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ON THE CONVERGENCE FOR NULL-ARRAYS OF THE POINT PROCESSES

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Abstract

The purpose of this paper is first to establish a representation of the Laplace transform for the regular infinitely divisible point processes, and then to give a sufficient and neccesary condition for convergence of the null-arrays toward a regular infinitely divisible point process.

§ 1. Notations

Let (X, d) be a separable complete metric space, and denote by $\mathscr{A}$ and $\mathscr{B}$ the Borel and bounded Borel sets in X respectively. A measure μ on $(X, \mathscr{A})$ is called locally finite if $\mu(A)<\infty$ for all $A\in\mathscr{B}$. Let M be the set of all locally finite measures on $(X, \mathscr{A})$ and $\mathscr{M}$ the smallest σ-algebra of subsets of M with respect to which the real functions $\mu\to\mu(A)$ $(\mu\in M)$ are measurable for all $A\in\mathscr{B}$. A measure μ in M is called a counting measure if $\mu(A)\in Z=\{0, 1, 2, \cdots,\}$ for all $A\in\mathscr{B}$. Denote by N the set of all counting measures, then $N\in\mathscr{M}$, and denote by $\mathscr{N}$ the σ-algebra $N\cap\mathscr{M}$. A probability measure on $(N, \mathscr{N})$ is called a random point distribution. Denote by $\mathscr{P}$ the set of all random point distributions. Suppose $P_1, P_2\in\mathscr{P}$. Then a distribution P on $(N, \mathscr{N})$ is called the convolution of P_1 and P_2, denoted by $P=P_1*P_2$, if it satisfies the condition

$$P(Y)=\iint \mathbf{1}_Y(\mu+\nu)P_1(d\mu)P_2(d\nu)$$

for all $Y\in\mathscr{N}$. In particular we denote by P^n the n-fold convolution of P with itself. A distribution on $(N, \mathscr{N})$ is called infinitely divisible if for each positive integer n there is a $Q\in\mathscr{P}$ such that $P=Q^n$. Let $\mathscr{P}_{IN}$ be the set of all infinitely divisible distributions. Denote by $\mathscr{F}_+$ the set of all nonnegative bounded functions on X with bounded support. By the Laplace transform of $P\in\mathscr{P}$ we mean the mapping

$$L_P(f)=\int e^{-\mu f}P(d\mu), \quad f\in\mathscr{F}_+,$$

of $\mathscr{F}_+$ into $[0, \infty)$, where $\mu f=\int f d\mu$.

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Proposition 1* ([2] Theorem 6.1). *Suppose $P\in\mathscr{P}$. Then P is infinitely divisible iff its Laplace transform has the f following form*

$$-\log L_P(f)=\int(1-e^{-\mu f})\tilde{P}(d\mu),\quad f\in\mathscr{F}_+,$$

where $\tilde{P}$, called the canonical measure of P, is a measure on N satisfying $\tilde{P}(\{0\})=0$ and $\tilde{P}(\mu:\mu(A)>0)<\infty$ for all $A\in\mathscr{B}$.

$P\in\mathscr{P}_{IN}$ is called a regular infinitely divisible distribution if $\tilde{P}(\mu:\mu(X)=\infty)=0$.

We denote by X^k ($k=0, 1, 2, \cdots,$) the k-times product of X, and by $\mathscr{A}^k$ the Borel sets of X^k. A measure G on $(X^k, \mathscr{A}^k)$ is called symmetric if

$$G(A_1\times A_2\times\cdots\times A_k)=G(A_{i_1}\times\cdots\times A_{i_k})$$

for all $A_1, \cdots, A_k\in\mathscr{B}$ and any permutation $(i_1, \cdots, i_k)$ of $(1, \cdots, k)$.

§ 2. The Laplace transform of regular infinitely divisible distribution

Theorem 1. *$P\in\mathscr{P}$ is a regular infinitely divisible distribution iff*

$$-\log L_P(f)=\sum_{k=1}^{\infty}\int_{X^k}\Big(1-\exp\Big(-\sum_{i=1}^{k}f(a_i)\Big)\Big)\gamma_k(da_1\times\cdots\times da_k),\quad f\in\mathscr{F}_+,\tag{1}$$

where γ_k, $k=1, 2, \cdots$ are symmetric locally finite measures on $(X^k, \mathscr{A}^k)$ such that

$$\sum_{k=1}^{\infty}\gamma_k(X^k-(A^c)^k)<\infty\tag{2}$$

for all $A\in\mathscr{B}$. Here A^c denotes the complement of A.

Proof Suppose (1) is the Laplace transform of some $P\in\mathscr{P}$ with γ_k, $k=1, 2, \cdots,$ satisfying (2), we define for integer $k>0$ a measure $\tilde{P}_k$ on N as follows

$$\tilde{P}_k=\gamma_k\psi_k^{-1},\quad k=1, 2, \cdots,$$

where ψ_k is a mapping from X^k into N such that

$$\psi_k(a_1, \cdots, a_k)=\delta_{a_1}+\cdots+\delta_{a_k},\quad \text{for } (a_1, \cdots, a_k)\in X^k,$$

Let $\tilde{P}=\sum_{k=1}^{\infty}\tilde{P}_k$. Then $\tilde{P}(\{0\})=0$ and

$$\tilde{P}(\mu:\mu(A)>0)=\sum_{k=1}^{\infty}\tilde{P}_k(\mu:\mu(A)>0)=\sum_{k=1}^{\infty}\gamma_k(X^k-(A^c)^k)<\infty$$

for all $A\in\mathscr{B}$. Since (1) can be rewritten in the form

$$-\log L_P(f)=\sum_{k=1}^{\infty}\int(1-e^{-\mu f})\tilde{P}_k(d\mu)=\int(1-e^{-\mu f})\tilde{P}(d\mu),$$

it is shown that P is a regular infinitely divisible distribution with canonical measure $\tilde{P}$.

Conversely, suppose P is regular infinitely divisible and $\tilde{P}$ is its canonical measure. Set

*) This theorem is proved in [2], where X is supposed to be a locally compact second countable Hausdorff topological space. It can be proved that the same result is still valid when X is a complete separable space.

$$N_k=\{\mu:\mu\in N,\ \mu(X)=k\},\quad \widetilde{P}_k(\cdot)=\widetilde{P}(\cdot\cap N_k),$$

$$N_{kj}=\{\mu:\mu\in N_k,\ \mu \text{ has } j \text{ different atoms}\},\quad j\leqslant k,\ j,\ k=1,\ 2,\ \cdots,$$

and

$$L=\{(a,\ \mu):\ a\in X,\ \mu\in N,\ \mu(a)>0\}.$$

By a measurable indexing relation $(a,\ \mu)\to i(a,\ \mu)$ we associate a positive integer $i(a,\ \mu)$ to each $(a,\ \mu)$ in L. (The definition of measurable indexing relation, see [1] pp. 229) Instead of $i(a,\ \mu)=s$, we write $a=z_s(\mu)$, so that $\mu\to z_s(\mu)$ is a measurable mapping of N_{kj} into X for all $k,\ j,\ s,\ s\leqslant j\leqslant k$.

Now we define a mapping φ_k of N_k into X^k as follows: if $n_1,\ \cdots,\ n_j\in Z_+,\ n_1+\cdots+n_j=k$ and $\mu\in\{\mu:\ \mu\in N_{kj},\ \mu=n_1\delta_{z_1(\mu)}+n_2\delta_{z_2(\mu)}+\cdots+n_j\delta_{z_j}(\mu)\}$, set

$$\varphi_k(\mu)=(\underbrace{z_1(\mu),\ \cdots,\ z_1(\mu)}_{n_1},\ \cdots,\ \underbrace{z_j(\mu),\ \cdots,\ z_j(\mu)}_{n_j})\in X^k,$$

Since φ_k is measurable, it induces a measure $\gamma'_k(\cdot)=\widetilde{P}_k\varphi_k^{-1}(\cdot)$ on X^k. Denoting by γ_k the symmetric version of γ'_k, it is not difficult to verify that γ_k is locally finite and $\widetilde{P}_k=\gamma_k\psi_k^{-1}$, hence the Laplace transform of P can be expressed in the form (1).

To prove (2), taking $f(a)=t\mathbf{1}_A(a)$ for $A\in\mathscr{B}$, we have

$$-\log L_P(f)=\int(1-e^{-\mu f})\widetilde{P}(d\mu)=\sum_{k=1}^{\infty}(1-e^{-tk})\widetilde{P}(\mu(A)=k),$$

and

$$-\log L_P(f)=\sum_{k=1}^{\infty}\Big(1-\exp\Big(-\sum_{i=1}^{k}f(a_i)\Big)\Big)\gamma_k(da_1\times\cdots\times da_k)$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\binom{k}{j}(1-e^{-tj})\gamma_k(A^j\times(A^c)^{k-j}),\quad t\geqslant 0.$$

Since

$$\sum_{k=1}^{\infty}\widetilde{P}(\mu(A)=k)=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\binom{k}{j}\gamma_k(A^j\times(A^c)^{k-j})=\sum_{k=1}^{\infty}\gamma_k(X^k-(A^c)^k),$$

it follows that $\sum_{k=1}^{\infty}\gamma_k(X^k-(A^c)^k)=\widetilde{P}(\mu(A)>0)<\infty$.

§ 3. Convergence of Null-Arrays

Let $P\in\mathscr{P}$. A set A in $\mathscr{B}$ is called a continuous set with respect to P if $P(\mu:\ \mu(\partial A)>0)=0$, where ∂A denote the boundary of A. Denote by $\mathscr{B}_P$ the class of all bounded continuous set of P.

A triangular array $(P_{n,j},\ n=1,\ 2,\ \cdots,\ 1\leqslant j\leqslant k_n)$ of distributions on $(N,\ \mathscr{N})$ is called a null-array if

$$\lim_{n}\max_{1\leqslant j\leqslant k_n}P_{n,j}(\mu:\ \mu(A)>0)=0$$

for all $A\in\mathscr{B}$.

Proposition 2 ([1] Theorem 3. 4. 2). *Let $P\in\mathscr{P}_{IN}$ and $(P_{n,j})$ be a null-array. Then*

$$\mathop{*}_{1\leqslant j\leqslant k} P_{n,j} \xrightarrow{W} P$$

iff

$$\mathop{*}_{1\leqslant j\leqslant k} (P_{n,j})_{A_1,\cdots,A_m} \xrightarrow{W} P_{A_1,\cdots,A_m}$$

for all $m\geqslant 1$ and disjoint $A_1, \cdots, A_m \in \mathscr{B}_P$, where $(P_{A_1,\cdots A_m})$ are the finite dimensional distributions of P, or equivalently, iff

$$\sum_{1\leqslant j\leqslant k_n} (P_{n,j})_{A_1,\cdots,A_m}((\cdot)) - (0, \cdots, 0)) \xrightarrow{W} \widetilde{P}_{A_1,\cdots,A_m}((\cdot) - (0, \cdots, 0)),$$

where $\widetilde{P}$ is the canonical measure of P and $(\widetilde{P}_{A_1,\cdots,A_m})$ are the finite dimensional measures of $\widetilde{P}$.

Theorem 2. *Let $(P_{n,j})$ be a null-array of distributions on $(N, \mathscr{N})$ and P be a regular infinitely divisible distribution having Laplace transform* (1). *Let*

$$\lambda(\cdot) = \sum_{k=1}^{\infty} \gamma_k(\cdot \times X^{k-1}).$$

Then $\mathop{*}_{1\leqslant j\leqslant k_n} P_{n,j} \xrightarrow{W} P$ *iff for all finite sequences $A_1, \cdots, A_m$. of pairwise disjoint sets in $\mathscr{B}_\lambda$* (the all bounded continuous sets of λ) *and all nonnegative integers $h_1, \cdots, h_m$, $h_1+\cdots+h_m>0$, the relations*

$$\lim_n \sum_{1\leqslant j\leqslant k_n} P_{n,j}(\mu\colon \mu(A_1) = h_1, \cdots, h\mu(A_m) = h_m)$$
$$= \sum_{h=0}^{\infty} \frac{(h_1+\cdots+h_m+h)!}{h_1!\cdots h_m!h!} \gamma_{h_1+\cdots+h_m+h}(A_1^{h_1}\times\cdots\times A_m^{h_m}\times((A_1\cup\cdots\cup A_m)^c)^h)$$

are satisfied.

Proof For any $A\in\mathscr{B}$, we have

$$\gamma_k(A\times X^{k-1}) \leqslant \widetilde{P}(\mu\colon \mu(A)>0, \mu(X)=k) \leqslant k\gamma_k(A\times X^{k-1}), \quad k=1, 2, \cdots,$$

hence

$$\lambda(A) = \sum_{k=1}^{\infty} \gamma_k(A\times X^{k-1}) \leqslant \widetilde{P}(\mu\colon \mu(A)>0),$$

and $\lambda(\partial A)=0 \Leftrightarrow \widetilde{P}(\mu\colon \mu(\partial A)>0)=0$, so that $\mathscr{B}_\lambda = \mathscr{B}_{\widetilde{P}}$.

Now let

$$e_1 = (1, 0, \cdots, 0), \cdots, e_m = (0, \cdots, 0, 1),$$

and $H=(h_1, \cdots, h_m)$ be in Z^m, $h_1+h_2+\cdots+h_m>0$. Then $H = h_1e_1+\cdots+h_me_m$.

If P is a regular infinitely divisible distribution having Laplace transform (1), then by definitions of ψ_k and γ_k, we obtain

$$\widetilde{P}_{A_1,\cdots,A_m}(H) = \widetilde{P}(\mu\colon \mu(A_1)=h_1, \cdots, \mu(A_m)=h_m)$$
$$= \sum_{h=0}^{\infty} \widetilde{P}(\mu\colon \mu(A_1)=h_1, \cdots, \mu(A_m)=h_m, \mu(X-(A_1\cup\cdots\cup A_m))=h)$$
$$= \sum_{h=0}^{\infty} \gamma_{h_1+\cdots h_m+h}\psi^{-1}_{h_1+\cdots+h_m+h}(\mu\colon \mu(A_1)=h_1, \cdots, \mu(A_m)=h_m, \mu(X-(A_1\cup\cdots\cup A_m))=h)$$
$$= \sum_{h=0}^{\infty} \frac{(h_1+\cdots+h_m+h)!}{h_1!\cdots h_m!h!} \gamma_{h_1+\cdots+h_m+h}(A_1^{h_1}\times\cdots\times A_m^{h_m}\times(X-(A_1\cup\cdots\cup A_m))^h).$$

Now the assertion of the theorem follows from Proposition 2.

Corollary. *$P\in\mathscr{P}_{IN}$ is called a G-P distribution if $\tilde{P}(\mu: \mu(X)>2)=0$* (see [3], [4]). *If P is a G-P distribution, then $\underset{1\leqslant j\leqslant k_n}{*} P_{n,j}\overset{W}{\Longrightarrow} P$ iff for arbitrary disjoint A, $B\in\mathscr{B}_\lambda$, the following conditions are satisfied*

1. $\lim\limits_n \sum\limits_{1\leqslant j\leqslant k_n} P_{n,j}(\mu: \mu(A)>2)=0;$
2. $\lim\limits_n \sum\limits_{1\leqslant j\leqslant k_n} P_{n,j}(\mu: \mu(A)=1,\ \mu(B)=0)=\gamma_1(A)+2\gamma_2(A\times(A\cup B)^c);$
3. $\lim\limits_n \sum\limits_{1\leqslant j\leqslant k_n} P_{n,j}(\mu: \mu(A)=1,\ \mu(B)=1)=2\gamma_2(A\times B);$
4. $\lim\limits_n \sum\limits_{1\leqslant j\leqslant k_n} P_{n,j}(\mu: \mu(A)=2)=\gamma_2(A\times A),$

where $\lambda(\cdot)=\gamma_1(\cdot)+\gamma_2((\cdot)\times X)$.

§ 4. The Stationary Case

Let $X=R^m$. We denote by $\mathscr{R}^m$ and $\mathscr{B}(R^m)$ the Borel σ-algebra and the class of all bounded Borel subsets in R^m respectively.

Lemma. *Let $\gamma_k(\cdot)$ $(k\geqslant 1)$ be a symmetric measure on $((R^m)^k, (\mathscr{R}^m)^k)$ satisfying the following conditions:*

1. *For all $t\in R^m$ and $A_1, \cdots, A_k\in\mathscr{R}^m$,*
$$\gamma_k((A_1+t)\times\cdots\times(A_k+t))=\gamma_k(A_1\times\cdots\times A_k),$$
where $A+t=\{s: s\in R^m,\ s-t\in A\}$;
2. *For all $A\in\mathscr{B}(R^m)$, $\gamma_k(A\times(R^m)^{k-1})<\infty$.*

Then $\gamma_k(A\times(R^m)^{k-1})=hL(A)$ for all $A\in\mathscr{B}(R^m)$ and some constant h, and $\gamma_k(\cdot)$ can be expressed in the form:
$$\gamma_k(\cdot)=(L\times C_k)\varphi_k(\cdot),$$
where L is the Lebesgue measure on R^m, C_k is a symmetric locally finite measure on $(R^m)^{k-1}$, and φ_k is the following mapping
$$\varphi_k: (x_1, \cdots, x_k)\to\left(\frac{x_1+\cdots+x_k}{\sqrt{k}}, \frac{x_1-x_2}{\sqrt{k}}, \cdots, \frac{x_1-x_k}{\sqrt{k}}\right).$$

Proof Let $A\in\mathscr{B}(R^m)$, From condition 1, it follows that
$$\gamma_k((A+t)\times(R^m)^{k-1})=\gamma_k(A\times(R^m)^{k-1})<\infty,$$
hence we have $\gamma_k(A\times(R^m)^{k-1})=hL(A)$ for some constant h.

Now if we denote $\tilde{\gamma}_k(\cdot)=\gamma_k\varphi_k^{-1}(\cdot)$, then
$$\tilde{\gamma}_k((A+t)\times B)=\tilde{\gamma}_k(A\times B)$$
for all $t\in R^m$, $A\in\mathscr{R}^m$, $B\in(\mathscr{R}^m)^{k-1}$. Therefore, for fixed $B\in(\mathscr{R}^m)^{k-1}$,
$$\tilde{\gamma}_k(\cdot\times B)=L(\cdot)C_k(B).$$
It is easy to verify that the set function $C_k(\cdot)$ is a symmetric locally finite measure on $(R^m)^{k-1}$, and hence
$$\tilde{\gamma}_k=L\times C_k,\quad \gamma_k=(L\times C_k)\varphi_k.$$

Combining Theorems 1 and 2 with lemma we can deduce the Laplace transform for the stationary regular infinitely divisible distribution P, therefore, the necessary and sufficient condition under which the null-array of distributions convenges to P can be obtained.

References

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