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ON THE RELATIVE POSITION OF LIMIT CYCLES FOR THE EQUATION OF TYPE $(\mathrm{II})_{l=0}$

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Abstrect

In this paper, we consider the relative position of limit cycles for the system

$$\begin{aligned}\frac{dx}{dt}&=\delta x-y+mxy-y^2,\\ \frac{dy}{dt}&=x+ax^2,\end{aligned}\tag{1}$$

under the condition

$$a<0,\ 0<\delta\leqslant m,\ m\leqslant\frac{1}{a}-a.\tag{2}$$

The main result is as follows:

(i) Under Condition (2), if $\delta=\frac{m}{2}+\frac{m^2}{4a}\equiv\delta_0$, then system $(1)_{\delta_0}$ has no limit cycles and on singular closed trajectory through a saddle point in the whole plane.

(ii) Under condition (2), the foci O and R' cannot be surrounded by the limit cycles of system (1) simultaneously.

In parper [1], while studying the centralized distribution of limit cycles for the equation of type $(\mathrm{II})_{l=0}$

$$\frac{dx}{dt}=\delta x-y+mxy-y^2,\quad \frac{dy}{dt}=x+ax^2\tag{1}$$

an interesting case is observed. If

$$a<0,\ 0<\delta\leqslant m,\ m\leqslant\frac{1}{a}-a\tag{2}$$

as $\delta\to m$, there exist two limit cycles around the focus O, and a semistable cycle appears abruptly around the other focus R', and then breaks into (at least) two limit cycles.

After the examples of the quadratic differential system with at least four cycles given by [2] and [3], Ye Yanqian[4] raised the following question: Under condition (2), whether (2, 2)-distribution appears to system (1). (i. e. there exist two limit cycles around each of the two foci simultaneously).

In this paper, we prove that under condition (2), the limit cycles of system (1)

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are concentrately distributed. The method used here is to reduce (1) to Liénard's equation by a series of transformations. Then in the case of

$$\delta=\frac{m}{2}+\frac{m^2}{4a}\equiv\delta_0,$$

there exists no limit cycle of system $(1)_{\delta_0}$ in the whole plane. Comparing (1) with $(1)_{\delta_0}$, we prove that the limit cycles of (1), under (2), are concentrately distributed.

From (2), obviously, $27a<4m^3$. Hence the cubic equation

$$f(x)\equiv x^3-mx^2+a=0 \tag{3}$$

has only one positive root k (>0).

Now follow the method used in [5] and transform (1) by a series of transformations:

let $$x_1=y-kx,\quad y_1=y,\quad d\tau=k^2dt \tag{4}$$

and let $$x_2=\alpha+\beta x_1,\quad y_2=y_1, \tag{5}$$

where $$\alpha=k^3-\delta k^2+k,\quad \beta=mk^2-2a>0, \tag{6}$$

Again let $$x_3=x_2,\quad y_3=b_{10}+b_{20}x_2+\beta y_2, \tag{7}$$

where $$b_{10}=\delta k^2-k-2a\frac{\alpha}{\beta},\quad b_{20}=\frac{a}{\beta}, \tag{8}$$

write $$b_{00}=\frac{a\alpha^2}{\beta}-(\delta k^2-k)\alpha. \tag{9}$$

we may prove $b_{00}\neq 0$ under condition (2) (see(19)).

Thus, let $$x_4=\frac{1}{b_{00}}x_3,\ y_4=y_3 \tag{10}$$

under the transformations (4), (5), (7) and (10), system (1) can be reduced to

$$\frac{dx}{dt}=1+xy,\quad \frac{dy}{dt}=a_{00}+a_{10}x+a_{20}x^2+a_{11}xy+a_{01}y+a_{02}y^2 \tag{11}$$

The relationship between the coefficients of (11) and (1) is as follows:

$$\left.\begin{aligned}
&a_{00}=\alpha k+\frac{a\alpha^2}{\beta}-b_{10}\left(k+2a\frac{\alpha}{\beta}\right)+\frac{a}{\beta}b_{10}^2+b_{00}b_{20}\\
&a_{10}=b_{00}\left[-\left(k+2a\frac{\alpha}{\beta}\right)-b_{20}\left(k+2a\frac{\alpha}{\beta}\right)+\frac{2a}{\beta}b_{10}(1+b_{20})\right]\\
&a_{20}=b_{00}^2\frac{a}{\beta}(1+b_{20})^2,\quad a_{11}=-b_{00}\frac{a}{\beta}\left(1+\frac{2a}{\beta}\right),\\
&a_{01}=k+\frac{2a}{\beta}(\alpha-b_{10}),\quad a_{02}=\frac{a}{\beta}.
\end{aligned}\right\} \tag{12}$$

Again making the transformation

$$Y=\frac{y}{x}+\frac{1}{x^2},\quad \omega=\frac{1}{x} \tag{13}$$

in (11) when $x\neq 0$, we obtain

$$\frac{d\omega}{dt}=\omega Y,\quad \frac{dY}{dt}=P_4(\omega)+P_2(\omega)Y+(1-a_{02})Y^2, \tag{14}$$

where $P_4(\omega)$ is a polynomial of degree four, and

$$P_2(\omega)=(1+2a_{02})\omega^2-a_{01}\omega-a_{11}. \tag{15}$$

Next using (14), when $\delta=\frac{m}{2}+\frac{m^2}{4a}\equiv\delta_0$, we prove that $(1)_{\delta_0}$ has no limit cycles in the whole plane. First, we prove several simple lemmas.

Lemma 1. *Under condition (2), the estimate value of the positive root k of equation (3) is*

$$m<k\leqslant -a \tag{16}$$

Proof From (2), we have $1-a(a+m)\leqslant 0$. Thus

$f(-a)=[1-a(a+m)]a\geqslant 0$. If $-a<k$, from $f(0)=a<0$, equation (3) must have another positive root in $(0, -a)$ besides k and this contradicts the hypothesis of Lemma 1. Again, $f(m)=a<0$ implies $m<k$.

Lemma 2. *Suppose condition (2) holds, when $\delta=\frac{m}{2}+\frac{m^2}{4a}\equiv\delta_0$, system $(1)_{\delta_0}$, i. e.*

$$\frac{dx}{dt}=\delta_0 x-y+mxy-y^2,\quad \frac{dy}{dt}=x+ax^2 \tag{17}$$

has no limit cycles and no singular closed trajectory passing through a saddle-point in the whole plane.

Proof By transformations (4), (5), (7) and (10), system (17) can be reduced to $(11)_{\delta_0}$. Therefore, the investigation of the limit cycles of (17) is reduced to those of $(11)_{\delta_0}$. Obviously, line $x=0$ is an arc without contact with $(11)_{\delta_0}$ (the corresponding line $\alpha+\beta(y-kx)=0$ is an arc without contact with (17)). Hence the limit cycles will not cross the line $x=0$. We need only consider the problem of limit cycles $(11)_{\delta_0}$ in semiplanes $x>0$ or $x<0$ (i. e. $x\neq 0$). Since we change $(11)_{\delta_0}$ in to $(14)_{\delta_0}$, there is no change in the number and configurations of the limit cycles. Since $\omega=0$ is a solution of $(11)_{\delta_0}$, then the limit cycles will not cross the line $\omega=0$. Following [5], when $\omega\neq 0$, by the transformation $Y=z|\omega|^{l-a_{02}}$ in (14), (14) can be reduced to

$$\frac{dz}{dt}=\operatorname{sgn}\omega[P_4(\omega)|\omega|^{2a_{02}-3}+zP_2(\omega)|\omega|^{a_{02}-2}]\equiv P(z,\omega),$$

$$\frac{d\omega}{dt}=z\equiv Q(x, y). \tag{18}$$

Therefore, the investigation of the limit cycles of $(11)_{\delta_0}$ corrsponds to those of $(18)_{\delta_0}$. For (18), we have

$$\frac{\partial P}{\partial z}+\frac{\partial Q}{\partial\omega}=\operatorname{sgn}\omega P_2(\omega)|\omega|^{a_{02}-2}.$$

Hence. we need only consider the determination sign of $P_2(\omega)$ in the following. For this purpose, we estimate the coefficients of $P_2(\omega)$. From (6) and applying (3), we have $\alpha=-a+k+(m-\delta)k^2>0$ (by (2), $m-\delta\geqslant 0$). From (9) and (16),

$$b_{00}=\frac{\alpha}{\beta}[k^3(a+k)-\delta k^5]<0. \tag{19}$$

Form (6), (8) and (12), we have

$$a_{11}=-b_{00}\frac{amk^2}{\beta^2}<0,$$

then
$$-a_{11}>0, \qquad 1+2a_{02}=\frac{mk^2}{\beta}>0. \tag{20}$$

Since a_{01} depends on δ we write $a_{01}(\delta)$. Substituting α, δ_0, b_{10} and $\beta=mk^2-2a$ into $a_{01}(\delta_0)$, then by an elementary computation, we show that

$$a_{01}(\delta_0)=k+\frac{2a}{\beta}(\alpha-b_{10})=\frac{1}{\beta^2}(k\beta^2+2a\beta k^3-2am\beta k^2-\beta m^2k^2+4a\beta k+4a^2\alpha)$$
$$=\frac{mk^3}{\beta^2}(a-mk^2+k^3)=0, \tag{21}$$
$$a_{10}=b_{00}(-a_{01}-b_{20}a_{01})=0. \tag{21$_1$}$$

(20) and (21) imply that $P_2(\omega)$ is positive definite in the region $\omega>0$, i. e. $\frac{\partial P}{\partial z}+\frac{\partial Q}{\partial \omega}$ is positive definite in $\omega>0$. Then, using Bendixson's criterion (see [7, Theorem 1.10]), we see that in the semi-plane $\omega>0$, system $(18)_{\delta_0}$ has no limit cycles and no singular closed trajectory passing through a saddle-point. Also, from (21) and $(21)_1$, $(14)_{\delta_0}$ is symmetrical with respect to line $\omega=0$. Hence, the conclusion of Lemma 2 is true in the whole plane.

Lemma 3. *The relationship between the coordinates of the finite singular points of systems* (1) *and* (11) *is given in the following formulas*

$$x_4=\frac{1}{b_{00}}[\alpha+\beta(y-kx)], \tag{22}$$
$$y_4=b_{10}+b_{20}[\alpha+\beta(y-kx)]+\beta y,$$

where x_4, y_4 are variables of (11). *Furthermore, under condition* (2) *and* $0<\frac{m}{2}+\frac{m^2}{2a}\leqslant\delta$, *focus R' of* (1) *corresponds to focus R' of* (18) *in the semi-plane* $\omega>0$,

$$R'\left(-\frac{1}{a}, \frac{-1}{2}\left[1+\frac{m}{a}+\sqrt{\left(1+\frac{m}{a}\right)^2-\frac{4\delta}{a}}\right]\right).$$

Proof Noting (4), (5), (7), (18), we obtain (22). Substituting the given coordinates of R' into (22), and let

$$r=\sqrt{\left(1+\frac{m}{a}\right)^2-\frac{4\delta}{a}}>0,$$

we then obtain

$$x_4=\frac{1}{b_{00}}\left[\frac{1}{a}(ak^3+ak-a\delta k^2-2ak+mk^3)-\frac{\beta(a+m)}{2a}\right]-\frac{\beta r}{2b_{00}}.$$

we need only estimate the brackets of the above formula. We have

$$[\cdots]=\frac{1}{a}(ak^3-a\delta k^2+mk^3-ak)+\frac{(a-k^3)(a+m)}{2a}=-k+\left(\frac{m}{2}+\frac{m^2}{2a}-\delta\right)k^2<0$$

Since $-k<0$, $b_{00}<0$, from condition $\frac{m}{2}+\frac{m^2}{2a}\leqslant\delta$ of Lemma 3, then $x_4>0$. Corrsponding to (11), R' belongs to the semi-plane $x>0$. Applying (13), system (11) can be reduced to (14) (or (18)). Thus R' belongs to the semi-plane $\omega>0$.

Theorem 1. *Under condition (2), foci O and R′ cannot be surrounded by the limit cycles of system (1) simultaneously, i. e. when* $\frac{m}{2}+\frac{m^2}{4a}\leqslant\delta\leqslant m$, *system (1) has no limit cycle around O; when* $0<\delta\leqslant\frac{m}{2}+\frac{m^2}{4a}$, *system (1) has no limit cycle around R′.*

Proof we now invesigate the cases of O and R' separately.

(A) In the first case $\delta_0\equiv\frac{m}{2}+\frac{m^2}{4a}\leqslant\delta$, there exists no limit cycle around O: when $\delta\equiv\delta_0$, the conclusiom is proved in Lemma 2. We need only consider the case of $\delta_0<\delta\leqslant m$. It is obvious that under condition (2), the limit cycles of systems (1) and (17) will not cross the line $x=-\frac{1}{a}$. Thus we need only invesigate the limit cycles in the semi-plane $x<-\frac{1}{a}$. When $x\neq 0$,

$$\left(\frac{dx}{dy}\right)_{(1)}-\left(\frac{dx}{dy}\right)_{(2)}=\frac{(\delta-\delta_0)_x}{(1+ax)x}=\frac{\delta-\delta_0}{1+ax}>0, \tag{23}$$

(1) and (17) have only two singular points O and $M(0,\ -1)$ in semiplan $x<\frac{1}{a}$. If there exists a limit cycle Γ around O, then $M(0,\ -1)\in\operatorname{int}\Gamma$. when $x\neq 0$, inequality (23) implies that trajectories of (17) cross cycle Γ, inwards. (when $x=0$, the continuity of a vector field implies the above conclusion). And when $\delta_0>0$, O is an unstable focus. This implies that (17) has a limit cycle around O, and it is contradictory to Lemma 2. Then (1) has no limit cycle around O in the semi-plane or the whole plane.

(B) In the second case where $0<\delta\leqslant\frac{m}{2}+\frac{m^2}{4a}\equiv\delta_0$, there exists no limit cycle of (1) around the other focus R'. We shall prove this in two steps as follows:

1. when $0<\delta\leqslant\frac{m}{2}+\frac{m^2}{2a}$, the conclusion is proved in paper [6];

2. when

$$\frac{m}{2}+\frac{m^2}{2a}\leqslant\delta\leqslant\frac{m}{2}+\frac{m^2}{4a}, \tag{24}$$

from (2) and (24), system (1) satisfies the condition of Lemma 3. It implies that R' belongs to semi-plane $\omega>0$. Next, we proceed to show that $P_2(\omega)$ has a definite sign in the semi-plane $\omega>0$. To this end, we estimate $a_{01}(\delta)$ and obtain

$$\begin{aligned}a_{01}(\delta)&\equiv k+\frac{2a}{\beta}(k^3+2k)+\frac{2a}{\beta}\left[\frac{2a}{\beta}(k^3+k)\right]-\frac{2a}{\beta}\left(1+\frac{2a}{\beta}\right)\delta k^2\\&\leqslant k+\frac{2a}{\beta}(k^3+2k)+\frac{2a}{\beta}\left[\frac{2a}{\beta}(k^3+k)\right]-\frac{2a}{\beta}\left(1+\frac{2a}{\beta}\right)\delta_0 k^2\equiv a_{01}(\delta_0)=0,\end{aligned}$$

i. e.

$$a_{01}(\delta)\leqslant 0,\quad -a_{01}(\delta)\geqslant 0. \tag{25}$$

We have proved that $-a_{11}>0$ (see (20)). Again from (20), (25), we know that $P_2(\omega)$ has a definite sign in the seim-plane $\omega>0$. i. e. $\frac{\partial P}{\partial z}+\frac{\partial Q}{\partial\omega}$ has a definite sign in the semi-plane $\omega>0$. Then (14) has no limit cycle around R' in $\omega>0$. And limit cycles of (14) cannot cross the line $\omega=0$, then system (14) or (1) has no limit cycles around R' in the whole plane. From (A) and (B), thus, the theorem is proved.

Remark. Under condition (2), as $\delta\to m$, when $\delta=\delta_2$, the limit cycles of (1) surround the focus O vanishes; when $\delta=\bar{\delta}$, a semi-stable cycle suddenly appears in the neighbourhood of R'. Paper [7] analysed this process, and raised the question of how to determine the magnitude relation between δ_2 and $\bar{\delta}$?

Using the above Theorem 1, it follows that $0<\delta_2<\bar{\delta}<m$.

Finally, considering system:

$$\frac{dx}{dt}=(\delta+l)x-y+alx^2+mxy-y^2,\quad \frac{dy}{dt}=x+ax^2, \tag{26}$$

we have

Theorem 2. *suppose condition* (2) *holds, when* $l>0$,

$$\frac{m}{2}+\frac{m^2}{4a}\leqslant\delta,$$

system (26) *has no limit cycles around focus* O.

Proof Since condition (2) is satisfied, system (1) has no limit cycles around O. Obviously, limit cycles will not cross the line $x=-\frac{1}{a}$. (1) and (26) has only singular points O and M in the semi-plane $x<-\frac{1}{a}$. From $\delta>0$, $l>0$, and O is unstable, when $x\neq0$,

$$\left(\frac{dx}{dy}\right)_{(26)}-\left(\frac{dx}{dy}\right)_{(1)}=\frac{(x+ax^2)l}{x+ax^2}=l>0$$

holds. Then as in the proof of Theorem 1, system (26) has no limit cycles around O.

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