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# ON DIRICHLET PROBLEMS FOR SECOND ORDER QUASILINEAR DEGENERATE ELLIPTIC EQUATIONS

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## Abstract

The purpose of this paper is to study the existence of the classical solutions of some Dirichlet problems for quasilinear elliptic equations

$$a_{11}(x, y, u)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x, y, u)\frac{\partial^2 u}{\partial x\partial y}+a_{22}(x, y, u)\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0,$$

where $a_{ij}(x, y, u)$ $(i, j=1, 2)$ satisfy

$$\lambda(x, y, u)|\xi|^2\leqslant\sum_{i,j=1}^{2}a_{ij}(x, y, u)\xi_i\xi_j\leqslant\Lambda(x, y, u)|\xi|^2$$

for all $\xi\in\mathbf{R}^2$ and $(x, y, u)\in\bar{\Omega}\times[0, +\infty)$, i. e., $\lambda(x, y, u)$、$\Lambda(x, y, u)$ denote the minimum and maximum eigenvalues of the matrix $[a_{ij}(x, y, u)]$ respectively, moreover

$$\lambda(x, y, 0)=0;\quad \Lambda(x, u, 0)=0;\quad \Lambda(x, y, u)\geqslant\lambda(x, y, u)>0,\quad (u>0).$$

Some existence theorems under the "natural conditions" imposed on $f(x, y, u, p, q)$ are obtained.

This paper is concerned with the Dirichlet problem on $\bar{\Omega}$ for the following quasilinear equation

$$a_{11}(x, y, u)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x, y, u)\frac{\partial^2 u}{\partial x\partial y}+a_{22}(x, y, u)\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0, \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^2$, $a_{ij}(x, y, u)$ $(i, j=1, 2)$ satisfy

$$\lambda(x, y, u)|\xi|^2\leqslant\sum_{i,j=1}^{2}a_{ij}(x, y, u)\xi_i\xi_j\leqslant\Lambda(x, y, u)|\xi|^2 \tag{2}$$

for all $\xi\in\mathbf{R}^2$ and $(x, y, u)\in\bar{\Omega}\times[0, +\infty)$, i. e., $\lambda(x, y, u)$、$\Lambda(x, y, u)$ denote the minimum and maximum eigenvalues of the matrix $[a_{ij}(x, y, u)]$ respectively. Moreover

$$\lambda(x, y, 0)=0,\quad \Lambda(x, y, 0)>0;\quad \Lambda(x, y, u)\geqslant\lambda(x, y, u)>0,\quad u>0. \tag{3}$$

Clearly, equation (1) is elliptic when $u>0$, but is degenerate when $u=0$.

In general, as we know[1], the Dirichlet problem for linear degenerate equations has only weak solutions. However, when the degeneracy occurs on no other than the

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boundary, it is possible to have classical solutions[2].

Jiang[3] studied the Dirichlet problem on $\Omega_1$ for quasilinear equations

$$K(u)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+a(x, y)\frac{\partial u}{\partial x}+b(x, y)\frac{\partial u}{\partial y}+c(x, y)u=f(x, y) \tag{4}$$

and

$$\frac{\partial^2 u}{\partial x^2}+K(u)\frac{\partial^2 u}{\partial y^2}+a(x, y)\frac{\partial u}{\partial x}+b(x, y)\frac{\partial u}{\partial y}+c(x, y)u=f(x, y), \tag{5}$$

where $\Omega_1$ is an open domain enclosed by the segment $\{(x, y)\,|\,0\leqslant x\leqslant 1,\ y=0\}$ and the curve $\Gamma_1$ joining the point $(0, 0)$ with the point $(1, 0)$ in the halfplane $\{(x, y)\,|\,y>0\}$, $K(u)$ satisfies

$$K(0)=0;\quad K(u)>0,\quad u>0. \tag{6}$$

Obviously, equations (4) and (5) are of type (1). Jiang showed that, when $c(x, y)\leqslant 0$ and $f(x, y)\leqslant 0$, equation (4), and equation (5) as well, together with the continuous boundary condition

$$u(x, y)\,|_{\partial\Omega_1}=\varphi_1(x, y), \tag{7}$$

$$\varphi_1(x, y)=0,\ (x, y)\in\partial\Omega_1\backslash\Gamma_1;\quad \varphi_1(x, y)>0,\ (x, y)\in\Gamma_1$$

has a classical solution. This solution is positive in the interior of $\Omega_1$, in other words, equations (4) and (5) are not degenerate in $\Omega_1$.

We[4] extended Jiang's result to more general equations in $n$-dimensional domain. In particular, for the equations on $\overline{\Omega}_1$

$$K(u)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0 \tag{8}$$

and

$$\frac{\partial^2 u}{\partial x^2}+K(u)\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0. \tag{9}$$

We proved that Dirichlet problems (8), (7) and (9), (7) have the classical solutions which are positive in the interior of $\Omega_1$ under the "natural conditions" imposed on $f(x, y, u, p, q)$.

In § 1 we first consider equation (8) and (9) on $\overline{\Omega}_2$ with the partially vinishing boundary value

$$u(x, y)\,|_{\partial\Omega_2}=\varphi_2(x, y), \tag{10}$$

where $\Omega_2$ is bounded by the straightlines $x=0$, $x=1$, $y=0$ and the open curve $\Gamma_2$ joining the point $(0, h)$ with the point $(1, h)$ in the halfplane $\{(x, y)\,|\,y>h\}$, $(h>0)$. $\varphi_2$ is defined and continuous on $\partial\Omega_2$ with

$$\varphi_2(x, y)=0,\ (x, y)\in\partial\Omega_2\backslash\Gamma_2;\quad \varphi_2(x, y)>0,\ (x, y)\in\Gamma_2.$$

We shall show that the Dirichlet problem for equation (8) on $\overline{\Omega}_2$ has a classical solution only under the "natural conditions" imposed on $f(x, y, u, p, q)$ as well. But equation (9) cannot be ensured to be undegenerate in the interior of $\Omega_2$. We shall give an example which shows that the solution of (9), (10) is not identically positive in $\Omega_2$. Furthermore we shall make an estimate about the support of the solution of the equation of type (9) with (10). Thus we can know where the degeneracy is caused. Hence we are sure that the coefficients $a_{ij}(x, y, u)$ must be restricted in order to make equation (1) undegenerate in the interior of $\Omega$.

In § 2 we shall discuss the Dirichlet problem for equation (1) under one of the following conditions:

(i) the boundary value vanishes only at a single point;

(ii) the boundary value vanishes on a straight segment;

(iii) the boundary value vanishes on two straight segments in different directions.

We shall prove the existence theorems for these Dirichlet problems under the "natural conditions" imposed on $f(x, y, u, p, q)$ and some restrictions imposed on $a_{ij}(x, y, u)$.

## § 1.

For problem (8), (10), we have the following result.

**Theorem 1.** *If*

1) $f(x, y, u, p, q)\in C(\overline{\Omega}_2\times\mathbf{R}\times\mathbf{R}^2)\cap C^1(\Omega_2\times\mathbf{R}\times\mathbf{R}^2)$,

$$f(x, y, 0, 0, 0)\geqslant 0, \quad f_u(x, y, u, 0, 0)\leqslant -c_0<0 \tag{11}$$

*for* $(x, y)\in\overline{\Omega}_2$ *and* $u\in\mathbf{R}$,

2) $|f(x, y, u, p, q)|\leqslant H(p^2+q^2+1)$,

$$|f_x(x, y, u, p, q)|,\ |f_y(x, y, u, p, q)|\leqslant H(p^2+q^2+1),$$
$$|f_u(x, y, u, p, q)|\leqslant H(p^2+q^2+1), \tag{12}$$
$$|f_p(x, y, u, p, q)|,\ |f_q(x, y, u, p, q)|\leqslant H(\sqrt{p^2+q^2}+1),$$

*for* $(x, y)\in\overline{\Omega}_2$, $0\leqslant u\leqslant M+1$, $-\infty<p,\ q<+\infty$, *where*

$$M=\max\{\max_{(x,y)\in\partial\Omega_2}\varphi_2(x, y),\ \max_{(x,y)\in\overline{\Omega}_2} f(x, y, 0, 0, 0)/c_0\},$$

*H is a positive constant,*

3) $K(u)\in C[0, \infty)\cap C^{1,\alpha}(0, \infty)$, (13)

4) $\Gamma_2$ *satisfies the out-ball condition,*

*the problem* (8), (10) *has a classical solution which is positive in* $\Omega_2$.

*Proof* Consider the equation

$$L_\varepsilon[u]\equiv K(u+\varepsilon)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0. \tag{14}$$

Define a fnnction $K_\varepsilon(u)\in C^{1,\alpha}(-\infty, +\infty)$ suth that

$$\frac{1}{2}\min_{\varepsilon\leqslant u\leqslant M+\varepsilon} K(u)\leqslant K_\varepsilon(u)\leqslant 2\max_{\varepsilon\leqslant u\leqslant M+\varepsilon} K(u),$$
$$K_\varepsilon(u)=K(u), \quad \varepsilon\leqslant u\leqslant M+\varepsilon.$$

According to an existence theorem[7] for quasilinear elliptic equations we know that the equation

$$K_\varepsilon(u+\varepsilon)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f\left(x,\ y,\ u,\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\right)=0 \tag{15}$$

with boundary condition (10) has a solution $u_\varepsilon(y, x)\in C(\bar{\Omega}_2)\cap C^{2,\alpha}(\Omega_2)$. Write (15) in the form

$$K_\varepsilon(u+\varepsilon)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\int_0^1 f_q\left(x,\ y,\ u,\ \frac{\partial u}{\partial y},\ \tau\frac{\partial u}{\partial y}\right)d\tau\cdot\frac{\partial u}{\partial y}$$
$$+\int_0^1 f_p\left(x,\ y,\ u,\ \tau\frac{\partial u}{\partial x},\ 0\right)d\tau\cdot\frac{\partial u}{\partial x}+\int_0^1 f_u(x,\ y,\ \tau u,\ 0,\ 0)d\tau\cdot u$$
$$=-f(x,\ y,\ 0,\ 0,\ 0).$$

By (11) and the maximum principle, we have

$$0\leqslant u_\varepsilon(x,\ y)\leqslant M,\quad (x,\ y)\in\bar{\Omega}_2. \tag{16}$$

Hence $u_\varepsilon(x, y)$ is also a solution of (14), (10).

Then we prove that for any sufficiently small $\delta>0$ there is a constant $\eta_0>0$, independent of $\varepsilon$ such that

$$u_\varepsilon(x,\ y)\geqslant\eta_0,\quad (x,\ y)\in\bar{\Omega}_2\cap\{2\delta\leqslant x\leqslant 1-2\delta,\ y\geqslant 2\delta\}. \tag{17}$$

Forthis purpose, we introduce

$$v(x,\ y)=\begin{cases}\int_{Kg(x,y)}^{Kg(x,\delta)} e^{-t^2}dt, & \delta<x<1-\delta,\\ 0, & x=\delta,\ 1-\delta,\end{cases}$$

where $g(x,\ y)=\dfrac{Y+1-y}{(x-\delta)(1-\delta-x)}$, $Y=\max\{y\,|\,(x,\ y)\in\bar{\Omega}_2\}$, and $K$ is a positive constant given later on. Set $S=\Omega_2\cap\{(x,\ y)\,|\,\delta<x<1-\delta,\ y>\delta\}$. Obviously $v(x, y)$ is continuous on $\bar{S}$ and it is easy to see that for $(x, y)\in S$ and $K>2$

$$\left|\frac{\partial v}{\partial x}\right|\leqslant\frac{1}{(x-\delta)(1-\delta-x)}kg(x,\ y)e^{-(Kg(x,y))^2},$$
$$\frac{\partial^2 v}{\partial x^2}\geqslant-\frac{2}{(x-\delta)(1-\delta-x)}kg(x,\ y)e^{-(Kg(x,y))^2}.$$

Consider

$$w(x,\ y)=u_\varepsilon(x,\ y)-\sigma v(x,\ y)\quad(0<\sigma<1).$$

Assuming that $w(x, y)$ has a negative minimum in $S$, we shall derive a contradiction. At the minimum point,

$$K(u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\geqslant 0,\quad \frac{\partial u_\varepsilon}{\partial x}=\sigma\frac{\partial v}{\partial x},\quad \frac{\partial u_\varepsilon}{\partial y}=\sigma\frac{\partial v}{\partial y}.$$

By (14), (11) and (12), at that point for $K>2$

$$K(u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-f(x,\ y,\ u_\varepsilon,\ \sigma v_x,\ \sigma v_y)-\sigma[K(u_\varepsilon+\varepsilon)v_{xx}+v_{yy}]$$
$$=-\sigma K(u_\varepsilon+\varepsilon)v_{xx}-\sigma v_{yy}-\sigma v_y\int_0^1 f_q(x,\ y,\ u_\varepsilon,\ \sigma v_y,\ \tau\sigma v_y)d\tau$$
$$-\sigma v_x\int_0^1 f_p(x,\ y,\ u_\varepsilon,\ \tau\sigma v_x,\ 0)d\tau-\sigma v\int_0^1 f_u(x,\ y,\ \tau u_\varepsilon,\ 0,\ 0)d\tau-f(x,\ y,\ 0,\ 0,\ 0).$$

$$\leqslant -\sigma[K(u_\varepsilon+\varepsilon)v_{xx}+v_{yy}-2H(v_x^2+v_y^2)-H(|v_x|+|v_y|)-Hv]$$

$$\leqslant -\sigma e^{-(Kg(x,y))^3}\Big\{\frac{3k^4(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}-K(u_\varepsilon+\varepsilon)\frac{2kg(x,y)}{(x-\delta)(1-\delta-x)}$$

$$-\frac{2Hk^2(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}-\frac{2Hk^2}{(x-\delta)^2(1-\delta-x)^2}-\frac{Hkg(x,y)}{(x-\delta)(1-\delta-x)}$$

$$-\frac{Hk^2}{(x-\delta)(1-\delta-x)}-Hve^{(k^3g(x,y))^3}\Big\}.$$

Since the function $e^{z^3}\int_z^{+\infty}e^{-t^3}dt\to 0$ (as $z\to+\infty$), there exists a constant

$$N=\sup_{0\leqslant z<+\infty}\left\{e^{z^3}\int_z^{+\infty}e^{-t^3}dt\right\}.$$

Thus, at the minimum point,

$$K(u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

$$\leqslant -\sigma e^{-(kg(x,y))^3}\Big\{\frac{3k^4(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}-\overline{A}\frac{2kg(x,y)}{(x-\delta)(1-\delta-x)}-\frac{2H(k^2g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}$$

$$-\frac{2Hk^2}{(x-\delta)^2(1-\delta-x)^2}-\frac{Hkg(x,y)}{(x-\delta)(1-\delta-x)}-\frac{Hk}{(x-\delta)(1-\delta-x)}-HN\Big\}$$

$$\leqslant -\sigma e^{-(Kg(x,y))^3}\Big\{\frac{kg(x,y)}{(x-\delta)(1-\delta-x)}\Big[\frac{k^3g(x,y)}{2(x-\delta)(1-\delta-x)}-2\overline{A}-H\Big]$$

$$+\frac{k^2(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}\Big[\frac{k^2}{2}-2H\Big]+\frac{k^2}{(x-\delta)^2(1-\delta-x)^2}\Big[\frac{k^2(g(x,y))^2}{2}-2H\Big]$$

$$+\frac{k}{(x-\delta)(1-\delta-x)}\Big[\frac{k^3(g(x,y))^2}{2(x-\delta)(1-\delta-x)}-H\Big]+\Big[\frac{k^4(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}-HN\Big]\Big\}$$

$$\leqslant -\sigma e^{-(kg(x,y))^3}\Big\{\frac{kg(x,y)}{(x-\delta)(1-\delta-x)}\Big[\frac{k^3}{2}-2\overline{A}-H\Big]$$

$$+\frac{k^2(g(x,y))^2}{(x-\delta)^2(1-\delta-x)^2}\Big[\frac{k^2}{2}-2H\Big]+\frac{k^2}{(x-\delta)^2(1-\delta-x)^2}\Big[\frac{k^2}{2}-2H\Big]$$

$$+\frac{k}{(x-\delta)(1-\delta-x)}\Big[\frac{k^3}{2}-H\Big]+[k^4-HN]\Big\}.$$

where $\overline{A}=\max\limits_{0\leqslant u\leqslant M+1}K(\varepsilon)$, take $k$ so large that the terms in the brace on the right-hand side are positive, it follows that $K(u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}<0$, we get a contradiction. Hence $w(x,y)$ cannot have a negative minimum is $S$. Since

$$w(x,y)=u_\varepsilon(x,y)-\sigma\cdot 0\geqslant 0,\quad (x,y)\in\partial S\cap\{x=\delta,\ 1-\delta,\ \text{or } y=\delta\}$$

and

$$w(x,y)=\varphi_2(x,y)-\sigma v(x,y)\geqslant\min_{(x,y)\in\Gamma_2\cap(\delta\leqslant x\leqslant 1-\delta)}\varphi_2(x,y)-\sigma\int_0^{+\infty}e^{-t^3}dt,$$

$$(x,y)\in\partial S\cap\Gamma_2,$$

taking $\sigma$ small enough, $w(x,y)$ is not negative on $\partial S$. It follows that $w(x,y)\geqslant 0$ on $\overline{S}$, i. e.,

$$u_\varepsilon(x,y)\geqslant\sigma v(x,y),\quad (x,y)\in\overline{S}.$$

By putting $\eta_0=\sigma v(2\delta,2\delta)$, (17) is established.

Set $\varepsilon=\frac{1}{n}$ $(n=1,2,\cdots)$ and denote the corresponding solution $u_\varepsilon(x,y)$ by $u_n(x,y)$.

Using the interior estimate of the solutions for quasilinear elliptic equations and the Schauder's interior estimate, we know that the $C^{2,\alpha}$-norms of $u_n(x, y)$, $(n=1, 2, \cdots)$ are uniformly bounded in any closed subdomain of $\Omega_2$. Then a subsequence of $\{u_n(x, y)\}$ can be selected such that it converges in any closed subdomain of $\Omega_2$ in $C^2$-norm sense. Without loss of generality we may assume that the subsequence is $\{u_n(x, y)\}$ itself. Denote its limit function by $u(x, y)$. Clearly $u(x, y)\in C^2(\Omega_2)$. Since $\underset{\overline{n}}{L_1}[u_n]=0$, it follows that

$$K(u)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0, \quad (x, y)\in\Omega_2.$$

From (16) and (17),

$$0<u(x, y)\leqslant M, \quad (x, y)\in\Omega_2.$$

In view of (13) and (17), using a well-known barrier, it is easy to show that $u(x, y)$ is continuous up to $\Gamma_2$ and satisfies boundary condition (10) on $\Gamma_2$. It remains to prove that $u(x, y)$ satisfies (10) on $\partial\Omega_2\backslash\Gamma_2$. To do so, we shall construct a continuous function $\widetilde{P}(x, y)$ such that

$$\widetilde{P}(x_0, y_0)=0, \tag{18}$$

$$\widetilde{P}(x, y)\geqslant u_n(x, y), \quad (x, y)\in T=\Omega_2\cap\{0<r<\delta_0\}, \tag{19}$$

where $(x, y)$ is a point on $\partial\Omega_2\backslash\Gamma_2$, $\delta_0$ is a positive constant,

$$r=\sqrt{(x-\bar{x})^2+(y-\bar{y})^2}-R,$$

$R$ is the redius of an out-ball which is tangent to $\partial\Omega_2$ at $(x_0, y_0)$ and centred at $(\bar{x}, \bar{y})$.

Thus, by (19) and (16), we have

$$\widetilde{P}(x, y)\geqslant u(x, y)>0, \quad (x, y)\in T.$$

From the continuity of $\widetilde{P}(x, y)$ and (18), $\lim\limits_{(x,y)\to(x_0,y_0)} u(x, y)=0$, namely, $u(x, y)$ satisfies (10) on $\partial\Omega_2\backslash\Gamma_2$.

Now let

$$\widetilde{P}(x, y)=P(r),$$

the function $P(z)$ is constructed as follows.

Define

$$l(z)=\max_{0\leqslant r\leqslant z}\varphi_2(x, y)+z, \quad z\geqslant 0,$$

$$h_0(z)=\begin{cases} l^{-1}(z), & z\geqslant M+2,\\ \inf\{y\,|\,(z, y)\in U^*\}, & 0\leqslant z\leqslant M+2,\\ 0, & z<0,\end{cases}$$

where $l^{-1}(z)$ is the inverse function of $l(z)$. $U=\{(z, y)\,|\,0\leqslant z\leqslant M+2,\ h(z)\leqslant y\leqslant h(M+2)\}$, $U^*$ denote the convex closure of $U$ in $\mathbf{R}^2$.

Next, define

$$h_1(z)=\int_{-\infty}^{+\infty} J_\alpha(z-t)h_0(t-\alpha)\,dt,$$

where $J_\alpha(t)$ is a mollifier with the radius $\alpha$.

Let
$$\overline{K}(u)=\begin{cases}\min\limits_{u\leqslant v\leqslant M+1}[K(v),\ 1], & 0\leqslant u\leqslant M+1,\\ \min[K(M+1),\ 1], & u>M+1,\end{cases}$$

and
$$B=H[h_1'(M+1)]^2+\frac{V}{R}h_1'(M+1)+H+1,$$

where
$$\tilde{\Lambda}=\max_{0\leqslant u\leqslant M+1}[K(u),\ 1].$$

Further define
$$\zeta(z)=\max\left[\frac{B}{\overline{K}(z)},\ \frac{d}{dz}(\ln h_1'(z))\right],\quad 0<z\leqslant M+1,$$
$$\chi(z)=\begin{cases}h_1'(M+1)e^{\int_{M+1}^{z}\zeta(t)dt}, & 0<z\leqslant M+1,\\ 0, & z=0,\end{cases}$$

and
$$h_2(z)=\int_0^z\psi(t)\,dt,\quad 0\leqslant z\leqslant M+1.$$

Finally, let $P(z)$ be the inverse function of $h_2(z)$, namely,
$$P(z)=h_2^{-1}(z),\quad 0\leqslant z\leqslant\delta_0,$$
where
$$\delta_0=h_2(M+1)>0.$$

It is not difficult to verify (see [4]) that
$$P(0)=0,\quad P(\delta_0)=M+1, \tag{20}$$
$$P(\sqrt{(x-\bar{x})^2+(y-\bar{y})^2}-R)\geqslant\varphi_2(x,\ y),\quad (x,\ y)\in\partial T\cap\partial\Omega_2, \tag{21}$$
$$P'(z)>0;\quad P''(z)<0,\quad 0<z<\delta_0. \tag{22}$$

Clearly the function $\tilde{P}(x,\ y)=P(r)$ satisfies (18), it remains to prove (19). Introduce
$$w(x,\ y)=\tilde{P}(x,\ y)-u_n(x,\ y).$$

If $w(x,\ y)$ has a negative minimum in $T$, then
$$\tilde{P}(x,\ y)\leqslant u_n(x,\ y),\quad K\left(u_n+\frac{1}{n}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\geqslant 0,$$
$$\frac{\partial u_n}{\partial x}=\frac{\partial\tilde{P}}{\partial x}=P'(r)\frac{x-\bar{x}}{\sqrt{(x-\bar{x})^2+(y-\bar{y})^2}},\quad \frac{\partial u_n}{\partial y}=\frac{\partial\tilde{P}}{\partial y}=P'(r)\frac{y-\bar{y}}{\sqrt{(x-\bar{x})^2+(y-\bar{y})^2}},$$

at the minimum point. By $L_{\frac{1}{n}}[u_n]=0$, (12), (22) and the construction of $P$, at that point,
$$K\left(u_n+\frac{1}{n}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=K\left(u_n+\frac{1}{n}\right)\frac{\partial^2\tilde{P}}{\partial x^2}+\frac{\partial^2\tilde{P}}{\partial y^2}+f\left(x,\ y,\ u_n,\ \frac{\partial u_n}{\partial x},\ \frac{\partial u_n}{\partial y}\right)$$
$$=K\left(u_n+\frac{1}{n}\right)\left[P''(r)\frac{(x-\bar{x})^2}{(x-\bar{x})^2+(y-\bar{y})^2}+P'(r)\frac{(y-\bar{y})^2}{(\sqrt{(x-\bar{x})^2+(y-\bar{y})^2})^3}\right]$$
$$+\left[P''(r)\frac{(y-\bar{y})^2}{(x-\bar{x})^2+(y-\bar{y})^2}+P'(r)\frac{(x-\bar{x})^2}{(\sqrt{(x-\bar{x})^2+(y-\bar{y})^2})^3}\right]$$
$$+f\left(x,\ y,\ u_n,\ \frac{\partial u_n}{\partial x},\ \frac{\partial u_n}{\partial y}\right)$$

$$\leqslant \min\left[K\left(u_n+\frac{1}{n}\right),\ 1\right]P''(r)+\max\left[K\left(u_n+\frac{1}{n}\right),\ 1\right]\frac{P'(r)}{\sqrt{(x-\bar{x})^2+(y-\bar{y})^2}}$$
$$+H[P'(r)^2+1]$$
$$\leqslant -\frac{\overline{K}(P(r))\zeta(P(r))}{\psi^2(P(r))}+\frac{\widetilde{A}}{R\psi(P(r))}+\frac{H}{\psi^2(P(r))}+H$$
$$\leqslant -\frac{1}{\psi'(P(r))}\left[B-H-\frac{\widetilde{A}}{R}h_1'(M+1)\right]+H$$
$$\leqslant -\frac{1}{[h_1'(M+1)]^2}\{H[h_1'(M+1)]^2+1\}+H<0,$$

then we get a contradiction. Hence $w(x, y)$ cannot have negative minimum in $T$. In view of (21) and (20), we have

$$w(x, y)=\widetilde{P}(x, y)-\varphi_2(x, y)\geqslant 0, \quad (x, y)\in\partial T\cap\{0\leqslant r<\delta_0\}$$

and

$$w(x, y)=P(\delta_0)-u_n(x, y)\geqslant 0, \quad (x, y)\in\partial T\cap\{r=\delta_0\},$$

namely, $w(x, y)\geqslant 0$ on $\partial T$. Thus $w(x, y)\geqslant 0$ on $\overline{T}$ and (19) follows.

However we cannot obtain the corresponding theorem for problem (9), (10). Applying the maximum principle for linear degenerate equations[5, 6], we know that the classical solution of (9), (10) is positive in $\Omega_2\cap\{y>h\}$. But it may vanish in $\Omega_2\cap\{0<y<h\}$, so that equation (9) may be degenerate in the interior of $\Omega_2$. For example, we consider the equation

$$\frac{\partial^2 u}{\partial x^2}+u^\alpha\frac{\partial^2 u}{\partial y^2}+a\frac{\partial u}{\partial x}-\frac{1}{\alpha}\frac{\partial u}{\partial y}=f(x, y) \quad \left(0<\alpha\leqslant\frac{1}{2}\right), \tag{23}$$

where $a$, $\alpha$ are constants and

$$f(x, y)=\begin{cases}0, & 0<y\leqslant 1,\\ -\dfrac{1}{\alpha}(y-1)^{\frac{1}{\alpha}-1}, & 1\leqslant y<1+\delta \quad (\delta>0).\end{cases}$$

Let $\Omega_2=\{(x, y)\,|\,0<x<1,\ 0<y<1+\delta\}$ and $\Gamma_2=\partial\Omega_2\cap\{1<y\leqslant 1+\delta\}$. Under the condition

$$u|_{\partial\Omega_2}=\varphi_2(x, y)=\begin{cases}0, & 0\leqslant y\leqslant 1,\\ (y-1)^{\frac{1}{\alpha}}, & 1\leqslant y\leqslant 1+\delta,\end{cases}$$

it is easy to see that equation (23) has the solution

$$u(x, y)=\begin{cases}0, & 0\leqslant y\leqslant 1,\\ (y-1)^{\frac{1}{\alpha}}, & 1\leqslant y\leqslant 1+\delta.\end{cases}$$

Now we give an estimate about the support of the solution of the equation

$$L(u)\equiv\frac{\partial^2 u}{\partial x^2}+K(u)\frac{\partial^2 u}{\partial y^2}+a(x, y)\frac{\partial u}{\partial x}+b(x, y)\frac{\partial u}{\partial y}+c(x, y)u=f(x, y) \tag{24}$$

with condition (10).

**Theorem 2.** *If*

1) $a(x, y),\ b(x, y),\ c(x, y),\ f(x, y)\in C(\overline{\Omega}_2)\cap C^1(\Omega_2)$,

$$c(x, y) \leqslant -c_0 < 0,\ b(x, y) \leqslant 0,\ f(x, y) \leqslant 0,\ (x, y) \in \Omega_2, \\ f(x, y) = 0,\ (x, y) \in \Omega_2 \cap \{0 < y < h\}, \tag{25}$$

2) *$K(u) \in C[0, \infty) \cap C^{1,\alpha}(0, \infty)$ satisfies (6) and the Lipschitz condition*

$$|K(u_1) - K(u_2)| \leqslant Q|u_1 - u_2|, \quad u_1, u_2 \in [0, M+1], \tag{26}$$

*where $Q = const.$, $M = \max\left[\max_{\overline{\Omega}_2} \frac{|f(x, y)|}{c_0}, \max_{\partial\Omega_2} \varphi_2(x, y)\right]$,*

3) *$h > h_0 = \sqrt{\frac{2MQ}{c_0}}$,*

4) *$u(x, y)$ is a classical solution of (24), (10),*

*then*

$$\operatorname{supp}\{u\} \subset \overline{\Omega}_2 \cap \{y \geqslant h - h_0\}.$$

*Proof* Applying the maximum principle, we have

$$0 \leqslant u(x, y) \leqslant M. \tag{27}$$

Introduce

$$v(y) = \begin{cases} \frac{c_0}{Q}[y - (h - h_0)]^2, & y \geqslant h - h_0, \\ 0, & y \leqslant h - h_0. \end{cases}$$

Obviously, $v'(y) \geqslant 0$ and $0 \leqslant v''(y) \leqslant \frac{c_0}{Q}$ $(y \neq h - h_0)$.

Let

$$v_n(y) = \int_{-\infty}^{+\infty} v(y - z) J_{\frac{1}{n}}(z)\, dz,$$

where $J_{\frac{1}{n}}(t)$ is a mollifier with the radius $\frac{1}{n}$ and satisfiies $J_{\frac{1}{n}}(-t) = J_{\frac{1}{n}}(t)$. Thus, we have

$$v_n(y) \in C^\infty(-\infty, +\infty), \quad v_n(y) \xrightarrow{1} v(y) \quad (n \to +\infty,\ 0 \leqslant y \leqslant h),$$

$$v_n(y) \geqslant v(y), \quad v_n'(y) \geqslant 0, \quad v_n''(y) \leqslant \frac{c_0}{Q}. \tag{28}$$

Next, consider the liner operator

$$\tilde{L}[w] \equiv \frac{\partial^2 w}{\partial x^2} + K(u)\frac{\partial^2 w}{\partial y^2} + a(x, y)\frac{\partial w}{\partial x} + b(x, y)\frac{\partial w}{\partial y} + \tilde{c}(x, y)w,$$

where

$$\tilde{c}(x, y) = c(x, y) + \frac{K(u) - K(v_n)}{u - v_n} v_n''(y) \leqslant -c_0 + Q \cdot \frac{c_0}{Q} = 0.$$

Let $\quad w(x, y) = v_n(y) - u(x, y), \quad (x, y) \in T = \Omega_2 \cap \{0 < y < h\}.$

By (24), (25), (26), (28) and (10), it is not difficult to show that $\tilde{L}[w] \leqslant 0$ in $T$ and $w(x, y) \geqslant 0$ on $\partial T$. Applying the maximum principle for degenerate elliptic equation we get $w(x, y) \geqslant 0$ on $\overline{T}$. i. e.,

$$v_n(y) \geqslant u(x, y), \quad (x, y) \in \overline{T}.$$

By letting $n \to +\infty$ and (25),

$$v(y) \geqslant u(x, y) \geqslant 0, \quad (x, y) \in \overline{T}.$$

Hence

$$u(x, y) = 0, \quad (x, y) \in \overline{\Omega}_2 \cap \{0 \leqslant y \leqslant h - h_0\}.$$

## §. 2

In this section we consider equation (1), i. e.,

$$L[u]=0,$$

where

$$L[u]\equiv a_{11}(x, y, u)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x, y, u)\frac{\partial^2 u}{\partial x\partial y}+a_{22}(x, y, u)\frac{\partial^2 u}{\partial y^2}$$
$$+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

Let $\Omega$ be a bounded domain in $\mathbf{R}^2$, $(x_0, y_0)$ is any point on $\partial\Omega$. Given the continuous boundary condition

$$u(x, y)|_{\partial\Omega}=\varphi(x, y), \tag{29}$$

where $\varphi(x, y)$ satisfies

$$\varphi(x_0, y_0)=0; \quad \varphi(x, y)>0, \quad (x, y)\in\partial\Omega\backslash\{(x_0), y_0)\},$$

we have

**Theorem 3.** *If $a_{ij}(x, y, u)\in C(\bar\Omega\times[0, \infty))\cap C^{1,\alpha}(\Omega\times(0, \infty))$ $(i, j=1, 2)$ satisfy (2) and (3), $f(x, y, u, p, q)$ satisfies the assumptions of Theorem 1 ($\Omega_2$ replaced by $\Omega$), $\partial\Omega$ satisfies the out-ball candition, moreover, if*

*1) there are constants $A$, $B>0$ such that*

$$A^2a_{11}(x, y, 0)\neq B^2a_{22}(x, y, 0), \quad (x, y)\in\bar\Omega, \tag{30}$$

*or 2)* $a_{12}(x, y, 0)\geqslant 0,\ (x, y)\in\bar\Omega,$ (31)

*or 3)* $a_{12}(x, y, 0)\leqslant 0,\ (x, y)\in\bar\Omega,$ (32)

*then problem (1), (29) has a classical solution which is positive in $\Omega$.*

*Proof* Without loss of generality, we may assume $(x_0, y_0)=(0, 0)$. As in the proof of Theorem 1, consider

$$L_\varepsilon[u]\equiv a_{11}(x, y, u+\varepsilon)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x, y, u+\varepsilon)\frac{\partial^2 u}{\partial x\partial y}+a_{22}(x, y, u+\varepsilon)\frac{\partial^2 u}{\partial y^2}$$
$$+f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)=0.$$

It is easy to prove that there is $u_\varepsilon(x, y)\in C(\bar\Omega)\cap C^{2,\alpha}(\Omega)$ which satisfies

$$L_\varepsilon[u_\varepsilon]=0, \quad u_\varepsilon(x, y)|_{\partial\Omega}=\varphi(x, y), \quad 0\leqslant u_\varepsilon(x, y)\leqslant M.$$

Now we show that $\{u_\varepsilon(x, y)\}$ has a uniform positive lower bound in any subdomain of $\Omega$.

(i) Assume that (30) holds.

Without loss of generality, we may suppose $AB\neq 0$. For any $\delta>0$, we introduce

$$v(x, y)=\begin{cases}\dfrac{\sqrt{\pi}}{2}-\displaystyle\int_0^{\frac{K}{Ax+By-\delta}}e^{-t^2}dt, & Ax+By>\delta,\\ 0, & Ax+By=\delta.\end{cases}$$

Consider the linear operator

$$A_\varepsilon[w] \equiv a_{11}(x, y, u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x^2}+2a_{12}(x, y, u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial x\partial y}+a_{22}(x, y, u_\varepsilon+\varepsilon)\frac{\partial^2 w}{\partial y^2}.$$

For $(x, y)\in S=\Omega\cap\{Ax+By>\delta\}$ and sufficiently large $K$, we have

$$\begin{aligned} A_\varepsilon[v] &= e^{-\frac{k^2}{(Ax+By-\delta)^2}}\Big[\frac{2k^3}{(Ax+By-\delta)^5}-\frac{2k}{(Ax+By-\delta)^3}\Big] \\ &\quad\times[a_{11}(x, y, u_\varepsilon+\varepsilon)A^2+2a_{12}(x, y, u_\varepsilon+\varepsilon)AB+a_{22}(x, y, u_\varepsilon+\varepsilon)B^2] \\ &\geqslant e^{-\frac{k^2}{(Ax+By-\delta)^2}}\cdot\frac{k^3}{(Ax+By-\delta)^5}\cdot I(x, y, u_\varepsilon+\varepsilon), \end{aligned}$$

where $\quad I(x, y, u)=a_{11}(x, y, u)A^2+2a_{12}(x, y, u)AB+a_{22}(x, y, u)B^2.$

By (3),

$$a_{12}^2(x, y, 0)=a_{11}(x, y, 0)\cdot a_{12}(x, y, 0);\quad a_{11}^2(x, y, 0)+a_{22}^2(x, y, 0)\neq 0,$$

it implies, from (30),

$$I(x, y, 0)>0,\quad (x, y)\in\overline{\Omega}.$$

By the continuity of $I(x, y, u)$, there are $I_0>0$ and $\beta>0$ such that

$$I(x, y, u)\geqslant I_0,\quad (x, y)\in\overline{\Omega},\ 0\leqslant u\leqslant\beta.$$

In view of (2) and (3)

$$I(x, y, u)\geqslant\lambda(x, y, u)(A^2+B^2)\geqslant K_0,\quad (x, y)\in\overline{\Omega},\ \beta\leqslant u\leqslant M+1,$$

where $K_0=\min\{I_0, \min\limits_{\substack{(x,y)\in\overline{\Omega}\\ \beta\leqslant u\leqslant M+1}}\lambda(x, y, u)(A^2+B^2)]>0.$

Thus

$$I(x, y, u)\geqslant K_0,\quad (x, y)\in\overline{\Omega},\ 0\leqslant u\leqslant M+1. \tag{33}$$

Consider

$$w(x, y)=u_\varepsilon(x, y)-\sigma v(x, y)\quad (0<\sigma<1).$$

If $w(x, y)$ takes a negative minimum in $S$, then, at the minimum point

$$A_\varepsilon[w]\geqslant 0;\quad \frac{\partial u_\varepsilon}{\partial x}=\sigma\frac{\partial v}{\partial x},\quad \frac{\partial u_\varepsilon}{\partial y}=\sigma\frac{\partial v}{\partial y}.$$

By $L_\varepsilon[u_\varepsilon]=0$, (11), (12) and (33), at that point,

$$\begin{aligned} A_\varepsilon[w] &= -f\Big(x, x, u_\varepsilon, \frac{\partial u_\varepsilon}{\partial x}, \frac{\partial u_\varepsilon}{\partial y}\Big)-\sigma A_\varepsilon[v] \\ &\leqslant -\sigma\{A_\varepsilon[v]-2H(v_x^2+v_y^2)-H(|v_x|+|v_y|)-Hv\} \\ &\leqslant -\sigma e^{-\frac{k^2}{(Ax+By-\delta)^2}}\Big\{K_0\frac{k^3}{(Ax+By-\delta)^5}-2H\frac{k^2(A^2+B^2)}{(Ax+By-\delta)^4} \\ &\quad -H\frac{k(A+B)}{(Ax+By-\delta)^2}-Hve^{\frac{k^2}{(Ax+By-\delta)^2}}\Big\}. \end{aligned}$$

Taking $K$ so large that the sum of the terms in the brace on the right-hand side is positive, we get a contradiction. Hence $w(x, y)$ cannot take a negative minimum in $S$. Furthermore, take $\sigma$ so small that $w(x, y)$ is non-negative on $\partial S$, then it follows that $w(x, y)\geqslant 0$ on $\overline{S}$.

Thus

$$u_\varepsilon(x, y)\geqslant\sigma v(x, y)\geqslant\sigma\Big(\frac{\sqrt{\pi}}{2}-\int_0^{\frac{k}{\delta}}e^{-t^2}dt\Big)>0,\quad (x, y)\in\overline{\Omega}\cap\{Ax+By\geqslant 2\delta\}.$$

In a similar way, we can prove that $\{u_s(x, y)\}$ has a positive lower bound on

$$\bar{\Omega}\cap\{Ax+By\leqslant-2\delta\},\quad \bar{\Omega}\cap\{Ax-By\geqslant2\delta\}$$

and $\bar{\Omega}\cap\{Ax-By\leqslant-2\delta\}$. Thus there is a positive constant $\eta_0$ indenpending of $s$ such that

$$u_s(x, y)\geqslant\eta_0,\quad (x, y)\in\bar{\Omega}\cap\{|Ax+By|\geqslant2\delta \text{ or } |Ax-By|\geqslant2\delta\}.$$

(ii) Assume that (31) or (32) holds.

For any $\delta>0$, we take $A_1\neq0$ such that

$$-\frac{\delta}{3}<A_1x<\frac{\delta}{3},\quad A_1a_{12}(x, y, 0)\geqslant0,\quad (x, y)\in\bar{\Omega}.$$

Introduce

$$v(x, y)=\begin{cases}\dfrac{\sqrt{\pi}}{2}-\displaystyle\int_0^{\frac{k}{A_1x+y-\delta/2}}e^{-t^2}dt, & A_1x+y>\dfrac{\delta}{2},\\ 0, & A_1x+y=\dfrac{\delta}{2}.\end{cases}$$

By taking $K$ sufficiently large and $\sigma>0$ sufficiently small, it is not difficult to verify that $u_s(x, y)\geqslant\sigma v(x, y)$ on $\bar{\Omega}\cap\left\{A_1x+y-\frac{\delta}{2}\geqslant0\right\}$, then it follows that

$$u_s(x, y)\geqslant\sigma\left(\frac{\sqrt{\pi}}{2}-\int_0^{\frac{k}{\delta}}e^{-t^2}dt\right)>0,\quad (x, y)\in\bar{\Omega}\cap\{y\geqslant2\delta\}.$$

Similarly we can get this estimate on $\bar{\Omega}\cap\{y\leqslant-2\delta\}$ and $\bar{\Omega}\cap\{|x|\geqslant2\delta\}$, hence

$$u_s(x, y)\geqslant\tilde{\eta}_0>0,\quad (x, y)\in\bar{\Omega}\cap\{|x|\geqslant2\delta \text{ or } |y|\geqslant2\delta\}.$$

By the above assertion, there is a sequence $\{u_n(x, y)\}$ which converges to $u(x, y)\in C^2(\Omega)$, so that

$$L[u]=0,\quad 0<u(x, y)\leqslant M,\quad (x, y)\in\Omega.$$

To complete the proof of Theorem 3, it suffices to show that $u(x, y)$ satisfies (29) at $(0, 0)$. Clearly we can apply the barrier $\tilde{P}(x, y)$ used in the proof of Theorem 1, only with the modification: in the expression of $\zeta(z)$

$$\bar{K}(u)=\begin{cases}\min\limits_{(x,y)\in\bar{\Omega},u\leqslant v\leqslant M+1}\lambda(x, y, v),\\ \min\limits_{(x,y)\in\bar{\Omega}}\lambda(x, y, M+1),\end{cases}$$

and in the expression of $B$

$$\tilde{\Lambda}=\max_{(x,y)\in\bar{\Omega},0\leqslant u\leqslant M+1}\Lambda(x, y, u).$$

Since equation (1) is degenerate when $u=0$, it is proper to impose the restrictions on $a_{ij}(x, y, 0)$ in Theorem 3. According to the maximum principle for linear degenerate elliptic equations, if the minmum of the solution of degenerate equation is attained at an inner point of the domain, it must propagate along some trajectories, namely, the minimum is attained everywhere on these trajectories, a class of which are determined by the row vectors of the second order coefficient matrix of the equation. The tangent directions of the trajectories coincide with the above vectors. Hence, if the solution $u$ of equation (1) with condition (29) attains the minimum 0 at a point in $\Omega$, then $u=0$ on the trajectory passing through the minimum point, so that the tangent direction of this trajectory is $(a_{11}(x, y, 0), a_{12}(x, y, 0))$ or $(a_{12}(x, y, 0), a_{22}(x, y, 0))$. By (3), we know this trajectory may be extended to the boundary $\partial\Omega$. In view of (29), either of the endpoints of the trajectory must fall at $(x_0, y_0)$. By (30) or (31) or (32), we stipulate the tangent direction so that there exists no above-mentioned trajectory. Thus, it is impossible that $u$ takes the minimum 0 in the interior of $\Omega$. Now we take the assumption (30) as an example. Obviously, (30) can be rewritten as

$$a_{11}^2(x, y, 0)A^2 \neq a_{12}^2(x, y, 0)B^2, \quad (\text{if } a_{11}(x, y, 0)\neq 0),$$

or

$$a_{12}^2(x, y, 0)A^2 \neq a_{22}^2(x, y, 0)B^2, \quad (\text{if } a_{22}(x, y, 0)\neq 0).$$

Namely, the vector $(a_{11}(x, y, 0), a_{12}(x, y, 0))$, or $(a_{12}(x, y, 0), a_{22}(x, y, 0))$, cannot take the directions $\pm(B, \pm A)$. But the tangent direction along a smooth curve whose two endpoints fall at a single point must change continuously over 180° and must take one of the directions $\pm(B, \pm A)$. Thus, such kind of trajectory (on which $u=0$) cannot exist. For the same reason, we need analogous assumptions in the following Theorems.

**Theorem 4.** *If* $a_{ij}(x, y, u)\in C(\bar{\Omega}_1\times[0, +\infty))\cap C^{1,\alpha}(\Omega_1\times(0, +\infty))$ $(i, j=1, 2)$ *satisfy* (2) *and* (3) ($\Omega$ *replaced by* $\Omega_1$), $f(x, y, u, p, q)$ *satisfy the assumptions of Theorem* 3 ($\Omega$ *replaced by* $\Omega_1$), $\Gamma_1$ *satisfies the out-ball condition, moreover, if*

1) $a_{22}(x, y, 0)\neq 0$, $(x, y)\in\bar{\Omega}_1$, (34)

*or* 2) $a_{12}(x, y, 0)\geqslant 0$, $(x, y)\in\bar{\Omega}_1$, (35)

*or* 3) $a_{12}(x, y, 0)\leqslant 0$, $(x, y)\in\bar{\Omega}_1$, (36)

*then problem* (1), (7) *has a classical solution which is positive in* $\Omega_1$.

*Proof* The proof is similar to that of Theorem 3 and it is sufficient to show that the solutions $u_\varepsilon(x, y)$ of the equations $L_\varepsilon[u]=0$ with condition (7) have a uniform positive lower bound in any subdomain of $\Omega_1$. For any $\delta>0$, we take a constant $A_1\neq 0$ such that

$$-\frac{\delta}{3}<A_1x<\frac{\delta}{3}, \quad (x, y)\in\bar{\Omega}_1$$

and

$$A_0^2a_{11}(x, y, 0)\neq a_{22}(x, y, 0), \quad (x, y)\in\bar{\Omega}_1, \quad \text{if (34) holds;}$$

$$A_1a_{12}(x, y, 0)\geqslant 0, \quad (x, y)\in\bar{\Omega}_1, \quad \text{if (35) or (36) holds.}$$

Introduce

$$v(x, y)=\begin{cases}\dfrac{\sqrt{\pi}}{2}-\displaystyle\int_0^{\frac{k}{A_1x+y-\delta/2}} e^{-t^2}dt, & A_1x+y>\dfrac{\delta}{2},\\ 0, & A_1x+y=\dfrac{\delta}{2}.\end{cases}$$

Set $S=\Omega_1\cap\left\{A_1x+y>\frac{\delta}{2}\right\}$. It is not difficult to verify that, by taking $K$ sufficiently large and $\sigma>0$ sufficiently small,

$$u_\varepsilon(x,\ y)\geqslant\sigma v(x,\ y),\quad (x,\ y)\in\bar{S}.$$

Then

$$u_\varepsilon(x,\ y)\geqslant\eta_0=\sigma\left(\frac{\sqrt{\pi}}{2}-\int_0^{\frac{K}{\delta}}e^{-t^2}dt\right)>0,\quad (x,\ y)\in\bar{\Omega}_1\cap\{y\geqslant2\delta\}.$$

At last let $\Omega_3$ be a domain bounded by the straightlines $x=0$, $y=0$ and the open curve $\Gamma_3$ connecting the point $(0,\ 1)$ and the point $(1,\ 0)$. Given the boundary condition

$$u(x,\ y)\,|_{\partial\Omega_3}=\varphi_3(x,\ y),\tag{37}$$

where $\varphi_3(x,\ y)$ is continuous on $\partial\Omega_3$ with

$$\varphi_3(x,\ 0)=\varphi_3(0,\ y)=0,\quad \varphi_3(x,\ y)>0,\ (x,\ y)\in\Gamma_3,$$

we have the following

**Theorem 5.** *If* $a_{ij}(x,\ y,\ u)\in C(\bar{\Omega}_3\times[0,\ +\infty))\cap C^{1,\alpha}(\Omega_3\times(0,\ +\infty))$, $(i,\ j=1,\ 2)$ *satisfy* (2) *and* (3) ($\Omega$ *replaced by* $\Omega_3$), $f(x,\ y,\ u,\ p,\ q)$ *satisfy the assumptions of Theorem* 3 ($\Omega$ *replaced by* $\Omega_3$), $\Gamma_3$ *satisfies the out-ball condition, moreover, if*

$$a_{12}(x,\ y,\ 0)\geqslant0,\quad (x,\ y)\in\bar{\Omega}_3,\tag{38}$$

*then problem* (1), (37) *has a classical solution which is positive in* $\Omega_3$.

*Proof* The proof is similar to that of Theorem 3. It suffices to show that the solutions $u_\varepsilon(x,\ y)$ of corresponding equation $L_\varepsilon[u]=0$ with boundary condition (37) have a uniform positive lower bound in any subdomain of $\Omega_3$. For any $\delta>0$, we choose positive constants $A$, $B(<1)$ such that

$$0\leqslant Ax<\frac{\delta}{3},\quad 0\leqslant By<\frac{\delta}{3},\quad (x,\ y)\in\bar{\Omega}_3.$$

Introduce

$$v(x,\ y)=\begin{cases}\dfrac{\sqrt{\pi}}{2}-\displaystyle\int_0^{\frac{k}{(x+By-\delta/2)(Ax+y-\delta/2)}}e^{-t^2}dt, & \left\{x+By>\frac{\delta}{2}\right\}\cap\left\{Ax+y>\frac{\delta}{2}\right\},\\ 0 & \left\{x+By=\frac{\delta}{2}\right\}\text{ or }\left\{Ax+y=\frac{\delta}{2}\right\}.\end{cases}$$

Set $S=\Omega_3\cap\left\{x+By>\frac{\delta}{2}\right\}\cap\left\{Ax+y>\frac{\delta}{2}\right\}$. It is easy to see that

$$A_\varepsilon[v]\geqslant e^{-\frac{k^2}{\left(x+By-\frac{\delta}{2}\right)^2\left(Ax+y-\frac{\delta}{2}\right)^2}}\left\{\frac{2k^3}{\left(x+By-\frac{\delta}{2}\right)^3\left(Ax+y-\frac{\delta}{2}\right)^3}I(x,\ y,\ u_\varepsilon+\varepsilon)\right.$$

$$-\frac{2k}{\left(x+By-\frac{\delta}{2}\right)\left(Ax+y-\frac{\delta}{2}\right)}\left[\frac{1}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\right]^2[a_{11}(x,\ y,\ u_\varepsilon+\varepsilon)$$

$$+2|a_{12}(x,\ y,\ u_\varepsilon+\varepsilon)|+a_{22}(x,\ y,\ u_\varepsilon+\varepsilon)],$$

where $A_\varepsilon$ is the operator defined in the proof of Theorem 3,

$$I(x, y, u)=a_{11}(x, y, u)\Big[\frac{1}{x+By-\frac{\delta}{2}}+\frac{A}{Ax+y-\frac{\delta}{2}}\Big]^2$$

$$+2a_{12}(x, y, u)\Big[\Big(\frac{1}{x+By-\frac{\delta}{2}}+\frac{A}{Ax+y-\frac{\delta}{2}}\Big)$$

$$\times\Big(\frac{B}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big)\Big]$$

$$+a_{22}(x, y, u)\Big[\frac{B}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2.$$

By (38), there is $\beta>0$ such that

$$2a_{12}(x, y, u)\geqslant-\min(A^2, B^2)\cdot\frac{\Lambda_0}{2}, \quad (x, y)\in\bar{\Omega}_3,\quad 0\leqslant u\leqslant\beta,$$

where $\Lambda_0=\min\limits_{(x,y)\in\bar{\Omega}_3,\,0\leqslant u\leqslant M+1}\Lambda(x, y, u)$. It follows that

$$I(x, y, u)\geqslant\min(A^2, B^2)\,[a_{11}(x, y, u)+a_{22}(x, y, u)]\cdot\Big[\frac{1}{x+By-\frac{\delta}{2}}$$

$$+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2-\min(A^2, B^2)\cdot\frac{\Lambda_0}{2}\cdot\Big[\frac{1}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2$$

$$\geqslant\frac{\Lambda_0}{2}\min(A^2, B^2)\Big[\frac{1}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2$$

for $(x, y, u)\in\bar{\Omega}_3\times[0, \beta]$. Hence

$$I(x, y, u)\geqslant K_0\Big[\frac{1}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2, \quad (x, y, u)\in\bar{\Omega}_3\times[0, M+1],$$

where $$K_0=\min\Big[\frac{\Lambda_0}{2}\min(A^2, B^2),\ \min_{(x,y)\in\bar{\Omega}_3,\,\beta\leqslant u\leqslant M+1}\lambda(x, y, u)\cdot(A^2+B^2)\Big].$$

Thus we have, for $(x, y)\in S$

$$A_s[v]\geqslant e^{-\frac{k^2}{\left(x+By-\frac{\delta}{2}\right)^3\left(Ax+y-\frac{\delta}{2}\right)^2}}\Big\{\Big[\frac{1}{x+By-\frac{\delta}{2}}+\frac{1}{Ax+y-\frac{\delta}{2}}\Big]^2$$

$$\times\Big[\frac{2K_0k^3}{\left(x+By-\frac{\delta}{2}\right)^3\left(Ax+y-\frac{\delta}{2}\right)^3}-\frac{8k\bar{\Lambda}}{\left(a+By-\frac{\delta}{2}\right)\left(Ax+y-\frac{\delta}{2}\right)}\Big]\Big\},$$

where $$\bar{\Lambda}=\max_{(x,y,u)\in\bar{\Omega}_3\times[0,M+1]}\Lambda(x, y, u).$$

By taking $k$ sufficiently large and $\sigma>0$ sufficiently small and applying the maximum principle, we get

$$u_\varepsilon(x, y) \geqslant \sigma v(x, y), \quad (x, y) \in \bar{S}.$$

Taking $\eta_0 = \sigma\left(\frac{\sqrt{\pi}}{2} - \int_0^{\frac{k}{\delta^2}} e^{-t^2} dt\right)$, we obtain

$$u_\varepsilon(x, y) \geqslant \eta_0, \quad (x, y) \in \bar{\Omega}_3 \cap \{x \geqslant 2\delta, \ y \geqslant 2\delta\}.$$

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