |\alpha|>0,$$

this means $Pu\sim f$ or $P_1[u]=[f]$, where $[u]$, $[f]$ denote corresponding equivalent class. The proof is complete.

Remark 3. Theorem 4 indicates that we can find a C^∞ function u such that Pu coincides with f at any finite points, so we can regard u as a kind of approximate solution. Unfortunately, we don't know outside the vicinities of these finite points how near to f the function Pu is.

Remark 4. We can generalize Theorem 4 to the case of system or Frobenius algebra of vector fields (see [5]). In the latter case, if $\mathbf{L}$ is a Frobenius algebra, which is spanned by m linearly independent elements $L_1, \cdots, L_m$, and $f_1, \cdots, f_m$ are C^∞ functions satisfing

$$L_{i_1}f_{i_2}=L_{i_2}f_{i_1}, \quad 1\leqslant i_1 i_2\leqslant m, \tag{8}$$

then $L_i u=f_i$ has an approximate solution in above-mentioned meaning.

For simplicity of notations we consider the case $m=2$, by means of a transformation of variables we can denote L_1, L_2 as $\frac{\partial}{\partial t_1}+\sum_{i=1}^{N}a_i\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial t_2}+\sum_{i=1}^{N}b_i\frac{\partial}{\partial x_i}$ $(N=n-2)$. At point M_j, we give the values of derivatives $(D_x^\alpha u)(M_j)$ arbitrarily, then from $L_i u=f_i$ we get $(D_{t_1}^l D_x^\alpha u)(M_j)$, $(D_{t_2}^l D_x^\alpha u)(M_j)$ $(1\leqslant l<\infty,\ |\alpha|<\infty)$ respectively. Furthermore, from $L_1L_2u=L_1f_2$ (or $L_2L_1u=L_2f_1$) we get $(D_{t_2}D_{t_1}^l D_x^\alpha u)(M_j)$ (or $(D_{t_1}D_{t_2}^l D_x^\alpha u)(M_j)$), and step by step, all derivatives of u at M_j can be obtained. The Frobenius condition $[L_{i_1}, L_{i_2}]=0$ and (8) ensure the process is reasonable, therefore, the result can be obtained like Theorem 4.

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