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THE STRUCTURE OF Π SPACES (I)

YAN SHAOZONG (严绍宗)
(*Institute of Mathematics Fudan University*)

Abstract

The spectral theory of selfadjoint operators and unitary operators in Hilbert space has been successfully generalized to Π_k space. However, there are only a few results for the spectral theory of selfadjoint operators and unitary operators in Π space. One of the important reasons is that the structure of Π space is more complex than that of Π_k space. This paper and the forthcoming paper "The structure of Π space(II)" will mainly be dealt with the structure of Π spaces, which will be used to further study the operators in Π spaces.

§ 1. Preliminary

In Π_k space, provided that a closed subspace L is non-degenerate, then $\Pi_k = L \oplus L^{\perp}$ holds[5]. This property of the structure of the space has a great influence upon the operator theory for Π_k spaces. But, in Π spaces, the above decomposition is no longer true[1]. In Π_k space, a closed, non-degenerate subspace L is still a Π_k-type space relative to the inner product of Π_k space, but in Π space, this is also no longer true. All of these facts constitutes one of the important reasons that there are only a few results for operator theory in Π space. In this section, we give a description of the operators defined in subspaces of Π space.

By Π we denote a Krein space[4] with an inner product $(\cdot, \cdot)$. Decompose $\Pi = H_+ \oplus H_-$, and call it a regular decomposition if $H_\pm$ are Hilbert spaces relative to $\pm(\cdot, \cdot)$ respectively[3]. The inner product and the norm induced by the regular decomposition are denoted by $[\cdot, \cdot]$ and $\|\cdot\|$. Write the orthogonal projections from a Hilbert space $(\Pi, [\cdot, \cdot])$ onto $H_\pm$ by $P_\pm$. Suppose that A is a linear operator from a Hilbert space $(H_-, -(\cdot, \cdot))$ into a Hilbert space $(H_+, (\cdot, \cdot))$. We call $L_A = \{\{x, Ax\} \mid x \in \mathscr{D}(A)\}$ a linear subspace induced by A, where $\{x, Ax\}$ represents the vector $x + Ax$.

Lemma 1.1. *The following propositions hold:*

(i) L_A *is a semi-positive (semi-negative) subspace of* Π *if and only if* A *is a contraction (extension).*

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(ii) *L_A is a closed subspace of Π if and only if A is a closed operator.*

(iii) *L_A is a positive(negative) subspace of Π if and only if for any $x\in\mathscr{D}(A)$, $x\neq 0$, $\|Ax\|<\|x\|$ ($\|Ax\|>\|x\|$) holds.*

(iv) *The maximal semi-positive or maximal semi-negative subspaces of Π must be closed subspaces.*

(v) *L_A is a maximal semi-negative subspace of Π if and only if A is a contraction, and $\mathscr{D}(A)=H_-$.*

(vi) *L_A is a closed, maximal negative subspace of Π if and only if $\mathscr{D}(A)=H_-$, and for any $x\in H_-$, $x\neq 0$, $\|Ax\|<\|x\|$ holds.*

(vii) *If L is a maximal semi-negative subspace of Π, then there must exist a contraction A, $\mathscr{D}(A)=H_-$, such that $L=L_A$.*

(viii) *If L is a maximal negative, closed subspace of Π, then there must be a constant α, $0\leqslant\alpha<1$, and an operator A, $\mathscr{D}(A)=H_-$, $\|A\|\leqslant\alpha$, such that $L=L_A$.*

It is not hard to prove this lemma. Some parts of the proof were also given in [1, 2].

We give an example that a maximal negative subspace is not closed as follow.

Example 1 $\Pi=l_-\oplus l_+$, where both l_- and l_+ are of l^2. Let $\{e_i^\pm\}$ be the complete orthonormal system of $l_\pm$. Let $z=e_1^++e_1^-$, $l'_\pm=\overline{\text{span}}\ \{e_i^\pm\,|\,i\geqslant 2\}$, and f be an unbounded linear functional defined on the whole space l'_-. Consider a subspace of Π,

$$L=\{x+f(x)z\,|\,x\in l'_-\}. \tag{1.1}$$

Since $z\overline{\in}L$ and $z\in\overline{L}$, so L is not a closed subspace of Π. It is easy to see that L is negative. Now we prove L is a maximal negative subspace. Suppose that this is false. There must be a negative subspace $L'\supset L$. Take $y\in L'$, $y\overline{\in}L$ arbitrarily. Write $y=y'_++ae_1^++be_1^-+y'_-$, $(y'_\pm\in l'_\pm)$. Since $y'_-+f(y'_-)z\in L$, thus

$$0\neq y-(y'_-+f(y'_-)z)=y'_++ae_1^++be_1^--f(y'_-)z\in L'. \tag{1.2}$$

It is evident that $ae_1^++be_1^--f(y'_-)z\neq 0$, and that $a\neq b$. Since f is unbounded, thus there exist $\{x_n\}\subset l'_-$, $\|x_n\|\leqslant\dfrac{1}{n}$, $f(x_n)=1$. Define a sequence in L'

$$\begin{aligned}h_n&=y'_++ae_1^++be_1^--f(y'_-)z-(b-f(y'_-))(x_n+f(x_n)z)\\&=y'_++(a-b)e_1^+-(b-f(y'_-))x_n.\end{aligned} \tag{1.3}$$

It is easy to see that for sufficient large n, h_n are positive vectors, which contradicts the hypothesis that L' is negative. This completes the proof.

If $\overline{\mathscr{D}(A)}=H_-$, by L_{A^*} we denote the subspace $\{\{A^*y,\ y\}\,|\,y\in\mathscr{D}(A^*)\subset H_+\}$.

Lemma 1.2. *Let A be a dense defined operator from H_- into H_+. The following propositions hold:*

(i) *$L_A^\perp=L_{A^*}$. $(L_A^\perp)^\perp=L_{A^{**}}$ if $\overline{\mathscr{D}(A^*)}=H_+$.*

(ii) *L_A is a non-degenerate subspace of Π if and only if $1\in\sigma_p(A^*A)$* (or $1\overline{\in}\sigma_p(AA^*)$).

(iii) $L_A \oplus L_A^\perp = \Pi$ *if and only if* $\mathscr{R}(I-A^*A) = H_-$, $\mathscr{R}(I-AA^*) = H_+$, *or* A *is a dense defined, closed operator, and* $1 \in \rho(AA^*) \cap \rho(A^*A)$.

(iv) L_A *is a maximal negative subspace of* Π *and* $\Pi = L_A \oplus L_A^\perp$ *is a regular decomposition if and only if* $\mathscr{D}(A) = H_-$, $\|A\| \leqslant \alpha < 1$ (α *is a constant*).

This lemma can be proved immediately. A part of its proof can also be found in [1, 2]. In [1], an example was given to illustrate that a maximal negative, closed subspace of Π needn't be a Hilbert space according to $-(\cdot, \cdot)$. In the following example, we give a positive subspace which is not a closed subspace of Π, but $(L, (\cdot, \cdot))$ is a Hilbert space.

Example 2 Let Π, $\{e_i^\pm\}$, z, and $l'_\pm$ be as in example 1. Let $\Phi_0 = \left\{ \sum_{i=2}^{n} x_i e_i^+ \mid n \geqslant 2. \right\}$ Denote the quotient space $\Phi = l'_+/\Phi_0$. Take a non-zero linear functional f in Φ. For any $x \in l'_+$, by $\tilde{x}$ we denote the equivalence class containing x. Put

$$L = \{f(\tilde{x})z + x \mid x \in l'_+\}.$$

It is evident that L is a positive subspace of Π, and it is not closed (since $z \in \bar{L}$, but $z \bar{\in} L$). Now we show that L is a Hilbert space relative to $(\cdot, \cdot)$. In fact, suppose that $\{f(\tilde{x}_n)z + x_n\} \subset L$, and it is a fundamental sequence. Since

$$(f(\tilde{x})z + x, f(\tilde{x})z + x) = (x, x). \tag{1.4}$$

$\{x_n\}$ is a fundamental sequence, hence there is $x \in l'_+$ such that

$$\lim_{n\to\infty} (x_n - x, x_n - x) = 0. \tag{1.5}$$

Thus $(f(\tilde{x}_n)z + x_n - (f(\tilde{x})z + x), f(\tilde{x}_n)z + x_n - (f(\tilde{x})z + x)) = (x_n - x, x_n - x) \to 0$, $(n \to \infty)$, i. e. $\{f(\tilde{x}_n)z + x_n\}$ converges to the vector of L, $f(\tilde{x})z + x$. Therefore L is a Hilbert space relative to $(\cdot, \cdot)$. This completes the proof.

Let L be a linear subspace of Π, $\Pi = H_- \oplus H_+$ be a regular decomposition. Denote $L_0^+ = \{\{0, y\} \mid \{0, y\} \in L\}$, $L_0^- = \{\{x, 0\} \mid \{x, 0\} \in L\}$.

Lemma 1.3. *The following propositions hold;*

(i) $L = L_0^- \dotplus L_A \dotplus L_0^+$, *where* A *is a linear operator from* H_- *into* H_+, $\mathscr{D}(A) \subset H_-$, *and* A *is an injection.*

(ii) *If* L *is a closed subspace, there must be a unique decompositon* $L = L_0^- \oplus L_A \oplus L_0^+$. *Besides, if there is a decomposition of subspace* L, $L = L_0^- \oplus L_A \oplus L_0^+$, *then* L *is closed if and only if* $L_0^\pm$, L_A *are all closed subspaces.*

(iii) L *is a closed subspace of* Π *if and only if* $L = (L^\perp)^\perp$.

(iv) *For any linear subspace* L *of* Π, $\bar{L} = (L^\perp)^\perp$ *holds. If* $\bar{L} \neq \Pi$, *there must be a non-zero vector* $y \in \Pi$ *such that* $y \perp \bar{L}$.

(v) *A closed subspace* L *is non-degenerate if and only if for decomposition* $L = L_0^- \oplus L_A \oplus L_0^+$, L_A *is non-degenerate.*

Proof (i) It is evident that $L_0^\pm = L \cap H_\pm$, $L_0^+ \cap L_0^- = \{0\}$. Put $\Phi = L/(L_0^+ \dotplus L_0^-)$. By the ultralimit induction, we choose representative elements x of $\tilde{x} \in \Phi$ such that all these representative elements constitute a linear space $L_1=\{x \mid x\in\tilde{x},\ \tilde{x}\in\Phi\}$. Thus $L=L_0^- \dotplus L_1 \dotplus L_0^+$. For any $x\in L_1$, we have a decomposition $x=x_-+x_+$. It is easy to see that $x_+=0$ if $x_-=0$. Thus there exists a linear injection A from $\mathscr{D}(A)\subset H_-$ into H_+, such that $L_1=L_A$.

(ii) If L is a closed subspace, $L_0^\pm$ are obviously closed, and $L=(L_0^+\oplus L_0^-)\oplus L_1'$, where $L_1'=\{x \mid [x,\ x_\pm]=0,\ x_\pm\in L_0^\pm,\ x\in L\}$. Similar to (i), $L_1'=L_A$, $\mathscr{D}(A)\perp L_0^-$, $\mathscr{R}(A)\perp L_0^+$. Since L is closed, hence L_A is also closed. For the closed subspace L, it is obvious that the decomposition $L=L_0^-\oplus L_A\oplus L_0^+$ is unique. Thus A is determined by L uniquely. On the contrary, if L can be decomposed into $L=L_0^-\oplus L_A\oplus L_0^+$, where A is an injection, $L_0^\pm$ and L_A are closed subspaces, then it is easy to see that L is closed.

(iii) Suppose that L is a closed subspace. From (ii), $L=L_0^-\oplus L_A\oplus L_0^+$. Put

$$\Pi^{(0)}=\overline{\mathscr{D}(A)}\oplus\overline{\mathscr{R}(A)} \tag{1.6}$$

Obviously, $\Pi^{(0)}$ is a Krein space, (1.6) is a regular decomposition of $\Pi^{(0)}$, and under the above decomposition of $\Pi^{(0)}$, A is a dense defined, closed operator from $\overline{\mathscr{D}(A)}$ into $\overline{\mathscr{R}(A)}$, as well as it has dense range. Thus, for $\Pi^{(0)}$ space, $(L_A^\perp)^\perp=(L_{A^*})^\perp=L_{A^{**}}=L_A$. From this it follows that for Π space

$$L^\perp=(H_-\ominus(L_0^-\oplus\overline{\mathscr{D}(A)}))\oplus L_{A^*}\oplus(H_+\ominus(L_0^+\oplus\overline{\mathscr{R}(A)})). \tag{1.7}$$

Hence $(L^\perp)^\perp=L_0^-\oplus L_A\oplus L_A^+=L$.

Conversely, if $(L^\perp)^\perp=L$, L is evidently a closed subspace.

(iv) From (iii), $\overline{L}=(\overline{L}^\perp)^\perp$. But for any linear subspace L, $L^\perp=\overline{L}^\perp$ holds, thus $\overline{L}=(L^\perp)^\perp$. Suppose that $\overline{L}\neq\Pi$. Make the decomposition $\overline{L}=L_0^-\oplus L_A\oplus L_0^+$, where for $\Pi^{(0)}=\overline{\mathscr{D}(A)}\oplus\overline{\mathscr{R}(A)}$, A is a dense defined closed injection from $\overline{\mathscr{D}(A)}$ into $\overline{\mathscr{R}(A)}$. If $L_A=\{0\}$, it is obvious that one of $H_-\ominus L_0^-$ and $H_+\oplus L_0^+$ must be nonempty, so that there is a non-zero vector $y\in\Pi$ such that $y\perp\overline{L}$. If $L_A\neq\{0\}$, then for any non-zero vector $y\in L_{A^*}\subset\Pi^{(0)}$, we have $y\perp\overline{L}$.

(v) A closed subspace L is degenerate if and only if $L\cap L^\perp\neq\{0\}$, i. e. there is a non-zero vector $z\in L$, $z\perp L$. Since $L_0^\pm$ is non-degenerate, hence we must have $z\in L_A$. Therefore, that L_A is degenerate is equivalent to that L is degenerate. This completes the proof.

It should be noted that in Lemma 1.3, we have $y\perp\overline{L}$, which does not mean that there exists a nonzero vector $y\overline{\in}L$ and $y\perp\overline{L}$ (however, which means this fact in a Π_κ space). The following is an example.

Example 3 Let $\Pi=l_-\oplus l_+$ as in example 1. $L=\{\{x,\ x\} \mid x\in l^2\}$. It is evident that L is a neutral, closed subspace of Π. From Lemma 1.1, it follows that L is a maximal semi-negative as well as a maximal semi-positive subspace. Obviously, $\overline{L}\neq\Pi$, and there is not nonzero vector $y\overline{\in}L$ satisfying $y\perp L$.

Example 3 illustrates that when we deal with an orthogonal set of subspaces, we should be more careful for Π space than for Π_k space.

§ 2. Complete subspaces

Definition. *Let $(\Pi, (\cdot, \cdot))$ be a Krein space, L be a closed linear subspace of Π. If $(L, (\cdot, \cdot))$ is still a Π-type space (i. e. a Krein space or a Pontrjagin space), then we call L a complete subspace of Π.*

An example is given in [1] to show that a closed subspace needn't be a Krein space relative to $(\cdot, \cdot)$. Example 2 of this paper illustrates that a subspace of Π which is a Krein space relative to $(\cdot, \cdot)$ needn't be closed.

Lemma 2. 1. *Let $(\Pi, (\cdot, \cdot))$ be a space with an indefinite metric. Let $\Pi^{(1)}$ and $\Pi^{(2)}$ be two linear subspaces, and $\Pi=\Pi^{(1)}\oplus\Pi^{(2)}$.*

(i) *If $(\Pi^{(i)}, (\cdot, \cdot))$ $(i=1, 2)$ are Π-type space, then $(\Pi, (\cdot, \cdot))$ is also a Π-type space;*

(ii) *If $(\Pi^{(i)}, (\cdot, \cdot))$ $(i=1, 2)$ are Π-type space, and L is a linear subspace of $\Pi^{(1)}$, then L is a closed* (or complete) *subspace of Π if and only if L is a closed (or complete) subspace of $\Pi^{(1)}$.*

Proof Suppose that there are regular decompositions

$$\Pi^{(i)}=H_-^{(i)}\oplus H_+^{(i)}, \quad i=1, 2. \tag{2.1}$$

It is easy to see that the following decomposition is also regular

$$\Pi=(H_-^{(1)}\oplus H_-^{(2)})\oplus(H_+^{(1)}+H_+^{(2)}). \tag{2.2}$$

From the above fact, the conclusions of lemma 2.1 (i), (ii) follow at once.

The following corollary is also evident.

Corollary 2.2. *Let $\Pi=H_-\oplus H_+$ be a regular decomposition, and L be a closed linear subspace. Then*

(i) *there exist four complete subspaces $\Pi^{(i)}$ $(i=0, 1, 2, 3)$ such that*

$$\Pi=\Pi^{(0)}\oplus\Pi^{(1)}\oplus\Pi^{(2)}\oplus\Pi^{(3)}, \tag{2.3}$$

which have the relations with the decomposition $L=L_0^-\oplus L_A\oplus L_0^+$ as follows

$$\Pi^{(0)}=\overline{\mathscr{D}(A)}\oplus\overline{\mathscr{R}(A)},\ \Pi^{(1)}=L_0^-,\ \Pi^{(2)}=L_0^+,$$
$$\Pi^{(3)}=(H_-\ominus(L_0^-\oplus\overline{\mathscr{D}(A)}))\oplus(H_+\ominus(L_0^+\oplus\overline{\mathscr{R}(A)})) \tag{2.4}$$

(ii) *if L_A is a complete subspace, then L is also a complete subspace.*

In fact, the converse proposition of corollary 2.2 (ii) is also true (see corollary 2.5).

Theorem 2.3. *L is a semi-negative (or semi-poistive) complete subspace of Π if and only if $\Pi=L\oplus L^\perp$ holds.*

Proof (Only consider the case that L is semi-negative) Since L is semi-negative and closed, so $L_0^+=\{0\}$, $L=L_0^-\oplus L_A$, and L^- is closed. By Lemma 2.1 and Corollary

2.2, it follows that L is a Hilbert space relative to $-(\cdot,\ \cdot)$ if and only if L_A is a Hilbert space relative to $-(\cdot,\ \cdot)$. Thus we may assume that $L=L_A$, and that under the regular decomposition $\Pi=H_-\oplus H_+$, A is a dense defined, closed injection from H_- into H_+, $\overline{\mathscr{R}(A)}=H_+$.

Necessity L_A is semi-negative $\Leftrightarrow \|Ax\|\leqslant\|x\|$ for any $x\in\mathscr{D}(A)$. Since A is dense defined and closed, thus $\mathscr{D}(A)=H_-$. Let $A=UR$ be the polar decompotition of A (obviously, U is a unitary operator from H_- onto H_+). From Lemma 1.2, $\Pi=L\oplus L^\perp$ will follow once it is shown that there is $\varepsilon_0>0$ such that

$$\|A\|=\|R\|\leqslant 1-\varepsilon_0. \tag{2.5}$$

Suppose that there is no ε_0 satisfying (2.5). Since L is a complete subspace, hence for any ε, $0<\varepsilon<1$,

$$(E_{1-0}-E_{1-\varepsilon})H_-\neq\{0\}, \tag{2.6}$$

where E_t $(0\leqslant t\leqslant 1)$ is the spectral family of R.

(I) Assume that $(H_-,\ -(\cdot,\ \cdot))$ is separable. By (2.6), it follows that there exists a finite scalar measure μ defined on the class $\mathscr{B}$ of Borel sets contained in $[0,1]$, such that for any ν, $1>\nu>0$, $\mu([1-\nu,\ 1))\neq 0$ holds, and $(H_-,\ -(\cdot,\cdot))$ is unitarily isomorphic to $L^2([0,\ 1],\ \mathscr{B},\ \mu;\ \mathscr{H})$, where $\mathscr{H}$ is a separable Hilbert space, $\mathscr{H}(t)$ $(0\leqslant t\leqslant 1)$ is a family of closed subspaces of $\mathscr{H}$, and $L^2([0,\ 1],\ \mathscr{B},\ \mu;\ \mathscr{H})$ is the space of all strongly measurable and square integrable (respect to μ) functions f, which are defined on $[0,\ 1]$, take values on $\mathscr{H}$, and $f(t)\in\mathscr{H}(t)$. The operator R on H_- is equivalent to

$$(\hat{R}f)(t)=tf(t),\ f\in L^2([0,\ 1],\ \mathscr{B},\ \mu;\ \mathscr{H}). \tag{2.7}$$

Take a strongly measurable function f on $[0,\ 1]$ satisfying the following conditions

$$\int_0^{1-\varepsilon}\|f(t)\|^2d\mu(t)<\infty,\quad \int_0^1\|f(t)\|^2d\mu(t)=\infty \tag{2.8}$$

$$\int_0^1(1-t^2)\|f(t)\|^2d\mu(t)<\infty \tag{2.9}$$

for any ε, $0<\varepsilon<1$. Using this function, define a sequence on

$$\mathscr{D}(A)=H_-,\ x_n(t)=\chi_n(t)f(t)\quad (n=1,\ 2,\ \cdots),$$

where $\chi_n(t)$ is the characteristic function of set $\left[0,\ 1-\frac{1}{n}\right]$. From (2.9), it is easy to see that the sequence $\{\{x_n,\ Ax_n\}\}$ in L_A is fundamental according to $-(\cdot,\ \cdot)$. But L_A is complete according to $-(\cdot,\cdot)$, thus there must exist $x(t)\in L^2([0,1],\mathscr{B},\mu;\ \mathscr{H})$ such that $\{x_n,\ Ax_n\}$ converges to $\{x,\ Ax\}$ according to $-(\cdot,\ \cdot)$, i. e.

$$\int_0^1(1-t^2)\|x_n(t)-x(t)\|^2d\mu(t)=-(\{x_n-x,\ A(x_n-x)\},$$
$$\{x_n-x,\ A(x_n-x)\})\to 0\quad (n\to\infty). \tag{2.10}$$

From the definition of $\{x_n(t)\}$ and (2.10), it follows that $x(t)=f(t)$, thus

$$\int_0^1\|x(t)\|^2d\mu(t)=\infty,$$

which contradicts to the assumption that $x(t)\in L^2([0, 1], \mathscr{B}, \mu; \mathscr{H})$.

(II) Suppose that $(H_-, -(\cdot, \cdot))$ is not separable. First, take a vector x_0 in H_- satisfying $\mu_{x_0}((1-\varepsilon, 1))=((E_{1-0}-E_{1-\varepsilon})x_0, x_0)\neq 0$ for any $0<\varepsilon<1$. By H_{x_0} we denote the smallest closed subspace containing x_0, which is invariant for R. Evidently $H_-=H_{x_0}\oplus(H_-\ominus H_{x_0})$. Consequently $H_+=UH_{x_0}\oplus U(H_-\ominus H_{x_0})$. We denote the restrictions of A on H_{x_0} and $H_-\ominus H_{x_0}$ by A_0 and A_1 respectively, so we have $L_A=L_{A_0}\oplus L_{A_1}$ where $\oplus$ is also the orthogonal sum on the Hilbert space $(L_A, -(\cdot, \cdot))$. Therefore, the limits of fundamental sequences in L_{A_0} relative to $-(\cdot, \cdot)$ is still in L_{A_0}. Thus a contradiction now follows by an argument like that given in (I).

The sufficiency is easy to prove. It follows from $\Pi=L_A\oplus L_A^\perp$ that L_A is closed. A is a contraction since L is semi-negative, thus $\mathscr{D}(A)=H_-$. By Lemma 1.2 (iii), it is easy to see that (2.5) holds. If $\{\{x_n, Ax_n\}\}$ is a fundamental sequence in L_A relative to $-(\cdot, \cdot)$, then $\{x_n\}$ must be a fundamental sequence in $(H_-, -(\cdot, \cdot))$ by (2.5), hence there is $x\in H_-$ such that $\{x_n, Ax_n\}\to\{x, Ax\}$, i. e. L_A is complete relative to $-(\cdot, \cdot)$. This completes the proof.

Corollary 2.4. *Let L be a semi-negative (or semi-positive) subspace of Π. If L is a complete subspace, then $L^\perp$ is also a complete subspace of Π.*

Proof (Only consider the case that L is semi-negative) From corollary 2.2, we have $L=L_0^-\oplus L_A\oplus L_0^+$,

$$\Pi=\Pi^{(0)}\oplus\Pi^{(1)}\oplus\Pi^{(2)}\oplus\Pi^{(3)},\ \Pi^{(2)}=L_0^+=\{0\},\ \Pi^{(1)}=L_0^-,$$

$$\Pi^{(0)}=\overline{\mathscr{D}(A)}\oplus\overline{\mathscr{R}(A)},\ \Pi^{(3)}=(H_-\ominus(L_0^-\oplus\mathscr{D}(A)))\oplus(H_+\ominus\overline{\mathscr{R}(A)}).$$

So it is easy to see that $L^\perp=\Pi^{(3)}\oplus L_{A^*}$, $L_{A^*}\subset\Pi^{(0)}$. Since L is a complete subspace, thus L_A is also a complete subspace. Therefore (2.5) holds. Since $\|A^*\|=\|A\|$, hence L_{A^*} is a complete positive subspace, and $L^\perp$ is complete by Corollary 2.2. The proof is completed.

Corollary 2.5. *A closed linear subspace of Π, $L=L_0^-\oplus L_A\oplus L_0^+$ is a complete subspace if and only if L_A is a complete subspace.*

Proof The sufficiency is given in Corollary 2.2 (ii). Now we show the necessity. It is evident that we need only show that L_A is a Π-type space according to $(\cdot, \cdot)$. By assumption, $(L, (\cdot, \cdot))$ is a Π-type space. The decomposition $L=L_0^-\oplus L_A\oplus L_0^+$ is apparently true in $(L, (\cdot, \cdot))$. Thus L_0^-, L_A, L_0^+ are all closed subspace of $(L, (\cdot, \cdot))$. Using Theorem 2.3 and Corollary 2.4 for L_0^- on $(L, (\cdot, \cdot))$, we see that both L_0^- and $L_A\oplus L_0^+$ are complete subspace of $(L, (\cdot, \cdot))$, so $L_A\oplus L_0^+$ is a Π-type space relative to $(\cdot, \cdot)$. Using Theorem 2.3 and Corollary 2.4 again for L_0^+ on Π-type space $(L_A\oplus L_0^+, (\cdot, \cdot))$, it follows at once that L_A is a Π-type space relative to $(\cdot, \cdot)$. Besides, since L_A is a closed subspace of Π, so L_A is a complete subspace of Π. This completes the proof.

Theorem 2.6 (The decomposition theorem of complete subspace). *A linear* *subspace L of Π is a complete subspace if and only if $\Pi=L\oplus L^{\perp}$.*

Proof From Corollary 2.2, it is easy to see that we may assume that $L=L_A$, $\overline{\mathscr{D}(A)}=H_-$, $\overline{\mathscr{R}(A)}=H_+$, and A is an injection.

Necessity. Since L is a complete subspace, thus it is a closed subspace, so A is a dense defined closed operator from H_- into H_+. We have the polar decomposition $A=UR$, where U is a unitary operator from H_- onto H_+ since A is aninjection and $\overline{\mathscr{R}(A)}=H_+$. Let $\mathfrak{H}_-=(E_1-E_{0-0})H_-$, $\mathfrak{H}_+=(E_\infty-E_1)H_-$ where $E_\lambda(-\infty\leqslant\lambda<\infty)$ is the spectral family of R. We denote the restriction of A on $\mathfrak{K}_-$ and, $\mathfrak{K}_+$ by A_- and A_+ respectively. Once we note U is an isometry, it can be seen that $L_A=L_{A_-}\oplus L_{A_+}$ and L_{A_-}, L_{A_+} are semi-negative and semi-positive subspaces respectively. Using Theorem 2.3 for $L_{A_\pm}$ on Π-type space $(L_A, (\cdot,\ \cdot))$, we see that $L_{A_\pm}$ are Π-type spaces relative to $(\cdot,\ \cdot)$, i. e. $L_{A_\pm}$ are Hilbert spaces relative to $\pm(\cdot,\cdot)$. Thus $L_A=L_{A_-}\oplus L_{A_+}$ is a regular decomposition of L_A. Besides, $A_\pm$ are obviously closed operators, so $L_{A_\pm}$ are closed subspaces of Π, which implies that $L_{A_\pm}$ are complete subspaces of Π. Using the proof of necessity in Theorem 2.3 for $L_{A_\pm}$ respectively, it is easy to see that there exists a constant s_0, $0<s_0<1$, such that

$$\|A_-\|\leqslant 1-s_0 \text{ and } \|A_+^{-1}\|\leqslant 1-s_0$$

i. e. 1 is a regular point of operator R, hence $1\in\rho(A^*A)\cap\rho(AA^*)$. It follows from Lemma 1.2 (iii), that $\Pi=L\oplus L^{\perp}$.

Sufficiency. Since $\Pi=L_A\oplus L_{A^*}$, we have $1\in\rho(A^*A)\subset\rho(AA^*)$ by Lemma 1.2(iii). Thus for the polar decomposition of dense defined closed operator A, $A=UR$, 1 is a regular point of R. From this it is easy to see that $L_{A_\pm}$ are Hilbert spaces relative to $\pm(\cdot,\ \cdot)$. Thus $L_A=L_{A_-}\oplus L_{A_+}$ is a regular decomposition of L_A, i. e. L_A is a Π-type space relative to $(\cdot,\ \cdot)$. Since $\Pi=L_A\oplus L_{A^*}$, we see that L_A is a closed subspace of Π. Thus, it is a complete subspace of Π. This completes the proof.

Remark. Theorem 2.6 had been obtained in [4]. But the proof there was purely depended on many methods of topological vector spaces. However, we prove this result in view of the description of operator. This simple and direct method is convenient to the latter discussion.

Theerem. 2.6 has the following apparent corollary.

Corollary 2.7. *Let Π', Π'' be two linear subspaces of Π, $\Pi=\Pi'\oplus\Pi''$. Suppose that L', L'' are linear subspaces of Π', Π''. Then $L'\oplus L''$ is a closed subspace of Π if and only if L' and L'' are closed linear subspaces of Π' and Π'' respectively.*

Generally speaking, L' and L'' are closed subspaces of Π, and $L'\perp L''$, so $L'\cap L''=\{0\}$. But, in this case, we can not assert that $L'\oplus L''$ is a closed subspace of Π.

Example 4. Let $\Pi=L_-\oplus L_+$, where $L_\pm$ are $L^2[0,\ 1]$. Let

$$(Af)(t)=tf(t),\quad f(t)\in L^2[0,\ 1].$$

Consider A as an operator from L_- into L_+. Take $L_1 = L_A$, and $L_2 = L_A^\perp$. It is evident that L_i ($i=1, 2$) are closed subspaces of Π, $L_1 \perp L_2$, and $L_1 \cap L_2 = \{0\}$. This time, $L_1 \oplus L_2$ is not a closed subspace of Π. Since for any vector $f \in L^2[0, 1]$ satisfying $(1-t^2)^{-1}f(t) \in L^2[0, 1]$, we have

$$\{f(t), 0\} = \{(1-t^2)^{-1}f(t), t(1-t^2)^{-1}f(t)\} + \{-t^2(1-t^2)^{-1}f(t), -t(1-t^2)^{-1}f(t)\}$$

i. e. $\{f, 0\} \in L_1 \oplus L_2$. However, the vectors satisfying the above condition are dense in $(L_-, -(\cdot, \cdot))$. If $L_1 \oplus L_2$ is closed, so $L_1 \oplus L_2 \supset L_-$, we can show $L_1 \oplus L_2 \supset L_+$. Thus $\Pi = L_A \oplus L_A^\perp$, which contradits to that $1 \in \sigma(A^*A)$.

The following corollary is also evident.

Corollary 2.8. *Let L_1, L_2 be two complete subspaces of Π. Then*

(i) *$L_1^\perp$ is a complete subspace,*

(ii) *A linear subspace $L \subset L_1$ and L is a complete subspace of Π if and only if L is a complete subspace of Π-type space $(L_1, (\cdot, \cdot))$,*

(iii) *$L = L_1 \oplus L_2$ is a complete subspace of Π if $L_1 \perp L_2$.*

(iv) *$L = L_1 \ominus L_1$ is a complete subspace of Π if $L_1 \subset L_2$.*

Generally speaking, Corollary 2.8 (iii) can not be generalized to the case of the orthogonal sum of a sequence $\{L_n\}$ (the counterexample can be given easily).

§ 3. Projection and complete subspaces

Let $(\Pi, (\cdot, \cdot))$ be a Krein space. If E is a linear operator defined in the whole space Π, and $E^2 = E$, $E^\dagger = E$, then E is called a projection on Π.

Evidently, projections are bounded linear operators in Π. If E is a projection, then $I-E$ is also a projection. Denote $E\Pi = \{x \mid Ex = x, x \in \Pi\}$. It is obvious that $\Pi = E\Pi \oplus (E\Pi)^\perp$. From Theorem 2.6, $E\Pi$ is a complete subspace. Conversely, for any complete subspace L, since $\Pi = L \oplus L^\perp$, we can uniquely introduce a projection E such that $E\Pi = L$. Thus there is a one to one correspondence between projections and complete subspace.

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