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A NOTE ON MORREY-NIKOLŚKII INEQUALITY

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Abstract

Let Q_0 be a Cube in $\mathbf{R}^n$ and $u(x)\in L^p(Q_0)$. Suppose that

$$\int_Q |u(x+t)-u(x)|^p dx \leqslant K^p |t|^{\alpha p} |Q|^{1-\beta/n}$$

for all parallel subcubes Q in Q_0 and for all t such that the integral makes sense with $K\geqslant 0$, $0<\alpha\leqslant 1$, $0\leqslant\beta\leqslant n$ and $p\geqslant 1$. If $\alpha p=\beta$, then $u(x)$ is of bounded mean oscillation on Q_0 (abbreviated to BMO (Q_0)), i. e.

$$\sup_{Q\subset Q_0} \frac{1}{|Q|}\int_Q |u(x)-u_Q|\,dx = \|u\|_* < \infty,$$

where u_Q is the mean value of $u(x)$ over Q.

1. In [1] J. Ross proved the following

Theorem A. *Let Q_0 be a cube in R^n and $u(x)\in L^p(Q_0)$. Suppose that*

$$\int_Q |u(x+t)-u(x)|^p dx \leqslant K^p |t|^{\alpha p} |Q|^{1-\frac{\beta}{n}} \tag{1.1}$$

for all parallel subcubes Q in Q_0 and for all t such that the integral is defined, the constants in (1.1) are assumed to satisfy $K\geqslant 0$, $0<\alpha\leqslant 1$, $0\leqslant\beta\leqslant n$ and $p\geqslant 1$. Then

(i) *If $\alpha p<\beta$, then the function $u(x)$ is in $L^r(Q_0)$ for all r satisfying $\frac{1}{r}>\frac{1}{p}-\frac{\alpha}{\beta}$, and*

$$\left(\int_{Q_0} |u(x)-u_{Q_0}|^r dx\right)^{\frac{1}{r}} \leqslant CK,$$

where u_{Q_0} is the average value of $u(x)$ on Q_0.

(ii) *If $\alpha p>\beta$, then the function $u(x)$ is Hölder continuous in Q_0 with exponent $\bar{\alpha}=\alpha-\frac{\beta}{p}$, and*

$$|u(x)-u(y)| \leqslant CK|x-y|^{\bar{\alpha}} \quad \text{for all } x,\ y\in Q_0.$$

The constant, C, depends only on $|Q_0|$, α, β, p, n and in the case (i) on r. Here $|Q|$ is the volume of the cube Q.

From the above theorem, it is natural to ask what happens if $\alpha p=\beta$. The aim of this note is to prove the following.

Theorem. *Let Q_0 be a cube in R^n and $u(x)\in L^p(Q_0)$. Suppose that*

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$$\int_Q |u(x+t)-u(x)|^p\,dx \leqslant K^p|t|^{\alpha p}|Q|^{1-\frac{\beta}{n}}$$

for all parallel subcubes, Q in Q_0 and for all t such that the integral is defined. The constants are assumed to satisfy $K\geqslant 0$, $0<\alpha\leqslant 1$, $0\leqslant\beta\leqslant n$ and $p\geqslant 1$. If $\alpha p=\beta$, then $u(x)$ is of bounded mean oscillation on Q_0 (abbreviated as BMO (Q_0)), i. e.,

$$\sup_{Q\subset Q_0}\frac{1}{|Q|}\int_Q |u(x)-u_Q|\,dx=\|u\|_*<\infty, \tag{1.2}$$

where Q is any parallel subcube in Q_0, and u_Q is the mean value of u over Q.

The BMO functions were first introduced by John and Nirenberg[2] in 1961. An important feature of these functions is the exponential nature of their distribution functions. That is

$$|\{x\in Q_0 \mid |u(x)-u_{Q_0}|>\alpha\}|\leqslant e^{-b\alpha/\|u\|_*}|Q_0| \quad (\alpha>0). \tag{1.3}$$

From this we get following result:

Corollary. *If $u(x)\in L^p(Q_0)$ and satisfies the condition (1.1), then $\exp\{\lambda|u(x)-u_{Q_0}|\}$ is integrable for $\lambda<b/\|u\|_*$.*

The proof of Theorem is given in section 2 based on a lemma, which enables us also to prove the second part of Theorem A easily, so in the section 3 we will give a simple proof of part (ii) of Theorem A.

2. Proof of Theorem.

Lemma. *Let*

$$I_n=[a_1,\ b_1]\times[a_2,\ b_2]\times\cdots\times[a_n,\ b_n]$$

be a n-dimensional cube, whose sides have length $h=b_k-a_k$ $(k=1, 2, \cdots, n)$, and let $f(x)\in L^p(I_n)$, $p\geqslant 1$. Then

$$\begin{aligned}\int_{I_n}\int_{I_n}&|f(x)-f(y)|^p\,dx\,dy\\ &=\sum_{\theta_1,\theta_2,\cdots,\theta_n=0}^{1}\int_0^h\cdots\int_0^h\int_{a_1}^{b_1-\xi_1}\cdots\int_{a_n}^{b_n-\xi_n}|f(t_1+\theta_1\xi_1,\ t_2+\theta_2\xi_2,\ \cdots,\ t_n+\theta_n\xi_n)\\ &\quad -f(t_1+\bar\theta_1\xi_1,\ t_2+\bar\theta_2\xi_2,\ \cdots,\ t_n+\bar\theta_n\xi_n)|^p\,dt_1dt_2\cdots dt_n d\xi_1 d\xi_2\cdots d\xi_n,\end{aligned} \tag{2.1}$$

where $\bar\theta_i=1-\theta_i$ $(i=1, 2, \cdots, n)$.

Proof of Lemma We first prove that, if $f(x)$, $g(x)\in L^p(a, b)$, then

$$\int_a^b\int_a^b|f(x)-g(y)|^p\,dx\,dy=\int_0^{b-a}du\int_a^{b-u}(|f(y+u)-g(y)|^p+|g(y+u)-f(y)|^p)dy. \tag{2.2}$$

In fact, the left-hand side is equal to

$$\begin{aligned}&\int_a^b dy\int_{a-y}^{b-y}|f(u+y)-g(y)|^p du\\ &=\int_0^{b-a}du\int_a^{b-u}|f(u+y)-g(y)|^p dy+\int_{a-b}^0 du\int_{a-u}^b|f(u+y)-g(y)|^p dy\\ &=\int_0^{b-a}\left\{\int_a^{b-u}|f(u+y)-g(y)|^p dy+\int_{a+u}^b|f(y-u)-g(y)|^p dy\right\}du\\ &=\int_0^{b-a}du\int_a^{b-u}(|f(y+u)-g(y)|^p+|g(y+u)-f(y)|^p)dy.\end{aligned}$$

Now, return to prove the lemma by induction. If $n=1$, (2.1) is a special case of (2.2) with $g(y)=f(y)$. Now, suppose that the lemma is true for $n=k-1$, $k\geq 2$ being a positive integer, then the left-hand side of (2.1) is equal to

$$\begin{aligned}
&\int_{a_k}^{b_k} dx_k \int_{a_k}^{b_k} dy_k \int_{I_{k-1}}\int_{I_{k-1}} |f(x_1, x_2, \cdots, x_{k-1}, x_k) \\
&\quad -f(y_1, y_2, \cdots, y_{k-1}, y_k)|^p dx_1 dx_2\cdots dx_{k-1} dy_1 dy_2\cdots dy_{k-1} \\
&= \int_{a_k}^{b_k} dx_k \int_{a_k}^{b_k} dy_k \sum_{\theta_1,\cdots,\theta_{k-1}=0}^{1} \int_0^h \cdots \int_0^h \int_{a_1}^{b_1-\xi_1}\cdots \int_{a_{k-1}}^{b_{k-1}-\xi_{k-1}} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \\
&\quad \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, x_k) - f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \\
&\quad \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, y_k)|^p dt_1\cdots dt_{k-1} d\xi_1\cdots d\xi_{k-1} \\
&= \sum_{\theta_1,\cdots,\theta_{k-1}=0}^{1} \int_0^h \cdots \int_0^h \int_{a_1}^{b_1-\xi_1}\cdots \int_{a_{k-1}}^{b_{k-1}-\xi_{k-1}} dt_1\cdots dt_{k-1}\, d\xi_1\cdots d\xi_{k-1} \\
&\quad \times \Big\{ \int_{a_k}^{b_k}\int_{a_k}^{b_k} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, x_k) \\
&\quad -f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, y_k)|^p dx_k dy_k \Big\}. \qquad (2.3)
\end{aligned}$$

By (2.2), the expression inside the brace can be written as

$$\begin{aligned}
&\int_0^h d\xi_k \int_{a_k}^{b_k-\xi_k} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, t_k+\xi_k) \\
&\quad -f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, t_k)|^p \\
&\quad +|f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, t_k) \\
&\quad -f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, t_k+\xi_k)|^p dt_k \\
&= \sum_{\theta_k=0}^{1} \int_0^h d\xi_k \int_{a_k}^{b_k-\xi_k} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, t_k+\theta_k\xi_k) \\
&\quad -f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, t_k+\overline{\theta}_k\xi_k)|^p dt_k. \qquad (2.4)
\end{aligned}$$

Combining (2.3) and (2.4), we get

$$\begin{aligned}
&\int_{I_k}\int_{I_k} |f(x)-f(y)|^p dx\, dy \\
&= \sum_{\theta_1,\cdots,\theta_{k-1}=0}^{1} \int_0^h \cdots \int_0^h \int_{a_1}^{b_1-\xi_1}\cdots \int_{a_{k-1}}^{b_{k-1}-\xi_{k-1}} dt_1\cdots dt_{k-1} d\xi_1\cdots d\xi_{k-1} \\
&\quad \times \sum_{\theta_k=0}^{1} \int_0^h d\xi_k \int_{a_k}^{b_k-\xi_k} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_k+\theta_k\xi_k) \\
&\quad -f(t_1+\overline{\theta}\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_k+\overline{\theta}_k\xi_k)|^p dt_k \\
&= \sum_{\theta_1,\cdots,\theta_k=0}^{1} \int_0^h \cdots \int_0^h \int_{a_1}^{b_1-\xi_1}\cdots \int_{a_k}^{b_k-\xi_k} |f(t_1+\theta_1\xi_1, t_2+\theta_2\xi_2, \cdots, t_{k-1}+\theta_{k-1}\xi_{k-1}, t_k+\theta_k\xi_k) \\
&\quad -f(t_1+\overline{\theta}_1\xi_1, t_2+\overline{\theta}_2\xi_2, \cdots, t_{k-1}+\overline{\theta}_{k-1}\xi_{k-1}, t_k+\overline{\theta}_k\xi_k)|^p dt_1\cdots dt_k d\xi_1\cdots d\xi_k.
\end{aligned}$$

This completes the proof of the lemma.

Now return to the proof of the Theorem. Given any cube

$$I_n=[a_1, a_1]\times[a_2, b_2]\times\cdots\times[a_n, b_n]\subset Q_0 \quad (b_i-a_i=h,\ i=1, 2, \cdots, n), \qquad (2.5)$$

we have

$$\frac{1}{|I_n|}\int_{I_n}|u(x)-u_{I_n}|\,dx=\frac{1}{|I_n|}\int_{I_n}\left|u(x)-\frac{1}{|I_n|}\int_{I_n}u(y)\,dy\right|dx$$
$$\leqslant\frac{1}{|I_n|^2}\int_{I_n}\int_{I_n}|u(x)-u(y)|\,dx\,dy\leqslant\frac{1}{|I_n|^{2/p}}\left(\int_{I_n}\int_{I_n}|u(x)-u(y)|^p\,dx\,dy\right)^{\frac{1}{p}}. \quad (2.6)$$

In virtue of (1.1) and lemma, the p-th power of the last multiple integral is not greater than

$$\sum_{\theta_1,\cdots,\theta_n=0}^{1}\int_0^h\cdots\int_0^h d\xi_1 d\xi_2\cdots d\xi_n\int_{a_1}^{b_1-\xi_1}\cdots\int_{a_n}^{b_n-\xi_n}|u(t_1+\theta_1\xi_1,\ t_2+\theta_2\xi_2,\ \cdots,\ t_n+\theta_n\xi_n)$$
$$-u(t_1+\bar\theta_1\xi_1,\ t_2+\bar\theta_2\xi_2,\ \cdots,\ t_n+\bar\theta_n\xi_n)|^p\,dt_1dt_2\cdots dt_n$$
$$=\sum_{\theta_1,\cdots\theta_n=0}^{1}\int_0^h\cdots\int_0^h d\xi_1 d\xi_2\cdots d\xi_n\int_{a_1+\theta_1\xi_1}^{b_1-\bar\theta_1\xi}\cdots\int_{a_n+\theta_n\xi_n}^{b_n-\bar\theta_n\xi_n}|u(u_1,\ u_2,\ \cdots,\ u_n)$$
$$-u(u_1+(1-2\theta_1)\xi_1,\ u_2+(1-2\theta_2)\xi_2,\ \cdots,\ u_n+(1-2\theta_n)\xi_n)|^p\,du_1du_2\cdots du_n. \quad (2.7)$$

Now, let δ_0 denote the half length of the side of the cube Q_0. Notice the fact that for $0<h<\delta_0/2$, there are at least n numbers in

$$a_1,\ b_1,\ a_2,\ b_2,\ \cdots,\ a_n,\ b_n,$$

say $a_{i_1}, a_{i_2}, \cdots, a_{i_s}; b_{j_1}, b_{j_2}, \cdots, b_{j_r}$ $(r+s=n;\ i_l\neq j_\sigma,\ l=1,\ \cdots,\ s;\ \sigma=1,\ \cdots,\ r)$, such that the rectangle

$$[c_1,\ d_1]\times[c_2,\ d_2]\times\cdots\times[c_i,\ d_i]\times\cdots\times[c_n,\ d_n]\subset Q_0 \quad (2.8)$$

with $c_i=a_i-h,\ d_i=b_i$, if $i=i_l$ $(l=1,\ \cdots,\ s)$; $c_i=a_i,\ d_i=b_i+h$, if $i=j_\sigma$ $(\sigma=1,\ \cdots,\ r)$. By (1.1) and the Minkowski's inequality, each inner multiple integral is not greater than

$$2^pK^p(\sqrt{n}\,h)^{\alpha p}h^{n-\beta}=2^pn^{\alpha p/2}K^ph^{\alpha p+n-\beta}, \quad (2.9)$$

provided that $h<\delta_0/2$, since $\alpha p=\beta$. From (2.6)–(2.9), it follows that

$$\frac{1}{|I_n|}\int_{I_n}|u(x)-u_{I_n}|\,dx\leqslant 2^{1+\frac{n}{p}}n^{\alpha/2}K\cdot h^{\alpha-\frac{\beta}{p}}=K_{n,\alpha,p}h^{\alpha-\frac{\beta}{p}}=K_{n,\alpha,p}, \quad (2.10)$$

$K_{n,\alpha,p}$ being a constant depending only on n, α, p.

This is what we want to prove.

3. The argument enables us to prove the part (ii) of Ross's Theorem A easily. In fact, under the hypothesis in (ii) of Theorem A, (2.10) gives the estimate

$$\frac{1}{|I_n|}\int_{I_n}|u(x)-u_{I_n}|\,dx\leqslant K_{n,\alpha,p}h^{\alpha-\frac{\beta}{p}}$$

The result now follows from Meyers[3] or Campanato[4].

Remark. Ross's proof for (ii) of his Theorem A does not work in the case $\alpha p=\beta$.

References

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