Chin. Ann. of Math.
5B (1) 1984

ON THE STRUCTURE OF HOMOGENEOUS COMPLETELY REDUCIBLE MODULE

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Abstract

In this paper we illuminate the equivalent relations between the structures of homogeneous completely reducible modules and vector spaces over division rings. Besides, we also give a method to simplify the proof of some important theorems concerning the homogeneous completely reducible modules in [1].

Introduction

It is well known that the notion of completely reducible module is a natural generalization of the notion of irreducible module. According to Schur's lemma we know that the notion of irreducible module is essentially the same as the notion of vector space over a division ring. It is of interest to study the relation between the notions of completely reducible modules and vector spaces over rings. Some basic theorems established in [1] illuminated the close relations between the structures of homogeneous completely reducible modules and vector spaces over division rings, but they didn't characterize the equivalent relations between the structures of homogeneous completely reducible modules and vector spaces.

The purpose of this paper is to illuminate the equivalent relations between the structures of homogeneous completely reducible modules and vector spaces over division rings. The Theorems 1, 2 and 3 established in this paper not only point out the equivalent relations between the structures of homogeneous completely reducible modules and vector spaces but also give a method to simplify the proofs of some important theorems concerning the homogeneous completely reducible modules in [1]. To avoid verbiage we only take two theorems of [1] as examples to expound our theory.

All concepts in this paper which are not specially explained are cited from [1]. Let $\mathfrak{M}=\sum_{j\in A}\oplus\mathfrak{M}_j$ be a completely reducible left module, and E_j be the projective element of $\mathfrak{M}$ such that $m_jE_j=m_j$ for $m_j\in\mathfrak{M}_j$ and $\mathfrak{M}_\alpha E_j=0$ for $j\neq\alpha$, $\alpha\in A$. Let

Manuscript received october 28, 1981.

$\{E_j\}_{j\in A}$ be the set of above stated projeetive E_j. Then for the sake of convenience we give the following

Definition 1. *We call the set* $\{\mathfrak{M}_j\}_{j\in A}$ *of above stated irreducible left modules* $\mathfrak{M}_i$ *of* $\mathfrak{M}$ *a left module-basis of* $\mathfrak{M}$ *and the set* $\{E_j\}_{j\in A}$ *the corresponding basis to* $\{\mathfrak{M}_j\}_{j\in A}$. *We have the anologeous concepts as above for completely reducible right module* $\mathfrak{M}$.

From now on module without specially explaining always means right module.

Lemma 1. *Let* $\mathfrak{A}$ *be a ring,* $\mathfrak{M}=\sum_{j\in A}\oplus\mathfrak{M}_j$ *be a completely reducible* $\mathfrak{A}$*-module, and* Γ *its centralizer of* $\mathfrak{M}$. *Let* Ω *be the centralizer of* $\mathfrak{M}$ *as left* Γ*-module. Then* Ω*-submodule* $\mathfrak{N}$ *of* $\mathfrak{M}$ *is irreducible if and only if* $\mathfrak{N}$ *as* $\mathfrak{A}$*-submodle is irreducible.*

Proof The sufficiency of the condition is already given in [1]. Now we need to prove the necessity of the condition. Let $x\Omega$ be an irreducible Ω-submodule of $\mathfrak{M}$, by [1] Γx is irreducible. Since $\mathfrak{M}=\sum_{j\in A}\oplus\mathfrak{M}_j$ is a completely reducible $\mathfrak{A}$-module, $\mathfrak{M}_j=x_j\Omega$ for every $x_j\neq 0$, $x_j\in\mathfrak{M}_j$. Hence $x=\sum_{j=1}^{n}x_ja_j$ for $a_j\in\mathfrak{A}$. Clearly $x_ja_j\mathfrak{A}$ is irreducible. By [1] Γx_ja_j is irreducible. Now let $\sum_{j=1}^{n}\Gamma x_ja_j=\sum_{j=1}^{s}\oplus\Gamma x_ja_j$, then $x=\sum_{j=1}^{s}r_jx_ja_j$, $r_j\in\Gamma$, $\mathfrak{M}=\sum_{j=1}^{s}\oplus\Gamma x_ja_j\oplus\mathfrak{N}_0$, where $\mathfrak{N}_0$ is left Γ-module. Let E_j be the projective element: $r_jx_ja_jE_j=r_jx_ja_j$, $r_ix_ia_iE_j=0$, $i\neq j$, $i, j=1,\cdots, s$, $\mathfrak{N}_0E_j=0$. Thus $x\Omega=(\sum_{j=1}^{s}r_jx_ja_j)\Omega=\sum_{j=1}^{s}r_jx_ja_j\Omega$, it follws that $x\Omega=r_jx_ja_j\Omega$. By the sufficient condition in theorem we know that $x_j\mathfrak{A}$ is also irreducible Ω-module, hence $x_ja_j\mathfrak{A}=x_ja_j\Omega$ and $r_jx_ja_j\Omega=r_jx_ja_j\mathfrak{A}=x\Omega$. Owing to $x=r_jx_ja_ja'_j$, $a'_j\in\mathfrak{A}$ we have $x\mathfrak{A}=r_jx_ja_ja'_j\mathfrak{A}=r_jx_ja_j\mathfrak{A}=x\Omega$. Therefore $x\mathfrak{A}$ is irreducible.

Theorem 1. *Let* $\mathfrak{M}$ *be a faithful homogeneous completely reducible* $\mathfrak{A}$*-module,* Γ *the centralizer of* $\mathfrak{M}$, *and let* Ω *be the centralizer of* $\mathfrak{M}$ *as left* Γ*-module. Write* $\mathfrak{M}=\sum_{j\in A}\oplus\mathfrak{M}_j=\sum_{i\in B}\oplus\mathfrak{N}_i$, *where* $\mathfrak{M}_j$ *is irreducible* Ω*-module,* $\mathfrak{N}_i$ *is irreducible left* Γ*-module. Suppose that* $\{e_j\}_{j\in A}$ *is the corresponding basis to a rigit module-basis* $\{\mathfrak{M}_j\}_{j\in A}$ *of* $\mathfrak{M}$ *and* $\{E_i\}_{i\in B}$ *is the corresponding basis to a left module-basis* $\{\mathfrak{N}_i\}_{i\in B}$ *of* $\mathfrak{M}$, *then we have the following results*:

(i) $e_j\Gamma e_j$ *is a division ring for any* $e_j\in\{e_i\}_{i\in A}$. *Write* $K=e_j\Gamma e_j$, $V=K\mathfrak{M}$, *then* $\mathfrak{M}_j=V=\sum_{i\in B}\oplus Ku_i$ *is a vector space over* K *and* Ω *the complete ring of* K*-linear transformations of* V.

(ii) $E_i\Omega E_i$ *is a division ring. Write* $\hat{K}=E_i\Omega E_i$, $\hat{V}=\mathfrak{M}\hat{K}$, *then* $\mathfrak{N}_i=\hat{V}=\sum_{j\in A}\oplus v_j\hat{K}$ *is a vector space over* $\hat{K}$ *and* Γ *is the complete ring of* $\hat{K}$*-linear transformations of* $\hat{V}$.

(iii) K *is isomorphic to* $\hat{K}$.

Proof We first prove (i).

Since $\mathfrak{M}=\sum_{i\in B}\oplus\mathfrak{N}_i$ is completely reducible left Γ-module, $\mathfrak{N}_i=\Gamma y_i$ for $i\in B$.

From the assumptions of the theorem we have $\mathfrak{M}E_i=\mathfrak{N}_i$, $\mathfrak{N}_\alpha E_i=0$ for $\alpha\neq i$. Now we want to prove that $E_i\Omega$ is a minimal right ideal of Ω. In fact, let $a\in\Omega$ and $E_ia\neq 0$, then from the irreduciblity of $\mathfrak{N}_i=\Gamma y_i$ it follows that $y_i\Omega$ is irreducible Ω-module (see [1] p. 25). Hence $y_i\Omega=y_iE_i\Omega=y_iE_ia\Omega$. Let $E_i\omega$ be any element of $E_i\Omega$, then $y_iE_i\omega=y_iE_iaa'$ for $a'\in\Omega$. Hence $y_i(E_i\omega-E_iaa')=0$. Because $\{E_j\}_{j\in B}$ is the corresponding basis to left module-basis $\{\mathfrak{N}_j\}_{j\in B}$, we have $\mathfrak{M}\,(E_i\omega-E_iaa')=0$. Thus $E_i\omega\in E_ia\Omega$, $E_i\Omega=E_ia\Omega$. This proves that $E_i\Omega$ is minimal. On the other hand if $E_i\Omega a=0$ for an element $a\in\Omega$, then $\mathfrak{M}a=\Gamma y_iE_i\Omega a=0$, hence $a=0$. This means that Ω is a primitive ring.

Let $\mathfrak{S}=\sum_{i\in B}\oplus\Omega E_i$, we want to show that $\mathfrak{S}$ is an ideal of Ω. In deed, let ω be any element of Ω, we need only to prove that $E_i\omega\in\mathfrak{S}$. Since $y_i\omega=\sum_{j=1}^{s}\gamma_jy_j$, $\gamma_j\in\Gamma$, then $E_i\omega=\sum_{j=1}^{s}E_i\omega E_j\in\mathfrak{S}$. Hence $\mathfrak{S}$ is socle of Ω. On the other hand let $\hat{K}=E_j\Omega E_j$ for element $E_j\in\{E_i\}_{i\in B}$. It is well known that $\hat{K}$ is a division ring. If we put $\mathbf{A}=E_j\Omega$, then

$$\mathbf{A}=E_j\Omega=\hat{K}\mathfrak{S}=\sum_{i\in B}\oplus\hat{K}\alpha_i, \tag{1}$$

where $\alpha_i=E_j\sigma_iE_i$, $\sigma_i\in\Omega$, $i\in B$ (see [2]). Hence $\mathbf{A}$ is a vector space over $\hat{K}$. Now we want to prove that Ω is a complete ring of $\hat{K}$-linear transformations of $\mathbf{A}$. In fact, if σ is a $\hat{K}$-transformation of $\mathbf{A}$, then $\alpha_i\sigma\in\mathbf{A}\subset\Omega$. Since $y_j\alpha_i=y_jE_j\sigma_iE_i\in\mathfrak{M}E_i=\Gamma y_i$, we have

$$\Gamma y_j\alpha_i=\Gamma y_i,\quad i\in B. \tag{2}$$

On the other hand, since $y_j(\alpha_i\sigma)\Omega=y_j\Omega=y_j\alpha_i\Omega$, there exists element $\omega_i\in\Omega$ such that $y_j(\alpha_i\sigma)=y_j\alpha_i\omega_i$. As above we can show $E_j\,((\alpha_i\sigma)-\alpha_i\omega_i)=0$. From $\alpha_i\sigma-\alpha_i\omega_i\in\mathbf{A}=E_j\Omega$ it follows that

$$\alpha_i\sigma=\alpha_i\omega_i,\quad i\in B. \tag{3}$$

Now we want to prove that there exists $\omega\in\Omega$ such that $\alpha_i\omega=\alpha_i\omega_i$, $i\in B$. For this purpose we make a correspondence λ: $\sum_k\gamma_ky_k\to\sum_k\gamma_ky_k\omega_k$, where ω_k satisfies relation (3) and $\gamma_k\in\Gamma$. It is clear that λ is a Γ-homomorphism of $\mathfrak{M}$. Hence $\lambda\in\Omega$. On the other hand, from (2) it follows that $y_j\alpha_k=\gamma_ky_k$, $\gamma_k\in\Gamma$. Thus for $k\in B$ we have

$$(y_j\alpha_k)\lambda=(\gamma_ky_k)\lambda=\gamma_k(y_k\lambda)=\gamma_ky_k\omega_k=y_j(\alpha_k\omega_k).$$

As above we can obtain $\alpha_k\lambda=\alpha_k\omega_k$ for $k\in B$. From (3) it follows that $\alpha_k\lambda=\alpha_k\sigma$ for $k\in B$. Hence $\sigma=\lambda\in\Omega$. This proves that Ω is the complete ring of $\hat{K}$-linear transformations of $\mathbf{A}$.

Since $\mathfrak{M}=\Gamma y_j\Omega$, $\mathfrak{M}=\sum_{i\in A}\oplus\gamma_iy_j\Omega$. Write $\mathfrak{M}_i=\gamma_iy_j\Omega$, $\mathfrak{M}_0=y_j\Omega$, then it may be assumed that $\mathfrak{M}_0\in\{\mathfrak{M}_i\}_{i\in A}$. Denote the corresponding basis to right module-basis $\{\mathfrak{M}_i\}_{i\in A}$ by $\{e_i\}_{i\in A}$, then there exists an element $e_0\in\{e_i\}_{i\in A}$ such that $e_0\mathfrak{M}=\mathfrak{M}_0$. Put $K=e_0\Gamma e_0$, then as above we can show that K is a division ring, and

$$\mathfrak{M}_0=e_0\mathfrak{M}=\sum_{i\in B}\oplus e_0\Gamma y_i=\sum_{i\in B}\oplus K\Gamma y_i=\sum_{i\in B}\oplus Ku_i, \tag{4}$$

where $u_i=e_0\gamma_i^* y_i$, $\gamma_i^*\in\Gamma$, $i\in B$(see [1]). Write $\tilde{\Omega}$ for the complete ring of K-linear transformations of $\mathfrak{M}_0$. We want to prove that $\tilde{\Omega}=\Omega$. In fact, since $\mathfrak{M}_0=y_j\Omega$, we have $\mathfrak{M}_0=y_i\tilde{\Omega}$. Because $\{E_i\}_{i\in B}$ is the corresponding basis to left module-basis $\{\Gamma y_i\}_{i\in B}$, it follows that $E_j\Omega=E_j\tilde{\Omega}$ from $y_jE_j=y_j$. But $\mathbf{A}=E_j\Omega=\sum_{i\in B}\oplus\hat{K}\alpha_i\subset\Omega\subseteq\tilde{\Omega}$, hence $\tilde{\Omega}$ is also the ring of $\hat{K}$-linear transformations of $\mathbf{A}$. It follows now that $\tilde{\Omega}\subseteq\Omega$. This proves $\tilde{\Omega}=\Omega$. It is well known that $K\cong\hat{K}$. This completes the proofs of (i) and (ii) of the theorem. Similarly, we can prove (ii) of the theorem.

As a corollary we can prove the following well-known theorem (see [1]).

Corollary 1. *Let $\mathfrak{A}$ be a homogeneous distinguished ring of endomophisms, let Γ be the centralizer. Then $\mathfrak{A}$ is isomorphic to the complete ring of linear transformations of a certain vector space $\mathfrak{M}_0$ over a division ring K and Γ is the centralizer of a certain right vector space $\mathfrak{N}_0$ over K. The dimensionality of $\mathfrak{M}_0$ is the same as the dimensionality of $\mathfrak{M}$ as left Γ-module while the dimensionality of $\mathfrak{N}_0$ is the same as that of $\mathfrak{M}$ as $\mathfrak{A}$-module.*

Proof From (4) and the hypothesis of $\mathfrak{A}$ we know that $\mathfrak{A}=\Omega$ is the complete ring of K-linear transformations of vector space $\mathfrak{M}_0=\sum_{i\in B}\oplus Ku_i$ over $K=e_0\Gamma e_0$. Evidently, the dimensionality of $\mathfrak{M}_0$ is the same as the dimensionality of $\mathfrak{M}$ as left Γ-module. Since $\{e_i\}_{i\in A}$ is the corresponding basis to right module-brsis $\{\mathfrak{M}_i\}_{i\in A}$, by the proof of Theorem 1 $\mathfrak{S}'=\sum_{i\in A}\oplus e_i\Gamma$ is socle of Γ. From the proof of Theorem 1 we know the Γ is the complete ring of $\hat{K}$-linear transformations of $\mathfrak{M}_0'=\mathfrak{M}E_j=\sum_{i\in A}\oplus v_j\hat{K}$, where $\hat{K}=E_j\Omega E_j$ is division ring. By [2] $\mathfrak{N}_0=\Gamma e_0=\sum_{j\in A}\oplus\beta_jK$, where $\beta_i=e_i\sigma_ie_0$, $\sigma_i\in\Omega$, $i\in A$, $K=e_0\Gamma e_0$, and Γ is the complete ring of K-linear transformations of right vector space $\mathfrak{N}_0$ over K. Clearly, the dimensionality of $\mathfrak{N}_0$ is the same as the dimensionality of $\mathfrak{M}$ as $\mathfrak{N}$-module.

Theorem 2. *Let $\mathfrak{M}$ be a faithful homogeneous comletely reducible $\mathfrak{A}$-module, Γ the centralizer of $\mathfrak{M}$, and Ω the centralizer of $\mathfrak{M}$ as left Γ-module. Then*

(i) *the centralizer of any irreducible $\mathfrak{A}$-submodule $\mathfrak{M}_0$ is the same as the centralizer of $\mathfrak{M}_0$ as Ω-module.*

(ii) *let $\mathfrak{A}'$ be a subring of Ω, $\mathfrak{M}_0$ an irreducible $\mathfrak{A}'$-submodule of $\mathfrak{M}$. Suppose that $\mathfrak{M}_0$ is Ω-module and the centralizer of $\mathfrak{M}_0$ as Ω-module is the same as the centralizer of $\mathfrak{M}_0$ as $\mathfrak{A}'$-module. Then $\mathfrak{M}$ is a faithful homogeneous completely reducible $\mathfrak{A}'$-module, and Γ is the centralizer of $\mathfrak{M}$ as $\mathfrak{A}'$-module.*

Proof We now prove (i).

Let $\mathfrak{M}=\sum_{i\in A}\oplus\mathfrak{M}_i$, $\{e_i\}_{i\in A}$ be the corresponding basis to right module-basis $\{\mathfrak{M}_i\}_{i\in A}$. Let $e_j\in\{e_i\}_{i\in A}$, $K=e_j\Gamma e_j$. Then by Theorem 1 K is a division ring and K. $\mathfrak{M}_j=\mathfrak{M}_j$. Denote by F the centralizer of $\mathfrak{M}_j$ as $\mathfrak{A}$-module, then it is clear that $K\subseteq F$.

Since $\mathfrak{M}_j$ is irreducible $\mathfrak{A}$-module, it follows that F is a division ring. We want to prove that $F=K$. If $[F:K]>1$, and $F=\sum_{\alpha\in I}\oplus Kf^{(\alpha)}$, then

$$\mathfrak{M}_j=\sum_{i\in J}\oplus Fu_i=\sum_{i\in J,\alpha\in I}\oplus Kv_i^{(\alpha)}, \tag{5}$$

$$v_i^{(\alpha)}=f^{(\alpha)}u_i,\ i\in J,\ \alpha\in I.$$

Since $K=e_j\Gamma e_j$, by (5) we may assume that

$$e_jv_i^{(\alpha)}=v_i^{(\alpha)}\quad \text{for } i\in J,\ \alpha\in I. \tag{6}$$

Denote the centralizer of $\mathfrak{M}$ as Γ-module by Ω, then Ω is the complete ring of K-linear transformations of $\mathfrak{M}_j$ as vector space over K. Let $\tilde{\Omega}$ be the complete ring of F-linear transformations of $\mathfrak{M}_j$ as vector space over F, then it is clear that $\mathfrak{A}\subseteq\tilde{\Omega}\subseteq\Omega$. Evidently

$$\mathfrak{M}=\Gamma\mathfrak{M}_j=\Gamma\Big(\sum_{i\in J,\alpha\in I}Kv_i^{(\alpha)}\Big)=\sum_{i\in J,\alpha\in I}\Gamma v_i^{(\alpha)}. \tag{7}$$

On the other hand $\mathfrak{M}_j$ as vector space over K has corresponding basis $\{E_i^{(\alpha)}\}_{i\in J,\alpha\in I}$ to left module-basis $\{Kv_i^{(\alpha)}\}_{i\in J,\alpha\in I}$ by (5), hence $\{E_i^{(\alpha)}\}_{i\in J,\alpha\in I}\subset\Omega$. By (7) we obtain

$$\mathfrak{M}=\sum_{i\in J,\alpha\in I}\Gamma v_i^{(\alpha)}=\sum_{i\in J,\alpha\in I}\oplus\Gamma v_i^{(\alpha)}. \tag{8}$$

Since $\mathfrak{M}_j$ is irreducible $\mathfrak{A}$-module, we have

$$v_i^{(\alpha)}\mathfrak{A}=\mathfrak{M}_j=v_i^{(\alpha+1)}\mathfrak{A}, \tag{9}$$

$$\mathfrak{M}=\Gamma v_i^{(\alpha)}\mathfrak{A}=\sum_k\oplus\gamma_kv_i^{(\alpha)}\mathfrak{A}=\sum_k\oplus\gamma_kv_i^{(\alpha+1)}\mathfrak{A},$$

where $\gamma_k\in\Gamma$.

Now we want to prove that there exists an element $a\in\mathfrak{A}$ such that $v_i^{(\alpha+1)}a\neq0$, $v_i^{(\alpha)}a=0$. If it is not true, then from $v_i^{(\alpha)}a=0$ it follows that $v_i^{(\alpha+1)}a=0$ for any element $a\in\mathfrak{A}$. Thus we make the following correspondence σ:

$$m=\sum_{k=1'}^{n'}\gamma_kv_i^{(\alpha)}a_k\longrightarrow\sum_{k=1'}^{n'}\gamma_kv_i^{(\alpha+1)}a_k, \tag{10}$$

where $a_k\in\mathfrak{A}$ and γ_k is given by (9), we note that a_k in (10) may be equal to identity 1. Now we prove that σ is single-valuedness. In fact, if $m=\sum_{k=1'}^{n'}\gamma_kv_i^{(\alpha)}a'_k$, then

$$\sum_k\gamma_kv_i^{(\alpha)}(a_k-a'_k)=0.$$

From (9) we obtain

$$\gamma_kv_i^{(\alpha)}(a_k-a'_k)=0,\quad k=1',\cdots,n'. \tag{11}$$

Now we discuss (11) in two cases: (i) $\gamma_ke_j\neq0$, (ii) $\gamma_ke_j=0$. If (i) is true, then $\Gamma\gamma_ke_j=\Gamma e_j$, since Γe_j is minimal left ideal. Hence there exists an element $s_k\in\Gamma$ such that $s_k\gamma_ke_j=e_j$. From (6) and (11) it follows that $v_i^{(\alpha)}(a_k-a'_k)=0$, $v_i^{(\alpha+1)}(a_k-a'_k)=0$, since $a_k-a'_k\in\mathfrak{A}$. Therefore $\gamma_kv_i^{(\alpha+1)}(a_k-a'_k)=0$, $k=1',\cdots,n'$. On the other hand if (ii) is true, then it is clear that $\gamma_kv_i^{(\alpha+1)}(a_k-a'_k)=0$ for $k=1',\cdots n'$. This proves that σ is single-valuedness. By (10) σ is bimodule homomorphism. Since Γ is the centralizer of $\mathfrak{M}$ as $\mathfrak{A}$-module, $\sigma\in\Gamma$. It follows from (10) that

$$\sigma v_i^{(\alpha)}=v_i^{(\alpha+1)}. \tag{12}$$

Since $\{E_i^{(\alpha)}\}\subset\Omega$, we have $\sigma v_i^{(\alpha)}E_i^{(\alpha)}=v_i^{(\alpha+1)}E_i^{(\alpha)}=0$ by (12). Hence $v_i^{(\alpha+1)}=0$. This is impossible. Therefore there exists an element $a\in\mathfrak{A}$ such that

$$v_i^{(\alpha)}a=0,\quad v_i^{(\alpha+1)}a\neq 0. \tag{13}$$

Now we consider vector space $\mathfrak{M}_j=\sum_{i\in J}\oplus Fu_i$ over F. For the corresponding basis $\{\tilde{e}_i\}_{i\in J}$ to left module-basis $\{Fu_i\}_{i\in J}$ of $\mathfrak{M}_j$ we obtain by (5)

$$\mathfrak{M}_j\tilde{e}_i=\sum_{\alpha\in I}\oplus Kv_i^{(\alpha)},$$

$$v_i^{(\alpha)}\tilde{e}_i=v_i^{(\alpha)},\quad \alpha\in I. \tag{14}$$

Since $\tilde{\Omega}$ is the complete ring of F-linear transformations of $\mathfrak{M}_j$, $\{\tilde{e}_i\}_{i\in J}\subset\tilde{\Omega}$ and $\tilde{e}_i\tilde{\Omega}$ is minimal right ideal of $\tilde{\Omega}$. By (13) and (14) $\tilde{e}_ia\neq 0$. Hence $\tilde{e}_ia\tilde{t}=\tilde{e}_i$, where $\tilde{t}\in\tilde{\Omega}$. From (13) it follows that $v_i^{(\alpha)}\tilde{e}_i=v_i^{(\alpha)}=v_i^{(\alpha)}\tilde{e}a\tilde{t}=0$. This is a contradiction which means that the above assumption $[F:K]>1$ is false. Hence $F=K$. This completes the proof of (i).

Now we prove (ii). Since $\mathfrak{M}'_0$ is irreducible $\mathfrak{A}'$-submodule as well as Ω-module of $\mathfrak{M}$, $\mathfrak{M}=\Gamma\mathfrak{M}'_0$. Let $\mathfrak{M}'_0=x\mathfrak{A}'=x\Omega$, then $\mathfrak{M}=\sum_{k\in J}\oplus\gamma_kx\Omega=\sum_{k\in J}\oplus\gamma_kx\mathfrak{A}'$. Clearly, $\mathfrak{M}_k=\gamma_kx\mathfrak{A}'$ is irreducible $\mathfrak{A}'$-module, hence $\mathfrak{M}$ is faithful homogeneous completely reducible $\mathfrak{A}'$-module. Denote by Γ' the centralizer of $\mathfrak{M}$ as $\mathfrak{A}'$-module. Then we prove that $\Gamma'=\Gamma$. Let $\{e_i\}_{i\in J}$ be the corresponding basis to right module-basis $\{\mathfrak{M}_i\}_{i\in J}$ of $\mathfrak{M}=\sum_{i\in J}\oplus\mathfrak{M}_i$, where $\mathfrak{M}_i=\gamma_ix\mathfrak{A}'=\gamma_ix\Omega$, $\mathfrak{M}'_0=x\mathfrak{A}'=x\Omega\in\{\mathfrak{M}_i\}_{i\in J}$. Evidently, $\{e_i\}_{i\in J}\subset\Gamma\subset\Gamma'$. By Theorem 1 $e_i\Gamma e_i\subset\Gamma$ and $e_i\Gamma'e_i\subset\Gamma'$ are division rings. Write $e_0\in\{e_i\}_{i\in J}$ such that $e_0x_0=x_0$ for $x_0\in\mathfrak{M}'_0$. By Theorem 1 the centralizer of $\mathfrak{M}'_0$ as $\mathfrak{A}'$-module is $e_0\Gamma'e_0$ and the centralizer of $\mathfrak{M}'_0$ as Ω-module is $e_0\Gamma e_0$. By the assumption of (ii) of this theorem, $e_0\Gamma'e_0=e_0\Gamma e_0$. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be socles of Γ and Γ' respectively. Then by the proof of Theorem 1 we know that $\mathfrak{S}=\sum_{i\in J}\oplus e_i\Gamma$, $\mathfrak{S}'=\sum_{i\in J}\oplus e_i\Gamma'$. Put $A=\Gamma e_0$, $A'=\Gamma'e_0$, $K=e_0\Gamma'e_0$, then we have

$$A'=\Gamma'e_0\Gamma'e_0=\mathfrak{S}'K=\sum\oplus e_i\Gamma'K=\sum_{i\in J}\oplus e_i\Gamma K=A. \tag{15}$$

By Theorem 1 and [2] Γ and Γ' are the complete rings of K-linearly transformations of A and A'respectively. Hence $\Gamma=\Gamma'$ by (15). This completes the proof.

Definition 2. *A ring $\mathfrak{A}$ is called $\aleph_\nu$-fold transitive ring of homogeneous completely reducible module if and only if there exists a faithful homogeneous completely reducible $\mathfrak{A}$-module $\mathfrak{M}$ such that if Γ is the centralizer of $\mathfrak{M}$ as $\mathfrak{A}$-module and $\mathscr{L}$ is the centralizer of $\mathfrak{M}$ as left Γ-module, then for a subset $\{x_i\}_{i\in I}$ of a set of generators of $\mathfrak{M}$ as Γ-module with card $I<\aleph_\nu$, and for $l\in\mathscr{L}$ there exists an $a\in\mathfrak{A}$ such that $x_ia=x_il$ for all $i\in I$. A ring $\mathfrak{A}$ is called $\aleph_\nu$-fold transitive ring of a vector space belonging to homogeneous completely reducible module if and only if there exists an irreducible $\mathfrak{A}$-submodule $\mathfrak{M}_0$ of $\mathfrak{M}$ as $\mathfrak{A}$-module such that $\mathfrak{A}$ is $\aleph_\nu$-fold transitive (see [2]) in $\mathfrak{M}_0$ over K which is the centralizer of $\mathfrak{M}_0$ as $\mathfrak{A}$-module.*

From this definition we can prove the following

Theorem 3. *A ring $\mathfrak{A}$ is $\aleph_\nu$-fold transitive ring of homogeneous completely reducible module if and only if $\mathfrak{A}$ is $\aleph_\nu$-fold transitive ring of a vector space belonging to homogeneous completely reducible module.*

Proof Sufficiency. Let $\{x_i\}_{i\in I}$ be a subset of a set of generators of $\mathfrak{M}$ as Γ-module with Card. $I<\aleph_\nu$, and $l\in\mathscr{L}$. We want to prove that there exists an element $a\in\mathfrak{A}$ such that $x_ia=x_il$ for all $i\in I$. In fact, by the assumption of this theorem there exists an irreducible $\mathfrak{A}$-submodule $\mathfrak{M}_0$ of $\mathfrak{M}$. By Theorem 2 the centralizer of $\mathfrak{A}$-module $\mathfrak{M}_0$ is $e_0\Gamma e_0=K$, where Γe_0 is minimal left ideal and $e_0^2=e_0$. Write

$$\mathfrak{M}=\sum_{i\in I}\oplus\Gamma x_i\oplus\sum_{j\in I'}\oplus\Gamma x_j,$$

then

$$\mathfrak{M}_0=K\mathfrak{M}=\sum_{i\in I\cup I'}\oplus K\Gamma x_i=\sum_{i\in I\cup I'}\oplus Kr_ix_i=\sum_{i\in I\cup I'}\oplus Ky_i, \tag{16}$$

where $y_i=r_ix_i$, $r_i\in\Gamma$. By the assumption $\mathfrak{A}$ is λ'_ν-fold transitive in vector space $\mathfrak{M}_0$ over K. Hence we can find an element $a\in\mathfrak{A}$ such that $y_ia=y_il$ for all $i\in I$. Therefore $r_i(x_ia-x_il)=0$ for $i\in I$. Let i be any element of I, then by Lemma 1 and [2] $x_i\mathfrak{A}$ is irreducible $\mathfrak{A}$-module, $\mathfrak{M}=x_i\mathfrak{A}\oplus\mathfrak{M}_1$. Hence there exists an element $\hat{e}_i\in\Gamma$ such that $\hat{e}_ix_ia'=x_ia'$, $a'\in\mathfrak{A}$; $\hat{e}_i\mathfrak{M}_1=0$. Owing to $r_i\hat{e}_ix_i=r_ix_i=y_i\neq0$ it follows that $r_i\hat{e}_i\neq0$. Since $\Gamma\hat{e}_i$ is minimal, there exists an element $s_i\in\Gamma$ such that $s_ir_i\hat{e}_i=\hat{e}_i$. Therefore $s_ir_i\hat{e}_i(x_ia-x_il)=x_ia-x_il=0$, $x_ia=x_il$ for all $i\in I$.

Necessity. Let $\mathfrak{M}_0$ be a vector space in (16) belonging to homogeneous completely reducible module, $\{y_i\}_{i\in I}$ be a set of K-linearly independent elements of $\mathfrak{M}_0$ with card. $I<\aleph_\nu$ and $\{z_i\}_{i\in I}$ be any subset of $\mathfrak{M}_0$. We want to prove that there exists an element $a\in\mathfrak{A}$ such that $y_ia=z_i$ for all $i\in I$. In fact, by Theorem 1 the centralizer $\mathscr{L}$ of $\mathfrak{M}$ as left Γ-module is the complete ring of K-linear transformations of $\mathfrak{M}_0$. Hence we have an element $l\in\mathscr{L}$ such that $y_il=z_i$ for $i\in I$.

From Theorem 1 it is easy to see that

$$\mathfrak{M}=\Gamma\mathfrak{M}_0=\sum_{i\in I\cup I'}\oplus\Gamma(e_0y_i)=\sum_{i\in I\cup I'}\oplus\Gamma y_i, \tag{17}$$

where $\{y_i\}_{i\in I\cup I'}$ is a K-basis of $\mathfrak{M}_0$ which is extended by $\{y_i\}_{i\in I}$. Since $y_i\in e_0\mathfrak{M}=\mathfrak{M}_0$, $e_0y_i=y_i$ for $i\in I$. By the assumption of this theorem there exists an element $a\in\mathfrak{A}$ such that $y_ia=y_il=z_i$, $i\in I$. This completes the proof.

From Theorem 3 we can prove immediately the following

Corollary 2 (Density theorem). *Let $\mathfrak{M}$ be a faithful homogeneous completely reducible $\mathfrak{A}$-module, Γ the centralizer and $\mathscr{L}$ the centralizer of $\mathfrak{M}$ as left Γ-module. Then $\mathscr{L}$ is the closure of $\mathfrak{A}$ in the finite topology.*

Proof Since $\mathfrak{A}$ is (finite) transitive in vector space $\mathfrak{M}_0$ of (16), $\mathfrak{A}$ is finite transitive ring of homogeneous completely reducible module. From Theorem 3 the assertion of this theorem follows immediately.

Using the above mentioned theorem we can derive some basic theorems in [1] in a different way. For example, we can prove the following two theorems as corollaries.

Corollary 3. *Let $\mathfrak{A}$ be a ring of endomorphisms acting in the commutative group $\mathfrak{M}$. Then $\mathfrak{A}$ is distinguished and homogeneous if and only if the following two conditions hold:* (1) *$\mathfrak{A}$ is isomorphic to the complete ring of linear transformations in a vector space over a division ring,* (2) *$\mathfrak{M}\mathfrak{F}=\mathfrak{M}$ for $\mathfrak{F}$ the socle of $\mathfrak{A}$.*

Proof The necessary condition follows immediately from Theorem 1. Now we prove the sufficiency of the condition. Let $A=E\mathfrak{A}$ be a minimal right ideal of $\mathfrak{A}$. By condition (2) $\mathfrak{M}=\mathfrak{M}E\mathfrak{A}=\sum_i \oplus x_i E\mathfrak{A}$, and $x_i E\mathfrak{A}$ is clearly irreducible $\mathfrak{A}$-module. Evidently, $\mathfrak{M}$ is a homogeneous completely reducible $\mathfrak{A}$-module. Now we want to show that $\mathfrak{M}$ is faithful. Let $\mathfrak{M}a=0$ for $a\neq 0$, $a\in\mathfrak{A}$, then $\mathfrak{M}=\mathfrak{M}\mathfrak{F}a\mathfrak{F}=0$. This is impossible. Hence $\mathfrak{M}$ is faithful. Let Γ be the centralizer of $\mathfrak{M}$ as $\mathfrak{A}$-module, Ω be the centralizer of $\mathfrak{M}$ as Γ-module, then $\mathfrak{A}\subseteq\Omega$. Write $\mathfrak{M}_i=x_i E\mathfrak{A}$. By Lemma 1 $\mathfrak{M}_i=x_i E\Omega$. Using Theorem 2 we know that the centralizer of $\mathfrak{M}_i$ as $\mathfrak{A}$-module is the same as the centralizer of $\mathfrak{M}_i$ as Ω-module. Hence $A=E\mathfrak{A}=E\Omega$ is also minimal right ideal. Put $K=E\mathfrak{A}E$, then $A=\sum_i \oplus K\alpha_i$ is a vector space over K. From the condition (1) and [2] it follows that $\mathfrak{A}$ is the complete ring of K-linear transformations of A. Therefore $\Omega\subseteq\mathfrak{A}$ and $\mathfrak{A}=\Omega$. This completes the proof.

References

[1] Jacobson, N., Structure of rings, *Amer. Math. Soci. Colloq. Publ.*, **37** (1956).
[2] Xu Yonghua, *Acata Math.*, **22** (1979), 204—218.