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ON THE ALGEBRAIC INDEPENDENCE OF CERTAIN POWER SERIES OF ALGEBRAIC NUMBERS

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Abstract

Let

$$f_\nu(z)=\sum_{k=1}^{\infty}a_{\nu,k}z^{\lambda_{\nu,k}}\quad(\nu=1,\cdots,s)$$

be s power series with algebraic coefficients $a_{\nu,k}$, convergence radii $R_\nu>0$ and sufficiently rapidly increasing integers $\lambda_{\nu,k}$. It is shown that under certain conditions depending only on $a_{\nu,k}$ and $\lambda_{\nu,k}$, (i) $f_1(\theta_1),\cdots,f_s(\theta_s)$ are algebraically independent for arbitrary algebraic numbers $\theta_1,\cdots,\theta_s$ with $0<|\theta_\nu|<R_\nu(\nu=1,\cdots,s)$; (ii) $f_\nu(\theta_\mu)(\nu=1,\cdots,s;\ \mu=1,\cdots,t)$ are algebraically independent for t different algebraic numbers $\theta_1,\cdots,\theta_t$ with $0<|\theta_t|<|\theta_{t-1}|<\cdots<|\theta_1|<\min\limits_{1\leqslant\nu\leqslant s}R_\nu$.

§ 1. Introduction

It is well known that the Liouville numbers $\sum_{k=1}^{\infty}g^{-k!}(2\leqslant g\in\mathbf{Z})$ are transendental. Let

$$f_0(z)=\sum_{k=1}^{\infty}a_kz^{\lambda_k}\tag{1}$$

where $a_k\in\mathbf{A}$ or $\mathbf{Q}$, be a power series. Cohn[1], Baron and Braune[2], Mahler[3] considered the transcendence of $f_0(\theta)$, where $\theta\in\mathbf{A}$, respectively. Cijsouw and Tijdeman[4] obtained the following result: Let $\lambda_k\in\mathbf{N}$ $(k=1,2,\cdots)$ be a strictly monotonic increasing sequence, and let $R_0>0$ be the radius of convergence of (1). If

$$\lim_{n\to\infty}(\lambda_n+\log M_n+\log A_n)D_n/\lambda_{n+1}=0\tag{2}$$

where $D_n=[\mathbf{Q}(a_1,\cdots,a_n):\mathbf{Q}]$, $A_n=\max\limits_{1\leqslant k\leqslant n}\overline{|a_k|}^*$ and M_n is the least common denominator of $a_1,a_2,\cdots,a_n$, then $f_0(\theta)$ is transcendental for every $\theta\in\mathbf{A}$ with $0<|\theta|<R_0$.

In this paper we consider the algebraic independence of the values of one or several power series which have the form of (1) in algebraic points. We will show that the condition such as (2) is also sufficient for the algebraic independence. The establishment of this result depends on the main lemma in § 4. Our result implies in particular, the algebraically independent results in [4, 5, 6].

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§ 2. Formulation of the results

We denote the sets of all integers, positive integers by $\mathbf{Z}$, $\mathbf{N}$, and the fiields of all rational, algebraic and complex numbers by $\mathbf{Q}$, $\mathbf{A}$ and $\mathbf{C}$, respectively. Let $\mathbf{Z}[z_1, \cdots, z_s]$ be the set of all polynomials in variable $z_1, \cdots, z_s$ with integer coefficients. Let $P(z)=a_0z^d+\cdots+a_d\in\mathbf{Z}[z]$ with $a_0\neq 0$. Then $\partial(P)=d$, $H(P)=\max\limits_{0\leqslant i\leqslant d}|a_i|$ are called the degree and the height of P, respectively. For $\alpha\in\mathbf{A}$, if $P(z)$ is its minimal polynomial, then $\partial(P)$ and $H(P)$ are called the degree and the height of α, respectively; and all the zeros $\alpha^{(1)}=\alpha$, $\alpha^{(2)}$, $\cdots$, $\alpha^{(d)}$ of $P(z)$ are called the conjugates of α. We denote $\max\limits_{1\leqslant i\leqslant d}|\alpha^{(i)}|$ by $\overline{|\alpha|}$. For α, $\beta\in\mathbf{A}$, we have $\overline{|\alpha+\beta|}\leqslant\overline{|\alpha|}+\overline{|\beta|}$ and $\overline{|\alpha\beta|}\leqslant\overline{|\alpha|}\,\overline{|\beta|}$. A number $m\in\mathbf{N}$ is called a denominator of algebraic number α, if $m\alpha$ is an algebraic integer. More generally, $M\in\mathbf{N}$ is called a denominator of the algebraic numbers α_1, $\cdots$, α_n, if $M\alpha_1$, $\cdots$, $M\alpha_n$ are all algebraic integers. For any $a\in\mathbf{C}$, we put $a^*=\max(1, |a|)$.

In tnis paper, we denote the positive costants independing on n by C_1, C_2, $\cdots$.

Let $s\geqslant 1$ be an integer. Suppose that

$$f_\nu(z)=\sum_{k=1}^{\infty}a_{\nu,k}z^{\lambda_{\nu,k}}\quad(\nu=1, \cdots, s) \tag{3}$$

are s power series satisfying the following three conditions:

(i) For $\nu=1$, $\cdots$, s, $\lambda_{\nu,k}\in\mathbf{N}$ $(k=1, 2, \cdots)$ are s strictly monotonic increasing sequences;

(ii) $a_{\nu,k}\in\mathbf{A}$ For $k=1, 2, \cdots$; $\nu=1, \cdots, s$;

(iii) The radii of convergence of the series $R_\nu(\nu=1, \cdots, s)$ are positive.

For $\nu=1$, $\cdots$, s, we put

$$D_{\nu,n}=[\mathbf{Q}(a_{\nu,1}, \cdots, a_{\nu,n}):\mathbf{Q}],$$

$$A_{\nu,n}=\max_{1\leqslant k\leqslant n}\overline{|a_{\nu,k}|}^*,$$

$M_{\nu,n}=$ the least common denominator of $a_{\nu,1}$, $\cdots$, $a_{\nu,n}$.

and

$$D_n=[\mathbf{Q}(a_{1,1}, \cdots, a_{1,n}, \cdots, a_{s,1}, \cdots, a_{s,n}):\mathbf{Q}],$$

$$A_n=\max_{1\leqslant\nu\leqslant s}A_{\nu,n},$$

$M_n=$ the least common denominator of $M_{\nu,n}(1\leqslant\nu\leqslant s)$.

Theorem 1. *Let $s\geqslant 1$. If series (1) satisfies condition (2). Then $f_0(\theta_1), \cdots, f_0(\theta_s)$ are algebraically independent for s numbers θ_1, $\cdots$, $\theta_s\in\mathbf{A}$ with $0<|\theta_s|<|\theta_{s-1}|<\cdots<|\theta_1|<R_0$.*

Remark 1. This theorem implies the result of Adams[5].

Theorem 2. Suppose that series (3) satisfy

$$\lim_{n\to\infty} \lambda_{\nu,n}/\lambda_{\mu,n}=0 \quad (1\leqslant\nu<\mu\leqslant s), \tag{4}$$

$$\lim_{n\to\infty}(\lambda_{s,n}+\log M_n+\log A_n)D_n/\lambda_{1,n+1}=0. \tag{5}$$

If $s\geqslant 1$ and if $\theta_1, \cdots, \theta_s\in\mathbf{A}$ are s arbitrary numbers with $0<|\theta_\nu|<R_\nu(\nu=1, \cdots, s)$ ($\theta_1, \cdots, \theta_s$ are not necessarily pairwise distinct), then $f_1(\theta_1), \cdots, f_s(\theta_s)$ are algebraically independent.

Corollary 1. *Suppose that series* (3) *satisfy* (4) *and* (5). *Then* $f_1(\theta), \cdots, f_s(\theta)$ *are algebraically independent for* $\theta\in\mathbf{A}$ *with* $0<|\theta|<\min\limits_{1\leqslant\nu\leqslant s} R_\nu$.

Remark 2. This corollary implies the result of Kneser[6].

Theorem 3. *Let* $s, t\geqslant 1$. *If series* (3) *satisfy* (4) *and* (5), *then* $f_\nu(\theta_\mu)$ $(\nu=1,\cdots,s;$ $\mu=1, \cdots, t)$ *are algebraically independent for* t *numbers* $\theta_1, \cdots, \theta_t\in\mathbf{A}$ *with* $0<|\theta_t|<|\theta_{t-1}|<\cdots<|\theta_1|<\min\limits_{1\leqslant\nu\leqslant s} R_\nu$.

§ 3. Preliminaries

Lemma 1. *Let* $\theta\in\mathbf{A}$ *be of degree* d *and height* h. *Let* m *be a denominator of* θ. *Then* $h\leqslant(2m\overline{|\theta|}^*)^d$.

Proof See [4].

Lemma 2. *Let* $\theta_\nu\in\mathbf{A}$ *be of degree* d_ν *and height* $h_\nu(\nu=1, \cdots, s)$, *and put* $d=[\mathbf{Q}(\theta_1, \cdots, \theta_s):\mathbf{Q}]$. *Suppose that*

$$P(z_1, \cdots, z_s)=\sum_{i_1=0}^{N_1}\cdots\sum_{i_s=0}^{N_s} p_{i_1\cdots i_s}z_1^{i_1}\cdots z_s^{i_s} \tag{6}$$

$\in\mathbf{Z}[z_1, \cdots, z_s]$ *is a non-zero polynomial and that* B *is a constant such that*

$$|p_{i_1\cdots i_s}|<B \text{ for all } i_1, \cdots, i_s. \tag{7}$$

Then $P(\theta_1, \cdots, \theta_s)=0$ or

$$|P(\theta_1, \cdots, \theta_s)|\geqslant\{(N_1+1)\cdots(N_s+1)B\}^{-d+1}\prod_{\nu=1}^{s}\{(d_\nu+1)h_\nu\}^{-N_\nu d/d_\nu}.$$

Proof See [7].

Lemma 3. *Let* $f_1(z), \cdots, f_s(z)$ *be power series in* (3) (*they are not necessarily pairwise distinct*). *If* $\theta_1, \cdots, \theta_s\in\mathbf{A}$ (*they are not necessarily pairwise distinct*) *with* $0<|\theta_\nu|<R_\nu(\nu=1, \cdots, s)$. *We put* $\pi_\nu=f_\nu(\theta_\nu)$, $\phi_{\nu,n}=\sum\limits_{k=1}^{n} a_{\nu,k}\theta_\nu^{\lambda_{\nu,k}}$, $\psi_{\nu,n}=\pi_\nu-\phi_{\nu,n}(\nu=1, \cdots, s)$. *If*

$$\lim_{n\to\infty}(\max_{1\leqslant\nu\leqslant s}\lambda_{\nu,n}+\log M_n+\log A_n)D_n/\min_{1\leqslant\nu\leqslant s}\lambda_{\nu,n+1}=0 \tag{8}$$

and if $P(z_1, \cdots, z_s)\in\mathbf{Z}[z_1, \cdots, z_s]$ *is a non-zero polynomial such that*

$$P(\pi_1, \cdots, \pi_s)=0, \tag{9}$$

then there exists $n_0=n_0(P)\in\mathbf{N}$ *such that*

$$P(\phi_{1,n}, \cdots, \phi_{s,n})=0 \text{ for } n\geqslant n_0. \tag{10}$$

Proof We write $P(z_1, \cdots, z_s)$ as the form (6). We put

$$K_n=\{(N_1+1)\cdots(N_s+1)B\}^{-d^{(n)}+1}\prod_{\nu=1}^{s}\{(d_\nu^{(n)}+1)h_\nu^{(n)}\}^{-N_\nu d^{(n)}/d_\nu^{(n)}}, \tag{11}$$

where B is defined by (7), and

$$d^{(n)}=[\mathbf{Q}(\phi_{1,n}, \cdots, \phi_{s,n}):\mathbf{Q}],$$
$$d_\nu^{(n)}=[\mathbf{Q}(\phi_{\nu,n}):\mathbf{Q}],$$
$$h_\nu^{(n)}=H(\phi_{\nu,n}).$$

Let the degree and the denominator of θ_ν be d_ν and m_ν, respectively $(\nu=1, \cdots, s)$.

It is clear that

$$d^{(n)}\leqslant d_1\cdots d_s D_n, \tag{12}$$
$$\overline{|\phi_{\nu,n}|}^{*}=nA_n\overline{|\theta_\nu|}^{*\lambda_{\nu,n}} \quad (\nu=1, \cdots, s).$$

By Lemma 1, we get

$$h_\nu^{(n)}\leqslant\{2m_\nu^{\lambda_{\nu,n}}M_n\overline{|\phi_{\nu,n}|}^{*}\}^{d_\nu^{(n)}}\leqslant\{2m_\nu^{\lambda_{\nu,n}}M_n(nA_n\overline{|\theta_\nu|}^{*\lambda_{\nu,n}})^{*}\}^{d_\nu^{(n)}}$$
$$\leqslant(2m_\nu^{\lambda_{\nu,n}}M_n nA_n\overline{|\theta_\nu|}^{*\lambda_{\nu,n}})^{d_\nu^{(n)}}.$$

Since $n\leqslant 2^n\leqslant 2^{\lambda_{\nu,n}}$, we have

$$h_\nu^{(n)}\leqslant(2M_nA_n)^{d_\nu^{(n)}}(2m_\nu\overline{|\theta_\nu|}^{*})^{d_\nu^{(n)}\lambda_{\nu,n}} \quad (\nu=1, \cdots, s). \tag{13}$$

From (12), (13) and the inequality $(d_\nu^{(n)}+1)\leqslant 2^{d_\nu^{(n)}}$, we obtain

$$K_n>((N_1+1)\cdots(N_s+1)B)^{-d^{(n)}}\cdot\prod_{\xi=1}^{s}\{4M_nA_n(2m_\nu\overline{|\theta_\nu|}^{*})^{\lambda_{\nu,n}}\}^{-N_\nu d^{(n)}}$$
$$>\exp\{-C_1(\log M_n+\log A_n+\max_{1\leqslant\nu\leqslant s}\lambda_{\nu,n})D_n\}. \tag{14}$$

On the other hand, from (9), we have

$$|P(\phi_{1,n}, \cdots, \phi_{s,n})|=|P(\phi_{1,n}, \cdots, \phi_{s,n})-P(\pi_1, \cdots, \pi_s)|$$
$$\leqslant C_2\max_{1\leqslant\nu\leqslant s}|\psi_{\nu,n}|,$$

for sufficiently large n. We choose ρ_ν such that

$$|\theta_\nu|<\rho_\nu<R_\nu \quad (\nu=1, \cdots, s).$$

Then $|a_{\nu,k}|<\rho_\nu^{-\lambda_{\nu,k}}(\nu=1, \cdots, s)$ for sufficiently large k, hence we have

$$|\psi_{\nu,n}|\leqslant(1-|\theta_\nu|\rho_\nu^{-1})^{-1}(|\theta_\nu|\rho_\nu^{-1})^{\lambda_{\nu,n+1}} \quad (\nu=1, \cdots, s).$$

Therefore we deduce that

$$|P(\phi_{1,n}, \cdots, \phi_{s,n})|\leqslant\exp(-C_3\min_{1\leqslant\nu\leqslant s}\lambda_{\nu,n+1}). \tag{15}$$

From (8), (14) and (15) it follows that for $n\geqslant n_0(P)$

$$|P(\phi_{1,n}, \cdots, \phi_{s,n})|<K_n.$$

Acoding to Lemma 2 we obtain (10).

Lemma 4. *Suppose that $\zeta_1, \cdots, \zeta_s\in\mathbf{C}$ are algebraically dependent, but $\zeta_1, \cdots, \zeta_{s-1}$ and $\zeta_2, \cdots, \zeta_s$ algebraically independent, respectively. Then*

(i) *There is a non-zero polynomial $P(z_1, \cdots, z_s)\in\mathbf{Z}[z_1, \cdots, z_s]$ such that $P(\zeta_1, \cdots, \zeta_s)=0$.*

(ii) *There exist a open neighborhood $V\subset\mathbf{C}^s$ of $(\zeta_1, \cdots, \zeta_s)$ and a open neighborhood $W\subset\mathbf{C}^{s-1}$ of $(\zeta_1, \cdots, \zeta_{s-1})$ and a holomorphic function $g:W\to\mathbf{C}$ such that the relation*

$$(z_1, \cdots, z_s) \in V, \; P(z_1, \cdots, z_s) = 0$$

is equivalent to the relation

$$(z_1, \cdots, z_{s-1}) \in W, \; z_s = g(z_1, \cdots, z_{s-1}).$$

(iii) *There are constants* ε, $C_0 > 0$ *such that for* $|z_1 - \zeta_1| \leqslant \varepsilon$,

$$|g(z_1, \zeta_2, \cdots, \zeta_{s-1}) - g(\zeta_1, \zeta_2, \cdots, \zeta_{s-1})| \geqslant C_0 |z_1 - \zeta_1|.$$

Proof Let m be the degree of ζ_s over $\mathbf{Q}(\zeta_1, \cdots, \zeta_{s-1})$. Then there is a non-zero polynomial $R(z_1, \cdots, z_s) \in \mathbf{Z}[z_1, \cdots, z_s]$ with $\partial_{z_s}(R) = m$ such that

$$R(\zeta_1, \cdots, \zeta_s) = 0.$$

Since $R \not\equiv 0$ and $\zeta_2, \cdots, \zeta_s$ are algebraically independent, the polynomial

$$r(z) = R(z, \zeta_2, \cdots, \zeta_s) \not\equiv 0$$

and $r(\zeta_1) = 0$. Therefore there is an integer $t \geqslant 1$ such that

$$\frac{\partial^t}{\partial z_1^t} R(\zeta_1, \cdots, \zeta_s) \neq 0, \; \frac{\partial^{t-1}}{\partial z_1^{t-1}} R(\zeta_1, \cdots, \zeta_s) = 0.$$

By the definition of m, it follows that the polynomial

$$P(z_1, \cdots, z_s) = \frac{\partial^{t-1}}{\partial z_1^{t-1}} R(z_1, \cdots, z_s)$$

satisfies (i), and that

$$\frac{\partial}{\partial z_1} P(\zeta_1, \cdots, \zeta_s) \neq 0, \; \frac{\partial}{\partial z_s} P(\zeta_1, \cdots, \zeta_s) \neq 0.$$

Applying the implicit function theorem to the function $P(z_1, \cdots, z_s)$, we get (ii).

From the relation

$$P(z_1, \cdots, z_{s-1}, g(z_1, \cdots, z_{s-1})) \equiv 0 \quad (\text{for } (z_1, \cdots, z_{s-1}) \in W),$$

we deduce

$$\frac{\partial}{\partial z_1} g(\zeta_1, \cdots, \zeta_{s-1}) \neq 0,$$

by derivation for z_1. Thus (iii) follows.

Remark. Lemma 4 is due to Durand (see the proof of theorem 2 of [8]).

§ 4. The main lemma

Lemma 5. *Let* $f_1(z), \cdots, f_s(z)$ *be power series in* (3) (they are not necessarily distinct). *If* $\theta_1, \cdots, \theta_s \in \mathbf{A}$ *with* $0 < |\theta_\nu| < R_\nu (\nu = 1, \cdots, s)$ (they are not necessarily distinct). *We put* $\pi_\nu = f_\nu(\theta_\nu)$, $\phi_{\nu,n} = \sum_{k=1}^{n} a_{\nu,k} \theta_\nu^{\lambda_{\nu,k}}$, $\psi_{\nu,n} = \pi_\nu - \phi_{\nu,n} (\nu = 1, \cdots, s)$. *If*

(i) *For any* ν $(1 \leqslant \nu \leqslant s-1)$, *there exists an infinit seqeunce of nature numbers* $\mathcal{N}_\nu = \{k_n(\nu), \; (n = 1, 2, \cdots)\}$ *such that*

$$\sum_{t=\nu+1}^{s} |\psi_{t, k_n(\nu)}| = O(|\psi_{\nu, k_n(\nu)}|) \quad (n \to \infty);$$

(if $s = 1$, then this condition is omitted).

(ii)

$$\lim_{n\to\infty}(\max_{1\leqslant\nu\leqslant s}\lambda_{\nu,n}+\log M_n+\log A_n)\,D_n/\min_{1\leqslant\nu\leqslant s}\lambda_{\nu,n+1}=0.$$

Then $\pi_1, \cdots, \pi_s$ *are algebraically independent.*

Proof We use the induction for s.

Let $s=1$. We assume that there is a non-zero polynomial $P(z)\in\mathbb{Z}[z]$ such that $P(\pi_1)=0$. Then from (ii) and Lemma 3 we get

$$P(\phi_{1,n})=0 \quad (\text{for } n\geqslant n_0). \tag{16}$$

$P(z)$ has Taylor expansion about the point $z=\pi_1$

$$P(z)=\sum_{i=t}^{l}a_i(z-\pi_1)^i, \tag{17}$$

where $t\geqslant1$ and the summation extends over i for which $a_i\neq0$. From (16) we deduce that $l>t$. From (16), (17) we obtain

$$|a_t|=\sum_{i=t+1}^{l}|a_i|\,|\psi_{1,n}|^{i-t}.$$

We choose ρ such that $|\theta_1|<\rho<R_1$. Then for sufficiently large n,

$$|\psi_{1,n}|\leqslant C_4(|\theta_1|^{-1}\rho)^{-\lambda_{1,n+1}}.$$

Since $|\theta_1|^{-1}\rho>1$, we obtain

$$\lim_{n\to\infty}|a_t|=0.$$

This is impossible. Thus the lemma holds for $s=1$.

Now let $s\geqslant2$. Suppose that lemma holds when the number of the series is less than s. We show lemma holds for s series. We assume that (i) and (ii) hold, but $\pi_1, \cdots, \pi_s$ are algebraically dependent. Since the expression in (ii) is the increasing function of the number of series, from the inductive assumption we deduce that $\pi_1, \cdots, \pi_{s-1}$ and $\pi_2, \cdots, \pi_s$ are algebraically independent, respectively. According to Lemma 4, there are a non-zero polynomial $P(z_1, \cdots, z_s)\in\mathbb{Z}[z_1, \cdots, z_s]$ and a holomorphic function $z_s=g(z_1, \cdots, z_{s-1})$ in a neighborhood W of $(\pi_1, \cdots, \pi_{s-1})$ such that

$$P(\pi_1, \cdots, \pi_s)=0 \tag{18}$$

$$P(z_1, \cdots, z_{s-1}, g(z_1, \cdots, z_{s-1}))\equiv0 \quad (\text{for } (z_1, \cdots, z_{s-1})\in W) \tag{19}$$

and there are constants $\varepsilon>0$ and $C_0>0$ such that

$$|g(z_1, \pi_2, \cdots, \pi_{s-1})-g(\pi_1, \pi_2, \cdots, \pi_{s-1})|\geqslant C_0|z_1-\pi_1| \tag{20}$$

for $|z_1-\pi_1|\leqslant\varepsilon$.

By Lemma 3, from (ii), (18) we deduce that

$$P(\phi_{1,n}, \cdots, \phi_{s,n})=0$$

for sufficiently large n. From (ii) of Lemma 4 and (19) we get

$$\pi_s=g(\pi_1, \cdots, \pi_{s-1}), \quad \phi_{s,n}=g(\phi_{1,n}, \cdots, \phi_{s-1,n}).$$

From (20), we obtain

$$|g(\pi_1, \cdots, \pi_{s-1})-g(\phi_{1,n}, \pi_2, \cdots, \pi_{s-1})|\geqslant C_0|\psi_{1,n}| \tag{21}$$

for sufficiently large n.

On the other hand, from the identity

$$\begin{aligned}\pi_s-\phi_{s,n}&=g(\pi_1,\ \cdots,\ \pi_{s-1})-g(\phi_{1,n},\ \cdots,\ \phi_{s-1,n})\\&=\{g(\pi_1,\ \pi_2,\ \cdots,\ \pi_{s-1})-g(\phi_{1,n},\ \pi_2,\ \cdots,\ \pi_{s-1})\}\\&+\{g(\phi_{1,n},\ \pi_2,\ \pi_3,\ \cdots,\ \pi_{s-1})-g(\phi_{1,n},\ \phi_{2,n},\ \pi_3,\ \cdots,\ \pi_{s-1})\}\\&+\cdots+\{g(\phi_{1,n},\ \cdots,\ \phi_{s-2,n},\ \pi_{s-1})-g(\phi_{1,n},\ \cdots,\ \phi_{s-2,n},\ \phi_{s-1,n})\},\end{aligned}$$

we deduce that for sufficiently large n

$$|g(\pi_1,\ \cdots,\ \pi_{s-1})-g(\phi_{1,n},\ \pi_2,\ \cdots,\ \pi_{s-1})|\leqslant C_5\sum_{\nu=2}^{s}|\psi_{\nu,n}|. \tag{22}$$

From (21) and (22), we have

$$|\psi_{1,n}|\leqslant C_0^{-1}C_5\sum_{\nu=2}^{s}|\psi_{\nu,n}|.$$

This is in contradiction with (i). Hence $\pi_1,\ \cdots,\ \pi_s$ are algebraically independent. Thus Lamma is proved.

§ 5. Proof of theorems

Proof of Theorem 1. In Lemma 5 we put $f_1=\cdots=f_s=f_0$. From (2) we know that the condition (ii) holds. Now we verify the condition (i).

We have $|\theta_1|>|\theta_2|>\cdots>|\theta_s|$. For any fixed $\nu(\nu=1,\ \cdots,\ s-1)$ we choose $\delta=\delta(\nu)$ such that

$$0<\delta<\min\left(\frac{|\theta_\nu|-|\theta_{\nu+1}|}{|\theta_\nu|+|\theta_{\nu+1}|}R_0,\ R_0-|\theta_\nu|\right), \tag{23}$$

and put

$$\rho_1=R_0-\delta,\ \rho_2=R_0+\delta.$$

Then

$$|\theta_s|<\cdots<|\theta_\nu|<\rho_1<R_0, \tag{24}$$

$$R_0<\rho_2. \tag{25}$$

From (23) we obtain

$$\rho_1>R_0-\frac{|\theta_\nu|-|\theta_{\nu+1}|}{|\theta_\nu|+|\theta_{\nu+1}|}R_0=\frac{2|\theta_{\nu+1}|}{|\theta_\nu|+|\theta_{\nu+1}|}R_0,$$

$$\rho_2<R_0+\frac{|\theta_\nu|-|\theta_{\nu+1}|}{|\theta_\nu|+|\theta_{\nu+1}|}R_0=\frac{2|\theta_\nu|}{|\theta_\nu|+|\theta_{\nu+1}|}R_0$$

Hence

$$\rho_1|\theta_{\nu+1}|^{-1}>\rho_2|\theta_\nu|^{-1}>1. \tag{26}$$

From (24), we have $|a_k|<\rho_1^{-\lambda_k}$ for sufficiently large k, hence

$$|\psi_{t,n}|\leqslant C_6(\rho_1|\theta_t|^{-1})^{-\lambda_{n+1}}\quad(t=\nu+1,\ \cdots,\ s),$$

therefore

$$|\psi_{t,n}|\leqslant C_6(\rho_1|\theta_{\nu+1}|^{-1})^{-\lambda_{n+1}}\quad(t=\nu+1,\ \cdots,\ s). \tag{27}$$

From (25), there is an infinite sequence of nature numbers

$$\mathcal{N}_\nu=\{k_n(\nu),\ (n=1,\ 2,\ \cdots)\}$$

such that

$$|a_{k_n(\nu)+1}|>\rho_2^{-\lambda_{k_n(\nu)+1}}\quad (n=1,\ 2,\ \cdots).$$

Hence for sufficiently large n

$$|\psi_{\nu,k_n(\nu)}|=\left|\sum_{k=k_n(\nu)+1}^{\infty} a_k\theta_\nu^{\lambda_k}\right|\geqslant|a_{k_n(\nu)+1}|\ |\theta_\nu|^{\lambda_{k_n(\nu)+1}}-\sum_{k=k_n(\nu)+2}^{\infty}\rho_1^{-\lambda_k}|\theta_\nu|^{\lambda_k}$$
$$\geqslant(\rho_2|\theta_\nu|^{-1})^{-\lambda_{k_n(\nu)+1}}-C_7(\rho_1|\theta_\nu|^{-1})^{-\lambda_{k_n(\nu)+2}}.$$

Since $\lim_{n\to\infty}\lambda_n/\lambda_{n+1}=0$, we have

$$|\psi_{\nu,k_n(\nu)}|\geqslant C_8(\rho_2|\theta_\nu|^{-1})^{-\lambda_{k_n(\nu)+1}},\tag{28}$$

for sufficiently large n.

From (26), (27), and (28) we deduce that (i) holds. Thus the theorem follows.

Proof of Theorem 2. As above it is enough to verify the condition (i). We choose ρ_0 and $\rho_\nu(\nu=1,\ \cdots,\ s)$ such that

$$|\theta_\nu|<\rho_\nu<R_\nu(\nu=1,\ \cdots,\ s),$$
$$\max_{1\leqslant\nu\leqslant s}R_\nu<\rho_0.$$

Then for sufficiently large n

$$|\psi_{\nu,n}|\leqslant C_9(\rho_\nu|\theta_\nu|^{-1})^{-\lambda_{\nu,n+1}}\quad(\nu=1,\ \cdots,\ s),\tag{29}$$

and for any ν there is $\mathcal{N}_\nu=\{k_n(\nu),\ (n=1,\ 2,\ \cdots)\}$ such that

$$|\psi_{\nu,k_n(\nu)}|\geqslant(\rho_0|\theta_\nu|^{-1})^{-\lambda_{\nu,k_n(\nu)+1}}-C_0(\rho_\nu|\theta_\nu|^{-1})^{-\lambda_{\nu,k_n(\nu)+2}},$$

for sufficiently large n. From (4) and (5) we get $\lim_{n\to\infty}\lambda_{\nu,n}/\lambda_{\nu,n+1}=0$, therefore

$$|\psi_{\nu,k_n(\nu)}|\geqslant C_{11}(\rho_0|\theta_\nu|^{-1})^{-\lambda_{\nu,k_n(\nu)+1}}\quad(\nu=1,\ \cdots,\ s).\tag{30}$$

From (4), (29) and (30) we deduce that (i) holds. Thus the proof is complete.

Proof of Theorem 3. Using Lemma 5, we consider st power series

$$\underbrace{f_1,\ \cdots,\ f_1}_{t};\cdots\cdots;\underbrace{f_s,\ \cdots,\ f_s}_{t}$$

and st complex numbers

$$\theta_1,\ \theta_2,\ \cdots,\ \theta_t;\cdots;\ \theta_1,\ \theta_2,\ \cdots,\ \theta_t.$$

We put

$$\alpha_1=f_1(\theta_1),\ \alpha_2=f_1(\theta_2),\ \cdots,\ \alpha_t=f_1(\theta_t),\ \alpha_{t+1}=f_2(\theta_1),\ \cdots,\ \alpha_{st}=f_s(\theta_t).$$

If $\alpha_r=f_\nu(\theta_\mu)$, then we define

$$\phi_{r,n}=\sum_{k=1}^{n}a_{\nu,k}\theta_\mu^{\lambda_{\nu,k}},$$
$$\psi_{r,n}=f_\nu(\theta_\mu)-\phi_{r,n}.$$

From (4) and (5) we know that the condition (ii) coresponding to series and numbers given here holds. Now we prove that the coresponding condition (i) also holds.

Notise that $|\theta_1|>|\theta_2|>\cdots>|\theta_t|$. For any fixed $\alpha_r=f_\alpha(\theta_\beta)$, we consider two cases.

(a) If $\beta<t$. For any $\alpha_l=f_\alpha(\theta_\mu)$, where $\mu=\beta+1, \cdots, t$, we choose δ such that

$$0<\delta<\min\left(\frac{|\theta_\beta|-|\theta_{\beta+1}|}{|\theta_\beta|+|\theta_{\beta+1}|}R_\alpha,\ R_\alpha-|\theta_\beta|\right),$$

and put

$$\rho_1=R_\alpha-\delta,\ \rho_2=R_\alpha+\delta.$$

Like (27), we have

$$|\psi_{l,n}|\leqslant C_{12}(\rho_1|\theta_\mu|^{-1})^{-\lambda_{\alpha,n+1}}\quad (\mu=\beta+1, \cdots, t), \tag{31}$$

and it is similar to (28), there is an infinite sequence of natural numbers

$$\mathcal{N}_r=\{k_n(r),\ (n=1, 2, \cdots,)\}$$

such that

$$|\psi_{r,k_n(r)}|\geqslant C_{13}(\rho_2|\theta_\beta|^{-1})^{-\lambda_{\alpha,k_n(r)+1}}. \tag{32}$$

Noticing $\rho_2|\theta_\beta|^{-1}<\rho_1|\theta_{\beta+1}|^{-1}<\cdots<\rho_1|\theta_t|^{-1}$, from (31), (32), we obtain

$$|\psi_{l,k_n(r)}|=o(|\psi_{r,k_n(r)}|)\quad (n\to\infty), \tag{33}$$

where l is defined by $\alpha_l=f_\alpha(\theta_\mu)$, $\mu=\beta+1, \cdots, t$.

(b) If $\alpha<s$. For any $\alpha_l=f_\nu(\theta_\mu)$, where $\nu=\alpha+1, \cdots, s$; $\mu=1, \cdots, t$, we choose ρ_3 such that

$$\max_{1\leqslant\mu\leqslant t}|\theta_\mu|<\rho_3<\min_{\alpha+1\leqslant\nu\leqslant s}R_\nu.$$

Similar to (29), we have

$$|\psi_{l,n}|\leqslant C_{14}(\rho_3|\theta_\mu|^{-1})^{-\lambda_{\nu,n+1}}\quad (\nu=\alpha+1, \cdots, s;\ \mu=1, \cdots, t). \tag{34}$$

From (4), we get $\lim\limits_{n\to\infty}\lambda_{\alpha,n+1}/\lambda_{\nu,n+1}=0(\nu=\alpha+1, \cdots, s)$. Hence from (32) and (34) we obtain

$$|\psi_{l,k_n(r)}|=o(|\psi_{r,k_n(r)}|)\quad (n\to\infty), \tag{35}$$

where l is defined by $\alpha_l=f_\nu(\theta_\mu)$, $\nu=\alpha+1, \cdots, s$; $\mu=1, \cdots, t$.

From (33) and (35) we can deduce that condition (i) holds. By Lemma 5 we obtain the result.

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