Chin. Ann. of Math.
5B (1) 1984

ON HYPONORMAL WEIGHTED SHIFT

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Abstract

It is shown in this paper that the necessary condition that a hyponormal weighted unilateral shift of norm one be unitarily equivalent to a Toeplitz operator is that the associated weights $\{a_n\}_0^\infty$ must stisfy $1-|a_n|^2=(1-|a_0|^2)^{n+1}$ $\forall n\geqslant 0$. As a consequence we obtain that the answer to Abrahamse's Problem 3 is that the Bergman shift is not unitarily equivalent to a Toeplitz operator.

A necessary contition that a hyponormal weighted unilateral shift be unitarily equivalent to a Toeplitz operator is shown in this paper. As a direct consequence of this condition we obtain that the answer to Abrahamse's Problem 3 is that the Bergman shift is not unitarily equivalent to a Toeplitz operator. Obviously, this problem should be considered before the Halmos question on subnormal Toeplitz operator (cf. [2, 3]).

Let Δ be the unit circle in the complex plane, H^2 be Hardy space. As usually, H^2 is identifed with a closed subspace of $L^2(\Delta)$. For φ in $L^\infty(\Delta)$, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by the equation $T_\varphi(f)=P(\varphi f)$ where P is the orthogonal projection from $L^2(\Delta)$ onto H^2.

Lemma 1. *If Toeplitz operator T_φ is a hyponormal weighted unilateral shift with weights $\{a_n\}_0^\infty$, i. e. there exists a orthonormal basis $\{e_n\}_0^\infty$ of H^2 such that*

$$T_\varphi e_n=a_n e_{n+1}, \quad n\geqslant 0. \tag{1}$$

Then e_0 is an outer function and $\varphi(t)=r\dfrac{te_0}{\bar{e}_0}$ a. e. $t\in\Delta$, where r is a constant.

Proof It is known from [4] Chap. 7 that either $\ker T_\varphi=\{0\}$ or $\ker T_\varphi^*=\{0\}$. The fact that $e_0\in\ker T_\varphi^*$ implies that $|a_n|>0(n\geqslant 0)$. For $e_0\in H^2$ we have (cf. [4] Chap. 6, also [5] Chap. 5) $e_0=gF$, where g is an inner function and F is an outer function. Hence $T_\varphi^* F=T_{\bar\varphi}(\bar g e_0)=T_{\bar g}T_{\bar\varphi}e_0=T_{\bar g}T_\varphi^* e_0=0$. This means $F=\rho e_0$ because dim $\ker T_\varphi^*=1$, where ρ is a constant. The first part of our lemma is thus proved.

It is easy to check that the hyponormality implies that

$$|a_n|\leqslant|a_{n+1}| \quad (n\geqslant 0). \tag{2}$$

It follows from [6] p.240 that the essential spectrum of T_φ $\sigma_e(T_\varphi)=\{\lambda:|\lambda|=1\}$. Here we assume $\|T_\varphi\|=1$. So we have $|\varphi(t)|=1$ a. e. $t\in\Delta$(cf. [6] p. 139). Now $T_\varphi^* e_0=0$

Manuscript received January 8, 1982, revised august 26, 1983.

implies $\bar{\varphi}e_0 \equiv \xi \in H^{2\perp}$, where $H^{2\perp}=L^2(\Delta)\ominus H^2=(I-P)L^2(\Delta)$. Noticing $\frac{\bar{e}_0}{t}\in H^{2\perp}$ and $\frac{\xi}{t}\in H^2$, we obtain that $T^*_\varphi\left(\frac{\xi}{t}\right)=P\left(\bar{\varphi}\,\frac{\xi}{t}\right)=P\left(\frac{\bar{e}_0}{t}\right)=0$. By dim ker $T^*_\varphi=1$ again, we have $e_0=r\,\frac{\xi}{t}$, where r is constant. This proves that $\varphi=\frac{e_0}{\xi}=r\,\frac{te_0}{\bar{e}_0}$ with $|r|=1$ a. e. $t\in\Delta$.

Lemma 2. *If Toeplitz operator T_φ is a hyponormal unilateral shift with weights $\{a_n\}_0^\infty$. Then either $|a_n|=\mathrm{const}=\|T\|(=1)$ or $|a_0|\neq|a_1|$.*

Proof We prove the lemma by contradiction. By contradictory assumption, there exists a positive integer l such that $|a_0|=|a_1|=\cdots=|a_l|<|a_{l+1}|$. Because e_l and e_{l-1} are linearly independent, we can choose complex number α_l with $|\alpha_l|\leqslant1$ such that

$$(\alpha_l e_{l-1}+\sqrt{1-|\alpha_l|^2}\,e_l)\,|_{z=0}=0. \tag{3}$$

This means $\frac{x_l}{t}\in H^2$, where $x_l=\alpha_l e_{l-1}+\sqrt{1-|\alpha_l|^2}\,e_l$. Hence

$$\left\|T_\varphi\left(\frac{x_l}{t}\right)\right\|=\|T_t^*T_\varphi x_l\|=|a_0|\,\|T_t^*(\alpha_l e_l+\sqrt{1-|\alpha_l|^2}\,e_{l+1})\|\leqslant|a_0|.$$

The following fact is easy to verify:

$$\begin{cases}\text{If } \|T_\varphi u\|=\inf\limits_{x\in H^2,\,\|x\|=1}\|T_\varphi x\| \text{ for some } u\in H^2 \text{ with } \|u\|=1,\\ \text{then } u\in\mathscr{M}_l=\operatorname{Span}\{e_0,\ e_1,\ \cdots,\ e_l\}.\end{cases} \tag{4}$$

We get from (4) that

$$\frac{x_l}{t}\in\mathscr{M}_l. \tag{5}$$

The same arguement shows that there exists $x_{l-1}\in\{\operatorname{Span}\{e_1,\cdots,e_l\}\ominus x_l\}$ with $\|x_{l-1}\|=1$ such that $\frac{x_{l-1}}{t}\in\mathscr{M}_l$. Repeating the proceedure above, we obatain $x_2, x_3, \cdots, x_l$ such that

$$\begin{cases}x_i\in\operatorname{Span}\{e_1,\ \cdots,\ e_n\},\\ \langle x_i,\ x_j\rangle=\begin{cases}0 & i\neq j,\\ 1 & i=j,\end{cases}\quad 2\leqslant i,\ j\leqslant l,\\ \frac{x_i}{t}\in\mathscr{M}_l.\end{cases} \tag{6}$$

where the symbol$\langle\cdot,\ \cdot\rangle$stands for the inner product in $L^2(\Delta)$. Similary, we can pick up $x_1\in\{\operatorname{span}\{e_0,\ e_1,\ \cdots,\ e_l\}\ominus\operatorname{span}\ \{x_2,\ x_3,\ \cdots,\ x_l\}\}$ with $\|x_1\|=1$ and $\frac{x_1}{t}\in\mathscr{M}_l$. All this shows there exists a pair of appropriate matrices α_l and β_l of order $l\times(l+1)$ such that

$$\frac{1}{t}\begin{pmatrix}x_1\\x_2\\\vdots\\x_l\end{pmatrix}=\frac{1}{t}\,\alpha_l\begin{pmatrix}e_0\\e_1\\\vdots\\e_l\end{pmatrix}=\beta_l\begin{pmatrix}e_0\\e_1\\\vdots\\e_l\end{pmatrix}. \tag{7}$$

From the construction of $x_i's$, it is easy to check that the matrix α_l has a form as follows

$$\alpha_l=\left(\begin{array}{c|c} \rho_l & \\ 0 & \\ \vdots & A \\ 0 & \end{array}\right) \tag{8}$$

where the matrix A_l is of order $l\times l$. Also $|\rho_0|\neq 1$, this is because e_0 is an outer function. Moreover the orthognality of $x_i's$ indicates that det $A_l\neq 0$. By appling T_φ^* to both sides of (7), we obtain that

$$\alpha_l T_t^*\begin{pmatrix}0\\ e_0\\ \vdots\\ e_{l-1}\end{pmatrix}=\frac{1}{a_0}\alpha_l T_t^* T_\varphi^*\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_l\end{pmatrix}=\frac{1}{a_0}T_\varphi^*\left\{\frac{1}{t}\alpha_l\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_l\end{pmatrix}\right\}$$

$$=\frac{1}{a_0}\beta_l T_\varphi^*\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_l\end{pmatrix}=\beta_l\begin{pmatrix}0\\ e_0\\ \vdots\\ e_{l-1}\end{pmatrix}. \tag{9}$$

It follows from (8) and (9) that

$$T_t^*\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_{l-1}\end{pmatrix}=A_l^{-1}B_l\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_{l-1}\end{pmatrix}\equiv C_l\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_{l-1}\end{pmatrix}, \tag{10}$$

where the matrix B_l is obtained from β_l by omitting the first column.

Applying the same arguement given in (3)—(7) to the system $\{e_0, e_1, \cdots, e_{l-1}\}$ we obtain vectors $\{y_i\}_1^{l-1}$such that

$$\begin{cases}\{y_i\}_2^{l-1}\subset\text{span}\{e_1, \cdots, e_{l-1}\},\\ y_1\in\{\text{span}\{e_0, e_1, \cdots, e_{l-1}\}\ominus\text{span}\{y_2, \cdots, y_{l-1}\}\},\\ \langle y_i, y_j\rangle=\begin{cases}0 & i\neq j,\\ 1 & i=j,\end{cases}\quad 1\leqslant i, j\leqslant l-1\\ \dfrac{y_i}{t}\in\mathscr{M}_l.\end{cases} \tag{11}$$

We now assert that

$$\frac{y_i}{t}\in\mathscr{M}_{l-1}\equiv\text{span}\{e_0, e_1, \cdots, e_{l-1}\}. \tag{12}$$

Indeed, it follows from (10) that $T_t^*\mathscr{M}_{l-1}\subseteq\mathscr{M}_{l-1}$. Hence $\frac{y_i}{t}=T_t^* y_i\in T_t^*\mathscr{M}_{l-1}\subseteq\mathscr{M}_{l-1}$ $(1\leqslant i\leqslant l-1)$. We now can repeat the argnement given in (6)—(10) to the system $\{e_0, e_1, \cdots, e_{l-1}\}$and get that $T_t^*\mathscr{M}_{l-2}\subseteq\mathscr{M}_{l-2}$. Hence we obtain at last that

$$T_t^*\mathscr{M}_0\subseteq\mathscr{M}_0, \text{i. e.} \tag{13}$$

$$T_t^* e_0 = \rho e_0, \tag{13'}$$

where ρ is a constant. Obviously $|\rho| \leqslant 1$. Let $e_0 = \sum_{n=\infty}^{\infty} c_n z^n$. Then it follows from (13') that $c_n = \rho^n c_0$ $(n \geqslant 0)$. Therefore we have $e_0 = \frac{c_0}{1-\rho z}$ The fact that $\|e_0\| = 1$ implies $|\rho| < 1$.

On the other hand, we get from Lemma 1 that $\varphi = r \frac{te_0}{\bar{e}_0} = r \frac{t-\bar{\rho}}{1-\rho t}$ $t \in \Delta$, i. e. φ is an inner function. Therefore T_φ is an isometry. This contradicts that $|a_l| \neq |a_{l+1}|$ for some $l \geqslant 1$.

Lemma 2 is tuns proved.

The following is our main result.

Theorem 3. *Let T be a weighted unilateral shift (in Hilbert space $\mathscr{H}$) with weights $\{a_n\}_0^\infty$. If $|a_l| \geqslant |a_{l+1}|$ $(l \geqslant 0)$ with $\lim_{l\to\infty} |a_l| = 1$. Then a necessary condition that T be unitarily equivalent to a Toeplitz operator is that $1-|a_n|^2 = (1-|a_0|^2)^{n+1}$, $\forall n \geqslant 0$.*

Proof T is unitarily equivalent to a Toeplitz operator, say $T_{\bar{\varphi}}$. So T_φ is also a weighted unilateral shift in H^2 with weights $\{a_n\}_0^\infty$. T_φ is hyponrmal by $|a_n| \leqslant |a_{n+1}|$ $(n \geqslant 0)$. As is shown in Lemma 2, $|\varphi| = 1$ a. e. $t \in \Delta$. Without loss of generality, we assume $0 < a_n \leqslant 1$ (cf. [6] p. 46), also $a_n < 1$ $(n \geqslant 0$, cf. [1, 7]). By the definition of Toeplitz operator, the system (1) is equivalent to the following

$$\begin{cases} \varphi e_n = a_n e_{n+1} + (1-a_n^2)^{\frac{1}{2}} \eta_n, \\ \bar{\varphi} e_{n+1} = a_n e_n + (1-a_n^2)^{\frac{1}{2}} \xi_n. \end{cases} \quad n \geqslant 0, \tag{14}$$

where $\{\eta_n\}_0^\infty$, $\{\xi_n\}_0^\infty \subset H^{2\perp}$ and $\|\eta_n\| = \|\xi_n\| = 1 (n \geqslant 0)$. Obviously $\langle \varphi e_i, \varphi e_j \rangle = \langle e_i, e_j \rangle$, $i, j \geqslant 0$. So we obtain from $a_n < 1$ $(n \geqslant 0)$ that

$$\langle \eta_l, \eta_k \rangle = \langle \xi_l, \xi_k \rangle = \begin{cases} 0 & l \neq k \\ 1 & l = k. \end{cases} \quad l, k \geqslant 0. \tag{15}$$

What we want to show in the next step is to give the explicity expression of η_l's and ξ_k's. To this end, we obtain from (14) that

$$e_n = \bar{\varphi}(a_n e_{n+1} + (1-a_n^2)^{\frac{1}{2}} \eta_n) = a_n^2 e_n + a_n (1-a_n^2)^{\frac{1}{2}} \xi_n + (1-a^2)^{\frac{1}{2}} \bar{\varphi} \eta_n, \quad n \geqslant 0, \text{ i. e. } \tag{16}$$

$$\bar{\varphi \eta_n} = -a_n \bar{\xi}_n + (1-a_n^2)^{\frac{1}{2}} \bar{e}_n, \ n \geqslant 0. \tag{17}$$

Let $d_n = \frac{\bar{\eta}_n}{t}$ and $\rho_n = \frac{\bar{\xi}_n}{t}$. Then (17) has the from

$$\varphi d_n = -a_n \rho_n + (1-a_n^2)^{\frac{1}{2}} \frac{\bar{e}_n}{t}, \ n \geqslant 0. \tag{17'}$$

It is evident that $\frac{\bar{e}_n}{t} \in H^{2\perp} (n \geqslant 0)$ and $\{d_n\}_0^\infty$ is a orthonormal vector family in H^2. Also we get from (17') that

$$\begin{cases} \|T_\varphi d_0\| = a_0 = \inf\limits_{x\in H^2, \|x\|=1} \|T_\varphi x\|, \\ \|T_\varphi d_l\| = a_l = \inf\limits_{x\in H^2, x\perp\{e_0,\cdots,e_{l-1}\}, \|x\|=1} \|T_\varphi x\|. \end{cases} \tag{18}$$

Then it can be easily verified by using (1), (15) and (18) that

$$\begin{pmatrix} d_l \\ d_{l+1} \\ \vdots \\ d_{l+\epsilon_l} \end{pmatrix} = D_l \begin{pmatrix} e_l \\ e_{l+1} \\ \vdots \\ e_{l+\epsilon_l} \end{pmatrix} l\geqslant 0, \tag{19}$$

where D_l is an unitary matrix of order $\epsilon_l\times\epsilon_l$ and ϵ_l is an integer, may be zero, such that $a_{l-1}<a_l=a_{l+1}=\cdots=a_{l+\epsilon_l}<a_{l+\epsilon_l+1}$. Substituting (19) into (17′) and comparing it with (14) we get immediately that

$$\begin{cases} -D_l^* \begin{pmatrix} \rho_l \\ \rho_{l+1} \\ \vdots \\ \rho_{l+\epsilon_l} \end{pmatrix} = \begin{pmatrix} e_{l+1} \\ e_{l+2} \\ \vdots \\ e_{l+\epsilon_l+1} \end{pmatrix}, \\ D_l^* \begin{pmatrix} \frac{\bar{e}_l}{t} \\ \frac{\bar{e}_{l+1}}{t} \\ \vdots \\ \frac{\bar{e}_{l+\epsilon_l}}{t} \end{pmatrix} = \begin{pmatrix} \eta_l \\ \eta_{l+1} \\ \vdots \\ \eta_{l+\epsilon_l} \end{pmatrix}, \end{cases} \tag{20}$$

where D_l^* is the complex congugate transpose of D_l. Actually, $D_l^*=D_l^{-1}$.

Only for computation convenience we assume $\epsilon_l=0$ $(l\geqslant 0)$ in what follws, i. e. $a_l<a_{l+1}(l\geqslant 0)$. We shall see below that $a_0<a_1$ is essential in the proof. However, this is always true because of Lemma 2. In this case the matrix D_l is reduced to a complex number r_l of modulus one. Also, (14) is reduced to

$$\begin{cases} \varphi e_n = a_n e_{n+1} + (1-a_n^2)^{\frac{1}{2}} \bar{r}_n e_{-(n+1)}, \\ \bar{\varphi} e_{n+1} = a_n e_n - (1-a_n^2)^{\frac{1}{2}} \bar{r}_n e_{-(n+2)}, \end{cases} \quad n\geqslant 0, \tag{21}$$

where $e_{-(n+1)}\equiv \frac{\bar{e}_n}{t}$ a. e. $t\in\Delta(n\geqslant 0)$.

For completing the proof, we need some of further assertions.

The function $\psi\equiv\varphi-\overline{r_0 r}\,(1-a_0^2)^{\frac{1}{2}}\bar{\varphi}$ is in $H^2\cap L^\infty(\Delta)$, where r is the constant given in $\bar{\varphi}e_0=re_{-1}$ (see Lemma 2).

In fact, we get from (21) and Lemma 2 that

$$\varphi e_0 = a_0 e_1 + \bar{r}_0 (1-a_0^2)^{\frac{1}{2}} e_{-1} = a_0 e_1 + \bar{r}_0 \bar{r}\,(1-a_0^2)^{\frac{1}{2}} \bar{\varphi} e_0, \tag{22}$$

i. e. $\psi e_0 = a_0 e_1$. Then the following subspace of H^2

$$V = \{x \in H^2 : \psi x \in H^2\}$$

is not empty and invariant under the multiplication by z. So $V = \chi H^2$ by Beurling's Theorem (cf. [4] chap. 6, also [5] chap. 7 or [6] p. 79), where χ is an inner function. But $e_0 \in V$ and e_0 is an outer function so $\chi = 1$. Therefore $\psi = \psi \cdot 1 \in H^2$.

Remark 1. The fact that $a_0 \neq a_1$ is essential in the proof of $\psi \in H^2$.

We now return to the proof of our theroem.

The Laurent operator L_φ, the multiplication by φ, is unitary in $L^2(\Delta)$ because $|\varphi| = 1$ a. e. $t \in \Delta$. We identify L_φ with its matrix representation in the coordinate system $\{t^n\}_{-\infty}^{\infty}$. It follows from i) that (cf. [6] p. 135 for Laurent matrix)

$$L_\varphi = \begin{array}{c} \begin{array}{ccccc} & -1 & 0 & 1 & \end{array} \\ \left(\begin{array}{ccccc} \ddots & \rho c_2 & & & \\ \ddots & \rho c_1 & \rho c_2 & \ddots & \\ \cdots c_1 \cdots & c_0 & \rho c_1 & \rho c_2 & \cdots \\ \cdots & c_1 & c_0 & \rho c_1 & \cdots \\ \cdots & c_2 & c_1 & c_0 & \rho e_1 \cdots \\ & \ddots & c_2 & c_1 & \\ & & & c_2 & \ddots \end{array}\right) \end{array} \begin{array}{c} \\ \\ \\ -1 \\ 0 \\ 1 \\ \\ \\ \end{array}$$

$$= \begin{array}{c} \left(\begin{array}{c|c} \Omega & \rho \overline{\mathscr{L}} \\ \hline \mathscr{L} & \Omega^T \end{array}\right) \\ \begin{array}{cc} -1 & 0 \end{array} \end{array} \begin{array}{c} -1 \\ 0 \end{array} \tag{23}$$

where $\rho = \overline{r_0 r}\,(1 - a_0^2)^{\frac{1}{2}}$, Ω^T is the transpose of Ω. Similarly, it follows from (21) that the operator L_φ has a matrix representation in the coordinate system $\{e_n\}_{-\infty}^{\infty}$ as follows

$$
L_\varphi \overset{\{e_n\}_{-\infty}^{\infty}}{\sim}
\begin{array}{c c}
\begin{array}{ccccc} -(n+2)\cdots & -2 & -1 & 0 & \cdots\ n \end{array} & \\
\left(\begin{array}{c|c|c|c|c|c}
\ddots & & & & & \\
\cdots\ a_n & \cdots & \cdots & \cdots & \bar{r}_n(1-a_n^2)^{1/2} & \cdots \\
& \ddots & & & \iddots & \\
\cdots & a_0 & \cdots & \bar{r}_0(1-a_2^0)^{1/2} & \cdots & \cdots \\
& & \ddots & \iddots & & \\
\cdots & \cdots & \bar{r} & \cdots & \cdots & \cdots \\
& & \iddots & \ddots & & \\
\cdots & -r_0(1-a_0^2)^{1/2} & \cdots & a_0 & \cdots & \cdots \\
& \iddots & & & \ddots & \\
\cdots -r_n(1-a_n^2)^{1/2} & \cdots & \cdots & \cdots & a_n & \cdots \\
\iddots & & & & & \ddots
\end{array}\right)
&
\begin{array}{c} -(n+1) \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \\ n+1 \end{array}
\end{array}
$$

$$
=\begin{array}{c}\left(\begin{array}{c|c} A^T & \gamma \\ \hline \xi & A \end{array}\right)\begin{array}{c} -1 \\ 0 \end{array} \\ \begin{array}{cc} -1 & 0 \end{array}\end{array} \equiv Q \quad \text{where} \quad \xi=\begin{pmatrix} \bar{r} & 0 \ \ 0 \ \cdots \\ 0 & \\ 0 & -\gamma^* \\ \vdots & \end{pmatrix} \tag{24}
$$

Let

$$
W\begin{pmatrix} \vdots \\ e_{-m} \\ \vdots \\ e_{-2} \\ e_{-1} \\ e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_m \\ \vdots \end{pmatrix}=\begin{pmatrix} \vdots \\ t^{-n} \\ \vdots \\ t^{-2} \\ t^{-1} \\ 1 \\ t \\ t^2 \\ \vdots \\ t^n \\ \vdots \end{pmatrix}. \tag{25}
$$

Then W is unitary ond $WH^2=H^2$, $WH^{2\perp}=H^{2\perp}$. Moreover, if $t^n=\sum_{j=0}^{\infty} w_{n,j}e_j$ $(n\geqslant 0)$, then $t^{-(n+1)}=\frac{1}{t}\overline{\left(\sum_{j=0}^{\infty} w_{n,j}e_j\right)}=\sum_{j=1}^{\infty} \bar{w}_{n,j-1}e_{-j}(n\geqslant 0)$.

Therefore, it is easy to verify that

$$W=\begin{array}{cc} \begin{matrix} \quad -1 & \quad 0 \end{matrix} & \\ \left(\begin{array}{c|c} \bar{R} & 0 \\ \hline 0 & R \end{array}\right) & \begin{matrix} -1 \\ 0 \end{matrix} \end{array} \tag{26}$$

It it well know that $W^*QW=L_\varphi$. So we have, by (23)—(26), that $\bar{\rho}R^*\xi\bar{R}=R^*\bar{\gamma}\bar{R}$. Obviously, this holds if and only if $\bar{\rho}\bar{r}=-r_0(1-\alpha_0^2)^{1/2}$ and $\bar{\rho}r_n(1-\alpha_n^2)^{1/2}=-r_{n+1}(1-\alpha_{n+1}^2)^{1/2}$. This leads to that $1-\alpha_n^2=(1-\alpha_0^2)^{n+1}\forall n\geqslant 0$. The proof of our theorem is thus completed.

As a consesuence of Theorem 3, we obtain the answer to Abrahamse's Problem 3 (cf. [1]).

Corollary 4. *The Bergman shift is not unitarily equivalent to a Toeplitz operator.*

Proof The Bergman shift (cf. [1] for the definition) is a subnormal weighted shift with weights $\alpha_n=\sqrt{\dfrac{n+1}{n+2}}$ $(n\geqslant 0)$. Then Theorem 3 implies Corollary 4.

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