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SINGULAR INTEGALS IN SEVERAL COMPLEX VARIABLES (IV)——THE DERIVATIVE OF CAUCHY INTEGRAL ON SPHERE

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Abstract

The aim of this paper is to study the boundary properties of the derivative of Cauchy integral on the sphere. Using the concept of the Hadamard principal value of singular integral of higher order introduced in [1], the author obtains the corresponding Plemelj formula expressed by Hadamard principal value.

§ 0. Introduction

In the case of one complex variable, if L is a smooth closed curve in $\mathbf{C}$, the Cauchy integral of f is

$$F(z)=\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta-z}\,d\zeta,$$

its derivative

$$F'(z)=\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta=\frac{1}{2\pi i}\int_L \frac{f'(\zeta)}{\zeta-z}\,d\zeta$$

is also a Cauchy integral. In the case of several complex variables, the derivative of Cauchy integral is no longer a Cauchy integral.

In what follows, B denotes the unit ball in $\mathbf{C}^n$, S denotes the boundary of B, i. e.

$$B=\{z=(z_1, \cdots, z_n)\,|\,|z_1|^2+\cdots+|z_n|^2<1\},$$
$$S=\{z=(z_1, \cdots, z_n)\,|\,|z_1|^2+\cdots+|z_n|^2=1\}.$$

Suppose f is integrable on S, the Cauchy integral of f in the ball is

$$F(z)=\frac{1}{\omega_{2n-1}}\int_S \frac{f(u)\dot{u}}{(1-z\bar{u}')^n}\quad (z\in B),$$

its partial derivative

$$\frac{\partial F(z)}{\partial z}=\frac{n}{\omega_{2n-1}}\int_S \frac{\bar{u}f(u)\dot{u}}{(1-z\bar{u}')^{n+1}} \tag{0.1}$$

is no longer a Cauchy integral, where $\frac{\partial F(z)}{\partial z}=\left(\frac{\partial F(z)}{\partial z_1}, \cdots, \frac{\partial F(z)}{\partial z_n}\right)$. When z

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approaches the point v of S, (0.1) is a higher order singular integral.

In [1], using the concept of Hadamaad principal value, we obtain the Plemelj formula of integral

$$\frac{1}{\omega_{2n-1}}\int_S \frac{f(u)\dot{u}}{(1-z\bar{u}')^{n+\frac{1}{2}}} \quad (z\in B).$$

The aim of present paper is to study the limit value of the derivative of Cauchy integral. Let

$$P_j=\{v=(v_1, \cdots, v_j, \cdots, v_n)\,|\,v\in S,\ v_j=0\},$$
$$Q_j=\{v=(v_1, \cdots, v_j, \cdots, v_n)\,|\,v\in S,\ v_j\neq 0\}.$$

We find that for the points of P_j, the limit value of $\frac{\partial F(z)}{\partial z_j}$ can be represented by the Cauchy principal value and it must be represented by the Hadamard principal value for the points of Q_j. By the uniformity of these two kinds of principal values, it can be represented uniformly by Hadamard principal value.

Using the same method, we can discuss the derivative of H-R integral and S-K integral of strictly pseudoconvex domain.

§ 1. Some lemmas

Lemma 1. *Suppose $z\in B$. Then*

$$\int_S \frac{\bar{u}_j\dot{u}}{(1-z\bar{u}')^m}=0, \tag{1.1}$$

where m is an arbitrary positive number.

Proof Let $z\bar{z}'=\rho^2$. Pick a unitary matrix U, such that $zU=\rho p_n$, where $p_n=(0, \cdots, 0, 1)$. Let $u=w\bar{U}'$ and $\bar{U}'=(\alpha_{ij})$. Then $u_j=\sum_{l=1}^{n}\alpha_{lj}w_l$ and

$$\int_S \bar{u}_j(1-z\bar{u}')^{-m}\dot{u}=\sum_{l=1}^{n}\int_S \bar{\alpha}_{lj}\bar{w}_l(1-\rho\bar{w}_n)^{m}\dot{w}. \tag{1.2}$$

When $l<n$, we have

$$\int_S \bar{w}_l(1-\rho\bar{w}_n)^{-m}\dot{w}=\int_{v\bar{v}'<1}\bar{v}_l\dot{v}\int_{-\pi}^{\pi}(1-\rho re^{i\theta})^{-m}d\theta=2\pi\int_{v\bar{v}'<1}\bar{v}_l\dot{v}=0. \tag{1.3}$$

When $l=n$, since

$$\int_{-\pi}^{\pi} re^{i\theta}(1-\rho re^{i\theta})^{-m}d\theta=\sum_{q=0}^{\infty}\frac{\Gamma(m+q)}{\Gamma(q+1)\Gamma(m)}\rho^q r^{q+1}\int_{-\pi}^{\pi}e^{i(q+1)\theta}d\theta=0,$$

we have

$$\int_S \bar{w}_n(1-\rho\bar{w}_n)^{-m}\dot{w}=0. \tag{1.4}$$

Inserting (1.3), (1.4) into (1.2), we obtain (1.1).

Lemma 2. *Suppose $z\in B$. Then*

$$\frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_j x_k}{(1-z\bar{u}')^{n+1}}\dot{u}=\frac{1}{2n}\delta_{kj}, \tag{1.5}$$

$$\frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_j y_k}{(1-z\bar{u}')^{n+1}}\dot{u}=\frac{1}{2ni}\delta_{kj}, \tag{1.6}$$

where $j,\ k=1,\ \cdots,\ n;\ x_k=\mathrm{Re}(u_k),\ y_k=\mathrm{Im}(u_k)$.

Proof Let

$$G(z)=\frac{1}{\omega_{2n-1}}\int_S \frac{x_k}{(1-z\bar{u}')^n}\dot{u}.$$

By Cauchy integral formula and Lemma 1, we have

$$G(z)=\frac{1}{2}\left\{\omega_{2n-1}^{-1}\int_S u_k(1-z\bar{u}')^{-n}\dot{u}+\omega_{2n-1}^{-1}\int_S \bar{u}_k(1-z\bar{u}')^{-n}\dot{u}\right\}=\frac{1}{2}z_k,$$

so $\frac{\partial G}{\partial z_j}=\frac{1}{2}\delta_{kj}$, i. e. (1.5) holds. The proof of (1.6) is similar.

Lemma 3. *Suppose* $f:\bar{B}\to\mathbf{R}$. *If* $\frac{\partial f}{\partial x_j},\ \frac{\partial f}{\partial y_j}$ $(j=1,\ \cdots,\ n)$ *are in* Lip α_j *and* Lipβ_j *on* $\bar{B}$ *respectively, then*

(i) *For* $j,\ k=1,\ \cdots,\ n-1$, *the integrals*

$$I_{x_k}^{(j)}=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_k+iy_k,\ \cdots,\ x_n+iy_n)-f(0,\ \cdots,\ 0,\ iy_k,\ \cdots,\ x_n+iy_n)$$
$$-\frac{\partial f}{\partial x_k}(p_n)x_k\Big](1-\bar{u}_n)^{-(n+1)}\dot{u},$$

$$I_{y_k}^{(j)}=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ iy_k,\ x_{k+1}+iy_{k+1},\ \cdots,\ x_n+iy_n)$$
$$-f(0,\ \cdots,\ 0,\ x_{k+1}+iy_{k+1},\ \cdots,\ x_n+iy_n)-\frac{\partial f}{\partial y_k}(p_n)y_k\Big](1-\bar{u}_n)^{-(n+1)}\dot{u},$$

$$I_{x_n}^{(j)}=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_n)-f(0,\ \cdots,\ 0,\ 1)-\frac{\partial f}{\partial x_n}(p_n)(x_n-1)\Big]$$
$$\times(1-\bar{u}_n)^{-(n+1)}\dot{u},$$

$$I_{y_n}^{(j)}=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_n+iy_n)-f(0,\ \cdots,\ 0,\ x_n)-\frac{\partial f}{\partial y_n}(p_n)y_n\Big]$$
$$\times(1-\bar{u}_n)^{-(n+1)}\dot{u}.$$

exist.

(ii) *Suppose* $z\in B$. Write

$$I_{x_k}^{(j)}(z)=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_k+iy_k,\ \cdots,\ x_n+iy_n)$$
$$-f(0,\ \cdots,\ 0,\ iy_k,\ \cdots,\ x_n+iy_n)-\frac{\partial f}{\partial x_k}(p_n)x_k\Big](1-z\bar{u}')^{-(n+1)}\dot{u},$$

$$I_{y_k}^{(j)}(z)=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ iy_k,\ \cdots,\ x_n+iy_n)$$
$$-f(0,\ \cdots,\ 0,\ x_{k+1}+iy_{k+1},\ \cdots,\ x_n+iy_n)-\frac{\partial f}{\partial y_k}(p_n)y_k\Big](1-z\bar{u}')^{-(n+1)}\dot{u},$$

$$I_{x_n}^{(j)}(z)=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_n)-f(0,\ \cdots,\ 0,\ 1)-\frac{\partial f}{\partial x_n}(p_n)(x_n-1)\Big]$$
$$\times(1-z\bar{u}')^{-(n+1)}\dot{u},$$

$$I_{y_n}^{(j)}(z)=\omega_{2n-1}^{-1}\int_S \bar{u}_j\Big[f(0,\ \cdots,\ 0,\ x_n+iy_n)-f(0,\ \cdots 0,\ x_n)-\frac{\partial f}{\partial y_n}(p_n)y_n\Big]$$
$$\times(1-z\bar{u}')^{-(n+1)}\dot{u},$$

then we have

$$(K-\lim_{z\to p_n}) I^{(j)}_{x_k}(z)=I^{(j)}_{x_k}, \quad (K-\lim_{z\to p_n}) I^{(j)}_{y_k}(z)=I^{(j)}_{y_k} \quad (j=1,\cdots,n-1,\ k=1,\cdots,n).$$

Proof Note the inequalities

$$|u_j|\leqslant\sqrt{2}\,|1-\bar v_n|^{\frac{1}{2}} \quad (j=1,\cdots,n-1).$$

Using the method of proving Lemma 5 in [1], Lemma 3 follows.

Lemma 4. *For $j=1,\cdots,n-1$, we have*

$$\int_{\sigma_\varepsilon}\bar u_j(1-\bar u_n)^{-(n+1)}\dot u=0,$$

where $\sigma_\varepsilon=\{u\,|\,u\in S,\ \alpha^2(1-|u_n|^2)^2+4\beta^2(\operatorname{Im}u_n)^2>\varepsilon^2,\ \alpha>0,\ \beta>0\}$.

Proof Let $u_n=re^{i\theta}$, $u_1=v_1,\cdots,u_{n-1}=v_{n-1}$. Then

$$\int_{\sigma_\varepsilon}\bar u_j(1-\bar u_n)^{-(n+1)}\dot u=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\bar v_j\dot v\left\{\int_{-(\pi-c)}^{-c}(1-re^{i\theta})^{-(n+1)}d\theta+\int_{c}^{\pi-c}(1-re^{i\theta})^{-(n+1)}d\theta\right\}$$

$$+\int_{v\bar v'>\frac{\varepsilon}{\alpha}}\bar v_j\dot v\int_{-\pi}^{\pi}(1-re^{i\theta})^{-(n+1)}d\theta=I_1+I_2,$$

where $C=\sin^{-1}\dfrac{\sqrt{\varepsilon^2-\alpha^2(1-r^2)^2}}{2\beta r}$. Denote the inner integral of I_1 by $\varphi(r)$. Let $v_j=e^{i\tau}\zeta_i$ and note that $r^2=1-v\bar v'$, we have

$$I_1=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\dot v_j\varphi(r)\dot v=\int_{\zeta\bar\zeta'<\frac{\varepsilon}{\alpha}}e^{-i\tau}\zeta_j\varphi(r)\dot\zeta=e^{-i\tau}I_1,$$

and it follows that $I_1=0$. We can prove $I_2=0$ by the same method. This ends the proof.

Lemma 5. *For $j=1,\cdots,n-1$, $k=1,\cdots,n$, we have*

$$\lim_{\varepsilon\to0}\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\bar u_j x_k(1-\bar u_n)^{-(n+1)}\dot u=\frac{1}{2n}\left[1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\right]\delta_{jk}, \tag{1.7}$$

$$\lim_{\varepsilon\to0}\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\bar u_j y_k(1-\bar u_n)^{-(n+1)}\dot u=\frac{1}{2ni}\left[1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\right]\delta_{jk}. \tag{1.8}$$

Proof It is clear that

$$\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\bar u_j x_k(1-\bar u_n)^{-(n+1)}\dot u=(2\omega_{2n-1})^{-1}\int_{\sigma_\varepsilon}\bar u_j u_k(1-\bar u_n)^{-(n+1)}\dot u+(2\omega_{2n-1})^{-1}$$

$$\int_{\sigma_\varepsilon}\bar u_j\bar u_k(1-\bar u_n)^{-(n+1)}\dot u. \tag{1.9}$$

Using the similar method as in proving Lemma 4, we can obtain the following equalities

$$\int_{\sigma_\varepsilon}\bar u_j\bar u_k(1-\bar u_n)^{-(n+1)}\dot u=0 \quad (j=1,\cdots,n-1;\ k=1,\cdots,n), \tag{1.10}$$

$$\int_{\sigma_\varepsilon}\bar u_j u_k(1-\bar u_n)^{-(n+1)}\dot u=0 \quad (j\neq k). \tag{1.11}$$

When $j=k<n$, without loss of generality, taking $j=1$, we have

$$(2\omega_{2n-1})^{-1}\int_{\sigma_\varepsilon}|u_1|^2(1-\bar{u}_n)^{-(n+1)}\dot{u}$$
$$=(2\omega_{2n-1})^{-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2\dot{v}\left\{\int_{-(\pi-c)}^{-c}(1-re^{i\theta})^{-(n+1)}d\theta+\int_{c}^{\pi-c}(1-re^{i\theta})^{-(n+1)}d\theta\right\}$$
$$+(2\omega_{2n-1})^{-1}\int_{v\bar{v}'>\frac{\varepsilon}{\alpha}}|v_1|^2\dot{v}\int_{-\pi}^{\pi}(1-re^{i\theta})^{-(n+1)}d\theta=I_1+I_2. \tag{1.12}$$

It is not hard to see that

$$I_1=\omega_{2n-1}^{-1}\operatorname{Im}\left\{\sum_{k=1}^{n}J_k-\sum_{k=1}^{n}H_k+J_0-H_0\right\}+\omega_{2n-1}^{-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(\pi-2c)\dot{v},$$

where

$$J_0=-\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2\log(1+re^{-ic})\dot{v},\quad J_k=\frac{1}{k}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(1+re^{-ic})^{-k}\dot{v},$$

$$H_0=-\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2\log(1-re^{ic})\dot{v},\quad H_k=\frac{1}{k}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(1-re^{ic})^{-k}\dot{v}.$$

Suppose $v=(t_1,\cdots,t_{2n-2})$. Using the sphere coordinates, we have

$$J_k=\frac{1}{k}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(\cos^2\varphi_1+\sin^2\varphi_1\cos^2\varphi_2)\sin^{2n-4}\varphi_1\sin^{2n-5}\varphi_2\cdots\sin\varphi_{2n-4}\,d\varphi_1\cdots d\varphi_{2n-3}$$
$$\times\int_0^{\sqrt{\frac{\varepsilon}{\alpha}}}\{1+(2\beta)^{-1}\sqrt{4\beta^2(1-s^2)+\alpha^2s^4-\varepsilon^2}-i(2\beta)^{-1}\sqrt{\varepsilon^2-\alpha^2s^4}\}^{-k}s^{2n-1}ds.$$

Set $\eta=\frac{\varepsilon}{\alpha}$, $s=\sqrt{\eta}\,t$. The inner integral becomes

$$(2\beta)^k\eta^n\int_0^1 t^{2n-1}\{2\beta+\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}\}^{-k}dt.$$

Since the integrand is bounded,

$$\lim_{\varepsilon\to0}J_k=0\quad(k=1,\cdots,n).$$

Similarly $\lim\limits_{\varepsilon\to0}J_0=0$.

It is the same as J_k that H_k may be written as

$$H_k=\frac{1}{k}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(\cos^2\varphi_1+\sin^2\varphi_1\cos^2\varphi_2)\sin^{2n-4}\varphi_1\sin^{2n-5}\varphi_2\cdots\sin\varphi_{2n-4}\,d\varphi_1\cdots d\varphi_{2n-3}$$
$$\times(2\beta)^k\eta^n\int_0^1 t^{2n-1}[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}]^{-k}dt.$$

Denote the inner integral by M_k, then

$$M_k=(2\beta)^k\eta^n\int_0^1 t^{2n-1}\eta^{-k}[(\beta t^2-i\alpha\sqrt{1-t^4})+O(\eta)]^{-k}dt.$$

When $k<n$, $\lim\limits_{\varepsilon\to0}M_k=0$, so $\lim\limits_{\varepsilon\to0}H_k=0$. When $k=n$,

$$\lim_{\varepsilon\to0}M_n=(2\beta)^n\int_0^1 t^{2n-1}(\beta t^2-i\alpha\sqrt{1-t^4})^{-n}dt=2^{n-1}\int_0^1 x^{n-1}(x-i\gamma\sqrt{1-x^2})^{-n}dx,$$

it is known in [2] that

$$\operatorname{Im}\left(\int_0^1 x^{n-1}(x-i\gamma\sqrt{1-x^2})^{-n}dx\right)=\frac{\pi}{2}\left(\frac{\beta}{\alpha+\beta}\right)^n.$$

Hence we have

$$\lim_{\varepsilon\to 0}\operatorname{Im}(M_n)=2^{n-2}\pi\left(\frac{\beta}{\alpha+\beta}\right)^n.$$

A direct computation shows

$$\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(\cos^2\varphi_1+\sin^2\varphi_1\cos^2\varphi_2)\sin^{2n-4}\varphi_1\sin^{2n-5}\varphi_2\cdots\sin\varphi_{2n-4}\,d\varphi_1\cdots d\varphi_{2n-3}$$
$$=\frac{2\pi^{n-1}}{\Gamma(n)},$$

so

$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_n)=\frac{\pi^n}{2n\Gamma(n)}\left(\frac{2\beta}{\alpha+\beta}\right)^n.$$

It is easy to prove that

$$\lim_{\varepsilon\to 0}\operatorname{Im}H_0=0,\quad \lim_{\varepsilon\to 0}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(\pi-2c)\dot{v}=0.$$

Substituting these results into the expression of I_1, we have

$$\lim_{\varepsilon\to 0}I_1=-\lim_{\varepsilon\to 0}(\omega_{2n-1})^{-1}\operatorname{Im}(H_n)=-\frac{1}{4n}\left(\frac{2\beta}{\alpha+\beta}\right)^n.$$

On the other hand

$$\lim_{\varepsilon\to 0}I_2=\pi\omega_{2n-1}^{-1}\int_{v\bar{v}'<1}|v_1|^2\dot{v}=\frac{1}{2n}.$$

Substituting these into (1.12), we obtain

$$\lim_{\zeta\to 0}(2\omega_{2n-1})^{-1}\int_{\sigma_\varepsilon}|u_1|^2(1-\bar{u}_n)^{-(n+1)}\dot{u}=\frac{1}{2n}\left[1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\right]. \tag{1.13}$$

Obviously, replacing u_1 by u_j, $(j=2,\cdots,n-1)$, (1.13) is also true. Substituting (1.10), (1.11) and (1.13) into (1.9), (1.7) is obtained. (1.8) can be proved in a similar way.

§ 2. The limit value of a part of special points

Employing the above Lemmas, we can prove the following

Theorem 1. *Suppose that f satisfies the conditions of Lemma 3, $F(z)$ isthe Cauchy integral of f:*

$$F(z)=\frac{1}{\omega_{2n-1}}\int_S\frac{f(u)\dot{u}}{(1-z\bar{u}')^n}.$$

Then, for $j=1,\cdots,n-1$, we have

$$(K-\lim_{z\to p_n})\frac{\partial F}{\partial z_j}(z)=n\omega_{2n-1}^{-1}\int_{S(\gamma)}\bar{u}_jf(u)(1-\bar{u}_n)^{-(n+1)}\dot{u}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\frac{\partial f}{\partial u_j}(p_n),$$

where $\int_{S(\gamma)}$ denotes the Cauchy principal value, defined as $\lim_{\varepsilon\to 0}\int_{\sigma_\varepsilon}$.

Proof Obviously

$$\frac{\partial F}{\partial z_j}=n\omega_{2n-1}^{-1}\int_S\bar{u}_jf(u)(1-z\bar{u}')^{-(n+1)}\dot{u}. \tag{2.1}$$

According to the expression (2.1) of f in [1], we have

$$\omega_{2n-1}^{-1}\int_S \bar{u}_j f(u)(1-z\bar{u}')^{-(n+1)}\dot{u}$$
$$=\sum_{k=1}^{n}[I_{x_k}^{(j)}(z)+I_{y_k}^{(j)}(z)]-\sum_{k=1}^{n-1}\frac{\partial f}{\partial x_k}(p_n)\omega_{2n-1}^{-1}\int_S \bar{u}_j x_k(1-z\bar{u}')^{-(n+1)}\dot{u}$$
$$+\sum_{k=1}^{n-1}\frac{\partial f}{\partial y_k}(p_n)\omega_{2n-1}^{-1}\int_S \bar{u}_j y_k(1-z\bar{u}')^{-(n+1)}\dot{u}+\frac{\partial f}{\partial y_n}(p_n)\omega_{2n-1}^{-1}\int_S \bar{u}_j y_n(1-z\bar{u}')^{-(n+1)}\dot{u}$$
$$+\frac{\partial f}{\partial x_n}(p_n)\omega_{2n-1}^{-1}\int_S \bar{u}_j(x_n-1)(1-z\bar{u}')^{-(n+1)}\dot{u}+f(p_n)\omega_{2n-1}^{-1}\int_S \bar{u}_j(1-z\bar{u}')^{-(n+1)}\dot{u}.$$

By Lemmas 1, 2, 3, we have

$$(K-\lim_{z\to p_n})\,\omega_{2n-1}^{-1}\int_S \bar{u}_j f(u)(1-z\bar{u}'_n)^{-(n+1)}\dot{u}$$
$$=\sum_{k=1}^{n}(I_{x_k}^{(j)}+I_{y_k}^{(j)})+\frac{1}{2n}\frac{\partial f}{\partial x_j}(p_n)+\frac{1}{2ni}\frac{\partial f}{\partial y_j}(p_n)$$
$$=\sum_{k=1}^{n}(I_{x_k}^{(j)}+I_{y_k}^{(j)})+\frac{1}{n}\frac{\partial f}{\partial u_j}(p_n). \tag{2.2}$$

On the other hand, by Lemmas 3, 4, 5, we also have

$$\lim_{\varepsilon\to 0}\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\bar{u}_j f(u)(1-\bar{u}_n)^{-(n+1)}\dot{u}$$
$$=\sum_{k=1}^{n}(I_{y_k}^{(j)}+I_{y_k}^{(j)})+\frac{\partial f}{\partial x_j}(p_n)\frac{1}{2n}\Big[1-\frac{1}{2}\Big(\frac{2\beta}{\alpha+\beta}\Big)^n\Big]$$
$$+\frac{\partial f}{\partial y_j}(p_n)\frac{1}{2ni}\Big[1-\frac{1}{2}\Big(\frac{2\beta}{\alpha+\beta}\Big)^n\Big]$$
$$=\sum_{k=1}^{n}(I_{x_k}^{(j)}+I_{y_k}^{(j)})+\frac{1}{n}\Big[1-\frac{1}{2}\Big(\frac{2\beta}{\alpha+\beta}\Big)^n\Big]\frac{\partial f}{\partial u_j}(p_n),$$

i. e.

$$\sum_{k=1}^{n}(I_{x_k}^{(j)}+I_{y_k}^{(j)})=\omega_{2n-1}^{-1}\int_{S(\gamma)}\bar{u}_j f(u)(1-\bar{u}_n)^{-(n+1)}\dot{u}-\frac{1}{n}\Big[1-\frac{1}{2}\Big(\frac{2\beta}{\alpha+\beta}\Big)^n\Big]\frac{\partial f}{\partial u_j}(p_n).$$

Inserting it into (2.2), the desired result is obtained.

Theorem 2. *Suppose f: $\bar{B}\to\mathbb{C}$, where $f_1=\operatorname{Re} f$ and $f_2=\operatorname{Im} f$ satisfy the conditions of Lemma 3 respectively, and $F(z)$ is the Cauchy integral of f. Let*

$$P_k=\{v=(v_1,\cdots,v_k,\cdots,v_n)\,|\,v\in S,\ v_k=0\}.$$

Then

$$(K-\lim_{z\to v})\frac{\partial F}{\partial z_k}(z)=n\omega_{2n-1}^{-1}\int_S \bar{u}_k f(u)(1-v\bar{u}')^{-(n+1)}\dot{u}+\frac{1}{2}\Big(\frac{2\beta}{\alpha+\beta}\Big)^n\frac{\partial f}{\partial u_k}(v)$$

holds for any $v\in P_k$. Here the integral

$$n\omega_{2n-1}^{-1}\int_S \bar{u}_k f(u)(1-v\bar{u}')^{-(n+1)}\dot{u}$$

is defined as

$$\lim_{\varepsilon\to 0}n\omega_{2n-1}^{-1}\Big\{\int_{\sigma_\varepsilon(v)}\bar{u}_k f_1(u)(1-v\bar{u}')^{-(n+1)}\dot{u}+i\int_{\sigma_\varepsilon(v)}\bar{u}_k f_2(u)(1-v\bar{u}')^{-(n+1)}\dot{u}\Big\},$$

where

$$\sigma_\varepsilon(v)=\{u\,|\,u\in S,\ \alpha^2(1-|v\bar{u}'|^2)^2+4\beta^2(\operatorname{Im} v\bar{u}')^2>\varepsilon^2\}.$$

Proof Take a unitary matrix U such that $vU=p_n$. Set $\overline{U}'=(\alpha_{ij})$, then $(v_1, \cdots, v_n)=(\alpha_{n1}, \cdots, \alpha_{nn})$, so $\alpha_{nk}=0$. Let $uU=w$, $zU=\zeta$, we have

$$(K-\lim_{z\to v}) n\omega_{2n-1}^{-1}\int_S \bar{u}_k f_1(u)(1-z\bar{u}')^{-(n+1)}\dot{u}$$

$$=\sum_{j=1}^{n-1}\bar{\alpha}_{jk}\left\{(K-\lim_{\zeta\to p_n}) n\omega_{2n-1}^{-1}\int_S \bar{w}_j f_1(w\overline{U}')(1-\zeta\bar{w}')^{-(n+1)}\dot{w}\right\}$$

$$=n\omega_{2n-1}^{-1}\int_S \bar{u}_k f_1(u)(1-v\bar{u}')^{-(n+1)}\dot{u}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{\alpha}_{jk}\alpha_{jl}\frac{\partial f_1}{\partial u_l}(v)$$

$$=n\omega_{2n-1}^{-1}\int_S \bar{u}_k f_1(u)(1-v\bar{u}')^{-(n+1)}\dot{u}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\frac{\partial f_1}{\partial u_k}(v).$$

We have the same equality for $f_2=\mathrm{Im}(f)$. The proof is completed.

Taking $\alpha=\beta$ in the above theorem, we obtain a simple formula

$$(K-\lim_{z\to v})\frac{\partial F}{\partial z_k}(z)=n\omega_{2n-1}^{-1}\int_S \bar{u}_k f(u)(1-v\bar{u}')^{-(n+1)}\dot{u}+\frac{1}{2}\frac{\partial f}{\partial u_k}(v),$$

where $v\in P_k$.

§ 3. Representing the boundary value by means of the Hadamard principal value

The aim of this section is to study the limit value of the derivative of Cauchy integral at any point on the sphere. We need the following Lemmas.

Lemma 6. *Suppose $z\in B$, then*

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_n x_k x_l}{(1-z\bar{u}')^{n+1}}\dot{u}=\frac{1}{2n}\delta_{nk}\delta_{nl}, \tag{3.1}$$

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_n x_k y_l}{(1-z\bar{u}')^{n+1}}\dot{u}=\frac{1}{2ni}\delta_{nk}\delta_{nl}, \tag{3.2}$$

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_n y_k y_l}{(1-z\bar{u}')^{n+1}}\dot{u}=-\frac{1}{2n}\delta_{nk}\delta_{nl}. \tag{3.3}$$

Proof Let

$$G(z)=\frac{1}{\omega_{2n-1}}\int_S \frac{x_k x_l}{(1-z\bar{u}')^n}\dot{u}.$$

Since $x_k=\frac{1}{2}(u_k+\bar{u}_k)$, we have

$$G(z)=\frac{1}{4\omega_{2n-1}}\int_S \frac{u_k u_l+\bar{u}_k u_l+u_k\bar{u}_l+\bar{u}_k\bar{u}_l}{(1-z\bar{u}')^n}\dot{u}. \tag{3.4}$$

By the Cauchy integral formula,

$$\frac{1}{\omega_{2n-1}}\int_S \frac{u_k u_l}{(1-z\bar{u}')^n}\dot{u}=z_k z_l.$$

Using the method of proving Lemma 1, we have

$$\frac{1}{4\omega_{2n-1}}\int_S \frac{\bar{u}_k u_l}{(1-z\bar{u}')^n}\dot{u}=\frac{1}{4\omega_{2n-1}}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{il}\bar{\alpha}_{jk}\int_S \frac{w_i\bar{w}_j}{(1-\rho\bar{w}_n)^n}\dot{w}$$

$$=\frac{1}{4\omega_{2n-1}}\sum_{j=1}^{n}\alpha_{jl}\bar{\alpha}_{jk}\int_S \frac{|w_j|^2}{(1-\rho\bar{w}_n)^n}\dot{w}.$$

An easy computation shows

$$\int_S \frac{|w_j|^2 \dot{w}}{(1-\rho \bar{w}_n)^n} = \frac{2\pi^n}{\Gamma(n+1)}, \quad (j=1, \cdots, n),$$

so

$$\frac{1}{4\omega_{2n-1}} \int_S \frac{\bar{u}_k u_l \dot{u}}{(1-z\bar{u}')^n} = \frac{1}{4n} \delta_{kl}.$$

It is easy to see that

$$\frac{1}{4\omega_{2n-1}} \int_S \frac{\bar{u}_k \bar{u}_l}{(1-z\bar{u}')^n} \dot{u} = \frac{1}{4\omega_{2n-1}} \sum_{i=1}^n \sum_{j=1}^n \bar{a}_{il} \bar{a}_{jk} \int_S \frac{\bar{w}_i \bar{w}_j}{(1-\rho w_n)^n} \dot{w} = 0$$

Substituting these results into (3.4), we obtain

$$G(z) = \frac{1}{4} z_k z_l + \frac{1}{2n} \delta_{kl}.$$

It follows that

$$\lim_{z \to p_n} \frac{\partial G}{\partial z_n}(z) = \frac{1}{2} \delta_{nk} \delta_{nl}.$$

This is just the equality (3.1). The proofs of (3.2), (3.3) are similar.

Lemma 7.

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_{2n-1}} \int_{\sigma_\varepsilon} \frac{\bar{u}_n (x_n - 1)}{(1-\bar{u}_n)^{n+1}} \dot{u} = \frac{1}{2n} + \frac{1}{2} \left(\frac{2\beta}{\alpha+\beta} \right)^{n-1} \frac{(n-1)\beta}{n(\alpha+\beta)}, \tag{3.5}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_{2n-1}} \int_{\sigma_\varepsilon} \frac{\bar{u}_n y_n}{(1-\bar{u}_n)^{n+1}} \dot{u} = \frac{1}{i} \left\{ \frac{1}{2n} - \frac{1}{2} \left(\frac{2\beta}{\alpha+\beta} \right)^{n-1} \frac{\beta + n\alpha}{n(\alpha+\beta)} \right\}, \tag{3.6}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_{2n-1}} \int_{\sigma_\varepsilon} \frac{\bar{u}_n x_k}{(1-\bar{u}_n)^{n+1}} \dot{u} = 0 \quad (k=1, \cdots, n-1), \tag{3.7}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_{2n-1}} \int_{\sigma_\varepsilon} \frac{\bar{u}_n y_k}{(1-\bar{u}_n)^{n+1}} \dot{u} = 0 \quad (k=1, \cdots, n-1). \tag{3.8}$$

Proof First we prove (3.6).

$$\frac{1}{\omega_{2n-1}} \int_{\sigma_\varepsilon} \frac{\bar{u}_n y_n \dot{u}}{(1-\bar{u}_n)^{n+1}} = \frac{1}{2i\omega_{2n-1}} \int_{v\bar{v}' < \frac{\varepsilon}{\alpha}} \dot{v} \left\{ 2\,\mathrm{Re} \int_c^{\pi - c} \frac{r^2 - r^2 e^{2i\theta}}{(1-re^{i\theta})^{n+1}} d\theta \right\}$$

$$+ \frac{1}{2i\omega_{2n-1}} \int_{v\bar{v}' > \frac{\varepsilon}{\alpha}} \dot{v} \int_{-\pi}^{\pi} \frac{r^2(1-e^{2i\theta})}{(1-re^{i\theta})^{n+1}} d\theta = I_1 + I_2,$$

$$\int_c^{\pi-c} \frac{r^2 - r^2 e^{2i\theta}}{(1-re^{i\theta})^{n+1}} d\theta = \frac{1}{i} \left\{ r^2 \int_{re^{ic}}^{-re^{-ic}} \frac{dz}{z(1-z)^{n+1}} - \int_{re^{ic}}^{-re^{-ic}} \frac{z dz}{(1-z)^{n+1}} \right\}$$

$$= \frac{1}{i} \left\{ \left[\frac{r^2-1}{n} \frac{1}{(1+re^{-ic})^n} - \frac{r^2+1}{n-1} \frac{1}{(1+re^{-ic})^{n-1}} \right.\right.$$

$$\left. + \sum_{k=1}^{n-2} \frac{r^2}{k} \frac{1}{(1+re^{-ic})^k} + r^2 (\log(-re^{-ic}) - \log(1+re^{-ic})) \right]$$

$$- \left[\frac{r^2-1}{n(1-re^{ic})^n} + \frac{r^2+1}{(n-1)(1-re^{ic})^{n-1}} + \sum_{k=1}^{n-2} \frac{r^2}{k(1-re^{ic})^k} \right.$$

$$\left.\left. + r^2 \log(re^{ic}) - r^2 \log(1-re^{ic}) \right] \right\} = \frac{1}{i}(B_1 - B_2),$$

then

$$I_1=\frac{1}{i\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\operatorname{Im}(B_1-B_2)\dot{v}$$
$$=\frac{1}{i\omega_{2n-1}}\operatorname{Im}\left\{\left(\sum_{k=1}^{n}J_k+J_0-J_0'\right)-\left(\sum_{k=1}^{n}H_k+H_0-H_0'\right)\right\},$$

where

$$J_n=\frac{1}{n}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{r^2-1}{(1+re^{-ic})^n}\dot{v},\quad J_{n-1}=\frac{1}{n-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{r^2+1}{(1+re^{-ic})^{n-1}}\dot{v},$$

$$J_k=\frac{1}{k}\int_{v\bar{v}'<\frac{1}{\alpha}}\frac{r^2\dot{v}}{(1-re^{-ic})^k},\quad (k=1,\ \cdots,\ n-1),$$

$$J_0=\int_{x\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log(-re^{-ic})\dot{v},\quad J_0'=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log(1+re^{-ic})\dot{v},$$

$$H_n=\frac{1}{n}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{r^2-1}{(1-re^{ic})^n}\dot{v},\quad H_{n-1}=\frac{1}{n-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{r^2+1}{(1-re^{ic})^{n-1}}\dot{v},$$

$$H_k=\frac{1}{k}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{r^2}{(1-re^{ic})^k}\dot{v},\quad (k=1,\ \cdots,\ n-2),$$

$$H_0=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log(re^{ic})\dot{v},\quad H_0'=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log(1-re^{ic})\dot{v}.$$

Using the method we used in the proof of Lemma 5, it is easy to prove

$$\lim_{\varepsilon\to 0}\operatorname{Im}(J_0)=\lim_{\varepsilon\to 0}\operatorname{Im}(J_0')=0,\quad \lim_{\varepsilon\to 0}J_k=0,\quad (k=1,\ \cdots,\ n).$$

On the other hand, we have

$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_0)=\lim_{\varepsilon\to 0}\operatorname{Im}(H_0')=0,$$

and

$$H_{n-1}=\frac{1}{n-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\frac{1+r^2}{(1-re^{ic})^{n-1}}\dot{v}=\frac{2\pi^{n-1}}{(n-1)\Gamma(n-1)}(2\beta)^{n-1}$$
$$\times\int_0^1\frac{(2-\eta t^2)\eta^{n-1}t^{2n-3}dt}{\eta^{n-1}[\beta t^2-i\alpha\sqrt{1-t^4}+O(\eta)]^{n-1}},$$

so

$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_{n-1})=\frac{2\pi^{n-1}}{\Gamma(n)}2^{n-1}\operatorname{Im}\left\{\int_0^1\frac{x^{n-2}dx}{(x-i\gamma\sqrt{1-x^2})^{n-1}}\right\}=\frac{\pi^n}{\Gamma(n)}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}.$$

Using the same method, we may prove

$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_n)=-\frac{\pi^n}{n\Gamma(n-1)}\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n,$$
$$\lim_{\varepsilon\to 0}H_k=0\quad (k=1,\ \cdots,\ n-1).$$

Hence

$$\lim_{\varepsilon\to 0}I_1=\frac{1}{i\omega_{2n-1}}\lim_{\varepsilon\to 0}\operatorname{Im}(-H_n-H_{n-1})=-\frac{1}{2i}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\frac{\beta+n\alpha}{n(\alpha+\beta)}.$$

The computation of I_2 is easy. Since

$$\int_{-\pi}^{\pi}\frac{r^2(1-e^{2i\theta})}{(1-re^{i\theta})^{n+1}}d\theta=\sum_{q=0}^{\infty}\frac{\Gamma(n+q+1)}{\Gamma(n+1)\Gamma(q+1)}r^{q+2}\int_{-\pi}^{\pi}e^{iq\theta}(1-e^{2i\theta})d\theta=2\pi r^2,$$

we have

$$\lim_{\varepsilon\to 0} I_2=\frac{1}{2i}\frac{2\pi}{\omega_{2n-1}}\frac{2\pi^{n-1}}{\Gamma(n-1)}\int_0^1(1-s^2)s^{2n-3}ds=\frac{1}{2ni}.$$

Thus

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n y_n}{(1-\bar{u}_n)^{n+1}}\dot{u}=\frac{1}{i}\left[\frac{1}{2n}-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\frac{\beta+n\alpha}{n(\alpha+\beta)}\right].$$

This is just the equality (3.6).

Employing the equalities

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(x_n-1)}{(1-\bar{u}_n)^{n+1}}\dot{u}=\lim_{\varepsilon\to 0}\frac{i}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n y_n}{(1-\bar{u}_n)^{n+1}}\dot{u}-\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n\dot{u}}{(1-\bar{u}_n)^{n}}$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n}{(1-\bar{u}_n)^{n}}\dot{u}=-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1},$$

we obtain (3.5) immediately. Using the same method as in the proof of Lemma 4, we can prove the equalities (3.7), (3.8).

Lemma 8. *For $i, j=1, \cdots, n$, $i+j<2n$, we have*

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n x_i x_j}{(1-\bar{u}_n)^{n+1}}\dot{u}=-\frac{1}{4n}\left(\frac{2\beta}{\alpha+\beta}\right)^{n}\delta_{ij}, \tag{3.9}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n y_i y_j}{(1-\bar{u}_n)^{n+1}}\dot{u}=-\frac{1}{4n}\left(\frac{2\beta}{\alpha+\beta}\right)^{n}\delta_{ij}. \tag{3.10}$$

Proof

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n x_i x_j}{(1-\bar{u}_n)^{n+1}}\dot{u}=\frac{1}{\psi\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(u_iu_j+\bar{u}_iu_j+u_i\bar{u}_j+\bar{u}_i\bar{u}_j)}{(1-\bar{u}_n)^{n+1}}\dot{u}. \tag{3.11}$$

It is easy to prove that the value of above integral is 0, when $i\neq j$. Now suppose $i=j<n$. Without loss of generality, take $i=j=1$. Using the method of proving Lemma 4, we have

$$\int_{\sigma_\varepsilon}\frac{\bar{u}_n u_1^2\dot{u}}{(1-\bar{u}_n)^{n+1}}=\int_{\sigma_\varepsilon}\frac{\bar{u}_n\bar{u}_1^2}{(1-\bar{u}_n)^{n+1}}\dot{u}=0. \tag{3.12}$$

On the other hand

$$\int_{\sigma_\varepsilon}\frac{\bar{u}_n|u_1|^2}{(1-\bar{u}_n)^{n+1}}\dot{u}=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2\dot{v}\left\{2\operatorname{Re}\int_c^{\pi-c}\frac{re^{i\theta}d\theta}{(1-re^{i\theta})^{n+1}}\right\}$$
$$+\int_{v\bar{v}'>\frac{\varepsilon}{\alpha}}|v_1|^2\dot{v}\int_{-\pi}^{\pi}\frac{re^{i\theta}d\theta}{(1-re^{i\theta})^{n+1}}=I_1+I_2.$$

Clearly $I_2=0$. The inner integral of I_1 equals

$$\frac{2}{n}\operatorname{Im}\{(1+re^{-ic})^{-n}-(1-re^{ic})^{-n}\},$$

so $I_1=\operatorname{Im}(J_n-H_n)$, where

$$J_n=\frac{2}{n}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(1+re^{-ic})^{-n}\dot{v},\quad H_n=\frac{2}{n}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}|v_1|^2(1-re^{ic})^{-n}\dot{v}.$$

It is known in Lemma 5 that

$$\lim_{\varepsilon\to 0}J_n=0,\quad \lim_{\varepsilon\to 0}\operatorname{Im}(H_n)=\frac{\pi^n}{n\Gamma(n)}\left(\frac{2\beta}{\alpha+\beta}\right)^n,$$

i. e.

$$\lim_{\varepsilon\to 0} I_1 = -\frac{\pi^n}{n\Gamma(n)}\left(\frac{2\beta}{\alpha+\beta}\right)^n,$$

so

$$\lim_{\varepsilon\to 0}\int_{\sigma_\varepsilon}\frac{\bar{u}_n|u_1|^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = -\frac{\pi^n}{n\Gamma(n)}\left(\frac{2\beta}{\alpha+\beta}\right)^n. \tag{3.13}$$

Substituting (3. 12) and (3.13) into (3.11), we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n x_1^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = -\frac{1}{4n}\left(\frac{2\beta}{\alpha+\beta}\right)^n.$$

Replacing x_1 by $x_j(j=2, \cdots, n-1)$, the above equality also holds. Thus (3.9) is proved. The proof of (3.10) is similar.

Lemma 9.

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = -\frac{1}{2n}, \tag{3.14}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n y_n^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = -\frac{1}{2n}. \tag{3.15}$$

Proof Since

$$\left|\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\right| \leqslant \frac{1}{|1-\bar{u}_n|^{n-1}},$$

$\int_S \frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u}$ exists and

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = \frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u}. \tag{3.16}$$

On the other hand

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n(x_n-1)^2}{(1-z\bar{u}')^{n+1}}\dot{u} = \frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u}. \tag{3.17}$$

Thus

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n(x_n-1)^2}{(1-z\bar{u}')^{n+1}}\dot{u} = \lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u}. \tag{3.18}$$

Let

$$G(z) = \frac{1}{\omega_{2n-1}}\int_S\frac{(x_n-1)^2}{(1-z\bar{u}')^n}\dot{u}.$$

It is easy to see that

$$G(z) = \frac{1}{4\omega_{2n-1}}\int_S\frac{u_n^2+\bar{u}_n^2+4-4u_n-4\bar{u}_n+2u_n\bar{u}_n}{(1-z\bar{u}')^n}\dot{u} = \frac{1}{4}\left(z_n^2+4-4z_n+\frac{2}{n}\right),$$

so

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n(x_n-1)^2}{(1-z\bar{u}')^{n+1}}\dot{u} = -\frac{1}{2n}. \tag{3.19}$$

(3.14) follows from (3.18) and (3.19).

Using the same method, we can prove

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n|1-\bar{u}_n|^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = -\frac{1}{n}.$$

Now (3.15) follows from the following equality

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n y_n^2}{(1-\bar{u}_n)^{n+1}}\dot{u} = \frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n|1-\bar{u}_n|^2}{(1-\bar{u}_n)^{n+1}}\dot{u} - \frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n(x_n-1)^2}{(1-\bar{u}_n)^{n+1}}\dot{u}.$$

Lemma 10.

$$\int_{\sigma_\varepsilon} \frac{\bar{u}_n x_k y_j}{(1-\bar{u}_n)^{n+1}} \dot{u} = 0 \tag{3.20}$$

for $k, j = 1, \cdots, n$, $k+j<2n$.

$$\lim_{\varepsilon\to 0}\int_{\sigma_\varepsilon} \frac{\bar{u}_n (x_n-1) y_j}{(1-\bar{u}_n)^{n+1}} \dot{u} = 0 \tag{3.21}$$

for $j=1, \cdots, n$.

Proof Since

$$\int_{\sigma_\varepsilon} \frac{\bar{u}_n x_k y_j \dot{u}}{(1-\bar{u}_n)^{n+1}} = \frac{1}{4i}\int_{\sigma_\varepsilon} \frac{\bar{u}_n (u_k u_j + \bar{u}_k u_j - u_k \bar{u}_j - \bar{u}_k \bar{u}_j)}{(1-u_n)^{n+1}} \mathring{u},$$

if $k\neq j$, then the above integral is equal to 0, if $k=j<n$, tnen

$$\int_{\sigma_\varepsilon} \frac{\bar{u}_n x_k y_k}{(1-\bar{u}_n)^{n+1}} \dot{u} = \frac{1}{4\dot{i}}\int_{\sigma_\varepsilon} \frac{\bar{u}_n (u_k^2 - \bar{u}_k^2)}{(1-\bar{u}_n)^{n+1}} \dot{u}.$$

Using the method used in proving Lemma 4, it is easy to prove that the above integral also equals 0. Hence (3.20) is true.

(3.21) holds evidently for $j=1, \cdots, n-1$. When $j=n$, since

$$\left|\frac{\bar{u}_n (x_n-1) y_n}{(1-\bar{u}_n)^{n+1}}\right| \leqslant \frac{1}{|1-\bar{u}_n|^{n-\frac{1}{2}}},$$

using the same method as in the proof of Lemma 9, we have

$$\lim_{z\to p_n} \frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_n (x_n-1) y_n}{(1-z\bar{u}')^{n+1}} \dot{u} = \lim_{\varepsilon\to 0} \frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon} \frac{\bar{u}_n (x_n-1) y_n}{(1-\bar{u}_n)^{n+1}} \dot{u}.$$

Thus

$$\lim_{z\to p_n} \frac{1}{\omega_{2n-1}}\int_S \frac{\bar{u}_n (x_n-1) y_n}{(1-z\bar{u}')^{n+1}} \dot{u} = \frac{1}{2ni} - \frac{1}{2ni} = 0,$$

by Lemma 2 and Lemma 6. Hence (3.21) is also ture.

Applying the above Lemmas, we have

Theorem 3. *Suppose* $f: \bar{B}\to \mathbf{R}$ *and* $f\in C^3(\bar{B})$. *Then*

$$\lim_{\varepsilon\to 0} \frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon} \frac{\bar{u}_n [f(u) - f(p_n)]}{(1-\bar{u}_n)^{n+1}} \dot{u}$$

exists and equals

$$c_n \frac{\partial f(p_n)}{\partial \bar{u}_n} + \left(\frac{1}{n} - 2c_n d_n + c_n\right)\frac{\partial f(p_n)}{\partial u_n} - \frac{c_{n+1}}{n}\sum_{j=1}^{n-1} \frac{\partial^2 f(p_n)}{\partial u_j \partial \bar{u}_j} - \frac{1}{n}\frac{\partial^2 f(p_n)}{\partial u_n \partial \bar{u}_n} + I,$$

where $c_n = \frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}$, $d_n = \frac{\beta+n\alpha}{n(\alpha+\beta)}$, *I is a number to be determined.*

Proof By the Taylor formula

$$\begin{aligned} f(u_1, \cdots, u_n) &= f(x_1+iy_1, \cdots, x_n+iy_n) \\ &= f(p_n) + \sum_{j=1}^{n-1} \frac{\partial f(p_n)}{\partial x_j} x_j + \frac{\partial f(p_n)}{\partial x_n}(x_n-1) + \sum_{j=1}^{n} \frac{\partial f(p_n)}{\partial y_j} y_j \\ &\quad + \frac{1}{2}\Big\{\sum_{i,j=1}^{n-1} \frac{\partial^2 f(p_n)}{\partial x_i \partial x_j} x_i x_j + 2\sum_{i=1}^{n-1} \frac{\partial^2 f(p_n)}{\partial x_i \partial x_n} x_i (x_n-1) + \frac{\partial^2 f(p_n)}{\partial x_n^2}(x_n-1)^2 \\ &\quad + 2\sum_{i=1}^{n} \frac{\partial^2 f(p_n)}{\partial y_i \partial x_n} y_i (x_n-1) + 2\sum_{i=1}^{n-1}\sum_{j=1}^{n} \frac{\partial^2 f(p_n)}{\partial x_i \partial y_j} x_i y_j + \sum_{i,j=1}^{n} \frac{\partial^2 f(p_n)}{\partial y_i \partial y_j} y_i y_j\Big\} \\ &\quad + R_3(u), \end{aligned} \tag{3.22}$$

where $R_3(u)$ is the remainder term of third order. Obviously

$$R_3(u)=O(|1-\bar{u}_n|^{3/2}),$$

so the integral $\int_S \bar{u}_n R_3(u)(1-\bar{u}_n)^{-(n+1)}\dot{u}$ exists, and we denote its value by I. By (3.22) and the Lemmas 7, 8, 9, and 10, we have

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{u}_n[f(u)-f(p_n)]}{(1-\bar{u}_n)^{n+1}}\dot{u}$$
$$=\Big[\frac{1}{2n}+c_n(1-d_n)\Big]\frac{\partial f(p_n)}{\partial x_n}+\frac{1}{i}\Big(\frac{1}{2n}-c_nd_n\Big)\frac{\partial f(p_n)}{\partial y_n}$$
$$+\frac{1}{2}\Big\{\sum_{j=1}^{n-1}\Big(\frac{\partial^2 f(p_n)}{\partial x_j^2}+\frac{\partial^2 f(p_n)}{\partial y_j^2}\Big)\Big(-\frac{c_{n+1}}{2n}\Big)-\frac{1}{2n}\Big(\frac{\partial^2 f(p_n)}{\partial x_n^2}+\frac{\partial^2 f(p_n)}{\partial y_n^2}\Big)\Big\}+I,$$

this is just the desired result.

Theorem 4. *Suppose f is as in Theorem 3, the limit*

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon(v)}\frac{\bar{u}_k[f(u)-f(v)]}{(1-v\bar{u}')^{n+1}}\dot{u},\quad (k=1,\cdots,n) \tag{3.23}$$

exist for any $v\in S$.

Proof Take a unitary matrix U such that $vU=p_n$, let $u=w\bar{U}'$, $\bar{U}'=(\alpha_{ij})$, then

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon(v)}\frac{\bar{u}_k[f(u)-f(v)]}{(1-v\bar{u}')^{n+1}}\dot{u}=\sum_{j=1}^{n-1}\bar{\alpha}_{jk}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{w}_j[f(w\bar{U}')-f(p_n\bar{U}')]}{(1-w_n)^{n+1}}\dot{w}$$
$$+\bar{\alpha}_{nk}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{w}_n[f(w\bar{U}')-f(p_n\bar{U}')]}{(1-w_n)^{n+1}}\dot{w}. \tag{3.24}$$

Since $\int_{\sigma_\varepsilon}\frac{\bar{w}_j}{(1-\bar{w}_n)^{n+1}}\dot{w}=0$, we have

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{w}_j[f(w\bar{U}')-f(p_n\bar{U}'))}{(1-w_n)^{n+1}}\dot{w}=\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\bar{w}_jf(w\bar{U}')}{(1-\bar{w}_n)^{n+1}}\dot{w}.$$

We have proved the existence of above limit in the proof of Theorem 1. By Theorem 3, tqe limit of the second term of the right hand of (3.24) exists. Thus the theorem is proved.

We call limit (3.23) the Hadamard principal value of singular integral

$$\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_kf(u)\dot{u}}{(1-v\bar{u}')^{n+1}},$$

and denote it by

$$P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_kf(u)}{(1-v\bar{u}')^{n+1}}\dot{u}=\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon(v)}\frac{\bar{u}_k[f(u)-f(v)]}{(1-v\bar{u}')^{n+1}}\dot{u}.$$

From Theorem 3, we have the following

Theorem 5. *Suppose f is as in Theorem 3, then*

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_nf(u)}{(1-z\bar{u}')^{n+1}}\dot{u}=P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_nf(u)}{(1-\bar{u}_n)^{n+1}}\dot{u}-c_n\frac{\partial f(p_n)}{\partial\bar{u}_n}$$
$$-(c_n-2c_nd_n)\frac{\partial f(p_n)}{\partial u_n}+\frac{c_{n+1}}{n}\sum_{j=1}^{n-1}\frac{\partial^2 f(p_n)}{\partial u_j\partial\bar{u}_j}.$$

Proof In the expression (3.22) of f, $R_3(u)=O(|1-\bar{u}_n|^{3/2})$. It is known in [1] that the following inequality

$$|1-z\bar{u}'|\geqslant\frac{1}{4\alpha}|1-\bar{u}_n|$$

holds for $z\in D_\alpha(p_n)$, $u\in S$. So

$$\left|\frac{\bar{u}_n R_3(u)}{(1-z\bar{u}')^{n+1}}\right|\leqslant\frac{K}{|1-\bar{u}_n|^{n-\frac{1}{2}}}.$$

By the Lebesgue theorem,

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n R_3(u)}{(1-z\bar{u}')^{n+1}}\dot{u}=\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n R_3(u)}{(1-\bar{u}_n)^{n+1}}\dot{u}=I.$$

Employing the expression (3.22) of f and Lemmas 1, 2 and 6, we find

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n f(u)\dot{u}}{(1-z\bar{u}')^{n+1}}=I+\frac{1}{n}\frac{\partial f(p_n)}{\partial u_n}-\frac{1}{n}\frac{\partial^2 f(p_n)}{\partial u_n\partial\bar{u}_n}\tag{3.25}$$

By Theorem 3,

$$I=P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_n f(u)}{(1-\bar{u}_n)^{n+1}}\dot{u}-c_n\frac{\partial f(p_n)}{\partial\bar{u}_n}-\left(\frac{1}{n}-2c_n d_n+c_n\right)\frac{\partial f(p_n)}{\partial u_n}$$
$$+\frac{c_{n+1}}{n}\sum_{j=1}^{n-1}\frac{\partial^2 f(p_n)}{\partial u_j\partial\bar{u}_j}+\frac{1}{n}\frac{\partial^2 f(p_n)}{\partial u_n\partial\bar{u}_n}.\tag{3.26}$$

Inserting (3.26) into (3.25), we obtain the desired result.

We now prove our main result.

Theorem 6. *Suppose f is as in Theorem 3, and $F(z)$ is the Cauchy integral of f, then*

$$(K-\lim_{z\to v})\frac{\partial F(z)}{\partial z}$$
$$=P\frac{n}{\omega_{2n-1}}\int_S\frac{\bar{u}f(u)\dot{u}}{(1-v\bar{u}')^{n+1}}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\left\{\frac{\partial f(v)}{\partial u}\left[\frac{2\beta}{\alpha+\beta}I-n\frac{\beta-\alpha}{\beta+\alpha}v'\bar{v}\right]\right.$$
$$\left.-n\frac{\partial f(v)}{\partial\bar{u}}\bar{v}'\bar{v}+\frac{2\beta}{\alpha+\beta}\bar{v}\left[\operatorname{tr}\left(\frac{\partial^2 f(v)}{\partial u\partial\bar{u}}\right)-v\frac{\partial^2 f(v)}{\partial u\partial\bar{u}}\bar{v}'\right]\right\}\tag{3.27}$$

holds for any $v\in S$. Here $\frac{\partial F}{\partial z}=\left(\frac{\partial F}{\partial z_1},\cdots,\frac{\partial F}{\partial z_n}\right)$, $\frac{\partial f}{\partial u}=\left(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_n}\right)$,

$$\frac{\partial f}{\partial\bar{u}}=\left(\frac{\partial f}{\partial\bar{u}_1},\cdots,\frac{\partial f}{\partial\bar{u}_n}\right),\quad\frac{\partial^2 f}{\partial u\partial\bar{u}}=\begin{pmatrix}\frac{\partial^2 f}{\partial u_1\partial\bar{u}_1},&\cdots,&\frac{\partial^2 f}{\partial u_1\partial\bar{u}_n}\\ \cdots&\cdots&\cdots\\ \frac{\partial^2 f}{\partial u_n\partial\bar{u}_1},&\cdots,&\frac{\partial^2 f}{\partial u_n\partial\bar{u}_n}\end{pmatrix}.$$

Proof (3.27) may be written as the component form

$$(K-\lim_{z\to v})\frac{\partial F(z)}{\partial z_k}=P\frac{n}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f(u)}{(1-v\bar{u}')^{n+1}}\dot{u}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^n\frac{\partial f(v)}{\partial u_k}$$
$$-\frac{n}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\bar{v}_k\left\{\frac{\beta-\alpha}{\beta+\alpha}\sum_{j=1}^n\frac{\partial f(v)}{\partial u_j}v_j\right.$$
$$+\sum_{j=1}^n\frac{\partial f(v)}{\partial\bar{u}_j}\bar{v}_j-\frac{1}{n}\frac{2\beta}{\alpha+\beta}\sum_{j=1}^n\frac{\partial^2 f(v)}{\partial u_j\partial\bar{u}_j}$$
$$\left.+\frac{1}{n}\frac{2\beta}{\alpha+\beta}\sum_{i,j=1}^n\frac{\partial^2 f(v)}{\partial u_i\partial\bar{u}_j}v_i\bar{v}_j\right\},\quad(k=1,\cdots,n).\tag{3.28}$$

As before, take a unitary matrix U such that $vU=p_n$. Let $\bar{U}'=(a_{ij})$, then $v_k=a_{nk}$ $(k=1,\cdots,n)$. Set $u=w\bar{U}'$, then

$$(K-\lim_{z\to v})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f(u)\dot{u}}{(1-z\bar{u}')^{n+1}}=\sum_{j=1}^{n}\bar{\alpha}_{jk}(K-\lim_{\zeta\to p_n})\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{w}_j f(w\bar{U}')}{(1-\zeta\bar{w}')^{n+1}}\dot{w}. \quad (3.29)$$

By Theorems 1 and 5, the right hand side of (3.29) is equal to

$$\sum_{j=1}^{n-1}\bar{\alpha}_{jk}\left\{\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{w}_j f(w\bar{U}')\dot{w}}{(1-w_n)^{n+1}}+\frac{c_{n+1}}{n}\frac{\partial f(p_n\bar{U}')}{\partial w_j}\right\}$$
$$+\bar{\alpha}_{nk}\left\{P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{w}_n f(w\bar{U}')\dot{w}}{(1-\bar{w}_n)^{n+1}}-\frac{\partial f(p_n\bar{U}')}{\partial w_n}c_n-(c_n-2c_nd_n)\frac{\partial f(p_n\bar{U}')}{\partial w_n}\right.$$
$$\left.+\frac{c_{n+1}}{n}\sum_{j=1}^{n-1}\frac{\partial^2 f(p_n\bar{U}')}{\partial w_j\partial\bar{w}_j}\right\}$$
$$=P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f(u)\dot{u}}{(1-v\bar{u}')^{n+1}}+\frac{c_{n+1}}{n}\sum_{i=1}^{n}\sum_{l=1}^{n}\bar{\alpha}_{jk}\alpha_{jl}\frac{\partial f(v)}{\partial u_l}$$
$$-\bar{\alpha}_{nk}\left(c_n-2c_nd_n+\frac{c_{n+1}}{n}\right)\sum_{j=1}^{n}\frac{\partial f(v)}{\partial u_j}v_j-\bar{\alpha}_{nk}c_n\sum_{j=1}^{n}\frac{\partial f(v)}{\partial\bar{u}_j}\bar{v}_j$$
$$+\frac{c_{n+1}}{n}\bar{\alpha}_{nk}\sum_{j=1}^{n}\frac{\partial^2 f(v)}{\partial u_j\partial\bar{u}_j}-\frac{c_{n+1}}{n}\bar{\alpha}_{nk}\sum_{i,j=1}^{n}\frac{\partial^2 f(v)}{\partial u_i\partial\bar{u}_j}v_j\bar{v}_j$$
$$=P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f(u)\dot{u}}{(1-v\bar{u}')^{n-1}}+\frac{c_{n+1}}{n}\frac{\partial f(v)}{\partial u_k}-\bar{\alpha}_{nk}\left\{\left(c_n-2c_nd_n+\frac{c_{n+1}}{n}\right)\sum_{j=1}^{n}\frac{\partial f(v)}{\partial u_j}v_j\right.$$
$$\left.+c_n\sum_{j=1}^{n}\frac{\partial f(v)}{\partial\bar{u}_j}\bar{v}_j-\frac{c_{n+1}}{n}\sum_{j=1}^{n}\frac{\partial^2 f(v)}{\partial u_j\partial\bar{u}_j}+\frac{c_{n+1}}{n}\sum_{i,j=1}^{n}\frac{\partial^2 f(v)}{\partial u_i\partial\bar{u}_j}v_i\bar{v}_j\right\}.$$

Inserting

$$c_n=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1},\quad c_n-2c_nd_n+\frac{c_{n-1}}{n}=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\frac{\beta-\alpha}{\beta+\alpha}$$

into the above equality, (3.28) is obtained.

Theorem 7. *Suppose f: $\bar{B}\to\mathbf{C}$, where $f_1=\operatorname{Re}f$ and $f_2=\operatorname{Im}f$ are in $C^3(\bar{B})$, and $F(z)$ is the Cauchy integral of f, then (3.27) also holds for any $v\in S$. Here*

$$P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f(u)\dot{u}}{(1-v\bar{u}')^{n+1}}$$

is defined as

$$P\frac{1}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f_1(u)\dot{u}}{(1-v\bar{u}')^{n+1}}+P\frac{i}{\omega_{2n-1}}\int_S\frac{\bar{u}_k f_2(u)\dot{u}}{(1-v\bar{u}')^{n+1}}.$$

The proof is obvious.

Taking $\alpha=\beta$ in Theorem 7, we have

$$(K-\lim_{z\to v})\frac{\partial F(z)}{\partial z}=P\frac{n}{\omega_{2n-1}}\int_S\frac{\bar{u}f(u)\dot{u}}{(1-v\bar{u}')^{n+1}}$$
$$+\frac{1}{2}\left\{\frac{\partial f(v)}{\partial u}-n\frac{\partial f(v)}{\partial\bar{u}}\bar{v}'v+\bar{v}\left[\operatorname{tr}\left(\frac{\partial^2 f(v)}{\partial u\partial\bar{u}}\right)-v\frac{\partial^2 f(v)}{\partial u\partial\bar{u}}\bar{v}'\right]\right\}.$$

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