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INVARIANT SETS AND THE HUKUHARA-KNESER PROPERTY FOR PARABOLIC SYSTEMS

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Abstraet

In this paper we are concerned with the nonlinear boundary value problem for parabolic system

$$\begin{cases} Lu=f(x, t, u, \nabla u), \ x\in\Omega, \ 0<t\leqslant T, \\ Bu=g(x, t, u), \ x\in\partial\Omega, \ 0\leqslant t\leqslant T, \\ u(x, 0)=h(x), \ x\in\bar{\Omega}, \end{cases} \tag{1}$$

where $Lu=(L_1u_1, \cdots, L_Nu_N)$ with L_k the second order uniformly parabolic operators which may be different from one another, and $Bu=(B_1u_1, \cdots, B_Nu_N)$ with B_k either the Dirichlet boundary operators or the regular oblique derivative ones. We have proved that a certain form of convex set is invariant for (1), that there exist solutions to (1) if $f=f(x, t, u, p)$ has an almost quadratic growth in p, and that the set of solutions possesses the Hukuhara-Kneser property.

§ 1. Introduction

The purpose of this paper is to discuss the invariant sets, the existence theory of solutions related to invariant sets, and the Hukuhara-Kneser property for the parabolic system

$$Lu=f(x, t, u, \nabla u)$$

with nonlinear boundary conditions. Here $u=(u_1, \cdots, u_N)$ and $f=(f_1, \cdots, f_N)$ are real vector functions, $Lu=(L_1u_1, \cdots, L_Nu_N)$ with each L_k a uniformly parabolic operator with real coefficients. Some of the boundary data concerned are of first kind, and the others are of second kind and regular oblique derivative conditions.

A set $D\subset R^N$ is said to be invariant for a parabolic boundary value problem if the given initial and or boundary values remaining in D imply that each solution remains in D. The existence of an invariant set provides an a priori bound for maximum norm, and it is helpful to solving the existence of solutions. The above parabolic boundary value problems, with reaction-diffusion systems as their main back-ground, have aroused great interest in recent-years. In the case that all the L_k's are the same for $k=1, \cdots, N$ and $f=f(x, t, u)$ is independent of ∇u, Weinberger[8]

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proved that a closed convex set D is invariant for the Dirichlet problem if f satisfies "weak tangent condition". Sharpening this result, Chueh et al.[3] considered the case that L_k may be different from one another. Bebernes et al[2]. generalized invariance results to include gradient dependent nonlinearities, but the boundary conditions they were concerned with are still linear. Talaga[7] investigated invariant sets for the parabolic systems, where all the $L_k's$ are the same, with nonlinear boundary data. In this paper we allow not only L_k to be different from one another, but nonlinear boundary conditions as well. Although Redheffer et al[6]. mentioned this kind of problem, they did not study it in depth. Besides the weak tangent condition, they required $f=f(x, t, u, p)$ to be Lipschitz continuous in u and p, and to satisfy the inequality $(f(x, t, u, \lambda p)-f(x, t, u, p))\times n(u)\leqslant 0$ for $\lambda>1$, where $n(u)$ is the outward normal to ∂D at u. In this paper we just require f to be Hölder-continuous and to satisfy tangent condition.

The existence results in this paper seem to be a little better than earlier ones for the above problems. Amann[1] studied only homogeneous linear boundary value problems for the same systems as in this paper. Talaga[7] considered only the Neumann problems for the systems where all the L_k' s are the same, and he imposed a severe restricion on f, i. e., f is assumed to be bounded in p. It is incidentally pointed out that the proof of his basic existence result-Corollary 3.4 is incorrect. In this paper we have cancelled the above restriction in [7] and replaced it by an almost necessary condition that f has an almost quadratic growth in p. The key to establishing better existence results is that the a priori bound for the norm in $H^{1+\alpha}(\bar{Q}_T)$ (i. e. $C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_T)$) has been deduced fromone of Solonnikov's results and the intperolation inequalities in $H^{k+\alpha}(\bar{Q}_T)$. The superfluous smoothness assumptions on f and the growth restriction in p on its Hölder constants in our earlier work[9] are cancelled here.

A set of solutions to a boundary value problem is said to possess the Hukuhara-Kneser property if it is compact and conneced in an appropriate Banach space. The uniqueness theorem deduced from [5, Th. 104, p.621] in this paper is better than that in [7, 9], thus improving the results concerning the Hukuhara-Kneser property in [7], and making it apply to nonlinear boundary value problems with oblique derivative conditions.

§ 2. Notation and General Hypotheses

For each positive integer k, R^k denotes the k dimensional Euclidean space. For each $y\in R^k$, $|y|$ denotes its Euclidean norm, i. e. $|y|=\left[\sum_{i=1}^{k} y_i^2\right]^{1/2}$.

Let Ω be a bounded domain in $R^n (n\geq 1)$ with boundary $S\in C^{2+\alpha}$. We write $Q_t=\Omega\times(0, t]$, $S_t=S\times[0, t]$ for each $t\in(0, T]$.

The Greek letters φ, ψ, etc., and English letters u, v, w, etc. are always used to denote scaler functions and vector functions with N components respectively in this paper.

For scalar functions, the spaces $L_q(Q_T)$, $W_q^{2,1}(Q_T)$, $W_q^{l,l/2}(S_T)$, $W_q^l(\Omega)$, $C(\bar{Q}_T)$, $C^{1,0}(\bar{Q}_T)$, $C^{2,1}(\bar{Q}_T)$, $H^{k+\alpha}(\bar{Q}_T)\equiv H^{k+\alpha,\frac{k+\alpha}{2}}(\bar{Q}_T)$, $H^{1+\alpha}(S_T)\equiv H^{1+\alpha,\frac{1+\alpha}{2}}(S_T)$, $C^{2+\alpha}(\Omega)$etc. used in this paper have the same meaning as those used in [5, Chs. I and II]. The norms in these spaces are denoted by $\|\cdot\|_q$, $\|\cdot\|_{2,1,q}$, $\|\cdot\|_{l,l/2,q}$, $\|\cdot\|_{l,q}$, $|\cdot|_0$, $|\cdot|_{1,0}$, $|\cdot|_{2,1}$, $|\cdot|_{k+\alpha}$ etc. respectively.

For a vector function, if each of its components belongs to some of the above spaces, then we say for short that it itself belongs to the space, and define the sum of the norms of all of its components as its norm.

Hypothesis(D). *$D=D^1\times\cdots\times D^r$, where D^k is an open bounded convex subset of R^{m_k} containing the origin for each $k\in\{1, \cdots, r\}$ with $m_1+\cdots+m_r=N$.*

Correspondingly, the expression of a function u: $R^n\to R^N$ often takes the form $u=(u^1, \cdots, u^r)$ in the sequel, where $u^k=(u_1^k, \cdots, u_{m_k}^k)\in R^{m_k}$ for $k=1, \cdots, r$. The Jecobian of u is denoted by

$$\begin{aligned}\nabla u&=(\nabla u^1, \cdots, \nabla u^r)=(\nabla u_1^1, \cdots, \nabla u_{m_1}^1;\cdots;\nabla u_1^r, \cdots, \nabla u_{m_r}^r)\\&=(\nabla u_1, \cdots, \nabla u_n)=(\partial u_j/\partial x_i), i=1, \cdots, n; j=1, \cdots, N.\end{aligned}$$

The matrix of variables $p=(p_{ij}), i=1, \cdots, n; j=1, \cdots, N$. will correspond to ∇u, and the m_k-vector p^{ik} $(i=1, \cdots, n; k=1, \cdots, r)$ to $\partial u^k/\partial x_i=\left(\frac{\partial u_1^k}{\partial x_i}, \cdots, \frac{\partial u_{m_k}^k}{\partial x_i}\right)$.

For $u_0^k\in\partial D^k$, an m_k-vector $n(u_0^k)$ is said to be the outward normal to ∂D^k at u_0^k if

$$n(u_0^k)\cdot(u_0^k-u^k)\geq 0$$

holds for each $u^k\in D^k$.

For $k\in\{1, \cdots, r\}$, L^k represents such a uniformly parabolic operator with real coefficients that for φ: $Q_T\to\mathbf{R}$

$$L^k\varphi=\varphi_t-\sum_{i,j=1}^n a_{i,j}^k(x, t)\varphi_{x_ix_j}+\sum_{i=1}^n a_i^k(x, t)\varphi_x$$

where (a_{ij}^k) is a symmetric, positive definite matrix. For $u=(u^1, \cdots, u^r)$ with u^k: $Q_T\to R^{m_k}$,

$$Lu=(L^1u^1, \cdots, L^ru^r),$$

where

$$L^ku^k=(L^ku_1^k, \cdots, L^ku_{m_k}^k).$$

Let s be a nonnegative integer not greater than r. Let $B^1, \cdots, B^s$ be the first kind of boundary operators, and $B^{s+1}, \cdots, B^r$ the boundary operators including probably regular oblique derivative ones, i. e., for ψ: $S_T\to R$,

$$B^k\psi=\begin{cases}\psi, & k=1,\cdots,s,\\ \sum_{i=1}^n b_i^k(x,t)\psi_{x_i}+b_0(x,t)\psi, & k=s+1,\cdots,r,\end{cases}$$

where $b_0^k(x,t)\geqslant 0$, $b^k(x,t)=(b_1^k(x,t),\cdots,b_n^k(x,t))$ satisfies the inequality $b^k(x,t)\cdot\nu(x)>0$ with $\nu(x)$ being the unit outward normal to s at x. For $u=(u^1,\cdots,u^r)$,

$$Bu=(B^1u^1,\cdots,B^ru^r),$$

where

$$B^ku^k=\begin{cases}u^k, & k=1,\cdots,s,\\ (B^ku_1^k,\cdots,B^ku_{m_k}^k) & k=s+1,\cdots,r.\end{cases}$$

In this paper we are concerned with the parabolic boundary value problem

$$\begin{cases}Lu=f(x,t,u,\nabla u), & (x,t)\in Q_T,\\ Bu=g(x,t,u), & (x,t)\in S_T,\\ u(x,0)=h(x), & x\in\bar{\Omega},\end{cases}\tag{1}$$

with f, g, and h satisfying the following hypotheses:

Hypothesis (R). *$f=(f^1,\cdots,f^r)$ with f^k: $\bar{Q}_T\times R^N\times R^{nN}\to R^{m_k}$ locally belonging to $H^{\alpha,\alpha/2,\alpha,\alpha}$ for $k=1,\cdots,r$. $g=(g^1,\cdots,g^r)$ with $g^k=g^k(x,t)$ independent of u and $g^k\in H^{2+\alpha}(S_T)$ when $k=1,\cdots,s$, and $g^k=g^k(x,t,u)$ having continuous partial derivatives with respect to x and u, and as a functon of (x,t), $g^k\in H^{1+\alpha}(\bar{Q}_T)$, $\partial g^k/\partial x_i$ and $\partial g^k/\partial u_j\in H^\alpha(\bar{Q}_T)$ for $i=1,\cdots,n$ and $j=1,\cdots,N$ when $k=s+1,\cdots,r$. $h=(h^1,\cdots,h^r)\in C^{2+\alpha}(\bar{\Omega})$.*

Hypothesis (C). *f, g and h satisfy the following compatibility conditions at $t=0$*

$$Bh(x)=g(x,0,h(x)),\ x\in S,$$

$$\frac{\partial g^k(x,0)}{\partial t}+L^kh^k(x)=f^k(x,0,h(x),\nabla h(x)),\ k=1,\cdots,s,\ x\in S.$$

Hypothesis (ST) **and** (WT). *Let $D\subset R^N$ satisfy Hypothesis (D). We say that f, with D, satisfies the strong tangent condition (ST) if for each $u_0=(u_0^1,\cdots,u_0^r)\in\bar{D}$ with any its component $u_0^k\in\partial D^k$ there exists an outward normal $n(u_0^k)$ to D^k at u_0^k such that*

$$f^k(x,t,u_0,p)\cdot n(u_0^k)<0\tag{2}$$

holds for $(x,t)\in\bar{Q}_T$ and those p whose elements satisfy $p^{ik}\cdot n(u_0^k)=0$, $i=1,\cdots,n$, and holds for $(x,t)\in S_T$ and those p whose elements satisfy

$$\sum_{i=1}^n b_i^k(x,t)p^{ik}+b_0^k(x,t)u_0^k=g^k(x,t,u_0).\tag{3}$$

If the strong tangent inequality (2) is replaced by the weak tangent inequality

$$f^k(x,t,u_0,p)\cdot n(u_0^k)\leqslant 0\tag{2'}$$

in the above definition, then we say that f, with D, satisfies the weak tangent condition (WT).

Hypothesis (T_g). *For $k=s+1,\cdots,r$ and each $(x,t)\in S_T$*

$$g^k(x,t,u_0)\cdot n(u_0^k)\leqslant 0,\ u_0^k\in\partial D^k,\ u_0\in\bar{D}.\tag{4}$$

Hypothesis (QG). *For each $(x,t)\in\bar{Q}_T$,*

$$|f(x, t, u, p)| \leqslant \mu(|u|)(1+|p|^{2-s}), \tag{5}$$

where $0<s<1$ *and* $\mu(\tau)$ *is a non decreasing function of* τ *for* $\tau \geqslant 0$.

§ 3. Invariant sets and existence of solutions

Theorem 1. *Let* f, g, h *and the coefficients of the operators* L *and* B *be continuous functions of their own variables and* $D \subset R^N$ *be an open, bounded, convex set satisfying* $HYP(D)$. *Let* f *and* g *satisfy the strong tangent condition* (ST) *and the weak tangent condition* (T_g) *respectively. If*

$$g^k: S_T \to D^k \text{ for } k=1, \cdots, s; \; h: \bar{\Omega} \to D$$

and $u \in C^{2,1}(\bar{Q}_T)$ *is a solution to the problem* (1), *then* $u: \bar{Q}_T \to D$; *i. e.,* D *is invariant for* (1).

Proof Assume it is not the case. Then there exists a time $t_0 \in (0, T]$ such that $u(x, t) \in D$ for $(x, t) \in \bar{\Omega} \times [0, t_0)$ and $u(x_0, t_0) = u_0 = (u_0^1, \cdots, u_0^r) \in \partial D$ for some $x_0 \in \bar{\Omega}$. In virtue of the construction of D, there exists $k \in \{1, \cdots, r\}$ such that $u_0^k = u^k(x_0, t_0) \in \partial D^k$ and $u^k(x, t) \in D^k$ for $(x, t) \in \bar{\Omega} \times [0, t_0)$.

Let $n(u_0^k)$ be the outward normal to D^k at u_0^k satisfying (2) and (4). Consider the function

$$\varphi(x, t) = u^k(x, t) \cdot n(u_0^k).$$

By the definitions of t_0 and u_0^k, $\varphi(x, t)$ attains its maximum $u_0^k \cdot n(u_0^k) \equiv M$ on $\bar{Q}_{t_0}$ at (x_0, t_0). Since D^k is convex, $M>0$.

Suppose that $x_0 \in \Omega$. Thus $\varphi_{x_i}(x_0, t_0) = 0$, i. e.

$$\frac{\partial u^k(x_0, t_0)}{\partial x_i} \cdot n(u_i^k) = 0, \; i=1, \cdots, n, \tag{6}$$

and

$$L^k \varphi(x_0, t_0) \geqslant 0. \tag{7}$$

On the other hand, by (6) and (2)

$$L^k \varphi(x_0, t_0) = f^k(x_0, t_0, u_0, \nabla u(x_0, t_0)) \cdot n(u_0^k) < 0$$

This contradicts (7).

Suppose that $x_0 \in S$. Since we assume that $g^j: S_T \to D^j$ for $j=1, \cdots, s$, k must be in $\{s+1, \cdots, r\}$. Therefore

$$\sum_{i=1}^n b_i^k(x_0, t_0) \partial u^k(x_0, t_0)/\partial x_i + b_0^k(x_0, t_0) u_0^k = g^k(x_0, t_0, u_0).$$

Using (2) again we get $L^k \varphi(x_0, t_0) < 0$. By the continuity of $L^k \varphi$, there exists an open ball B with center (x_0, t_0) such that $L^k \varphi < 0$ on $B \cap Q_{t_0}$. In addition, it is easy to see from the convexity of D^k and the above argument that $\varphi(x, t) < M$ for $(x, t) \in B \cap Q_{t_0}$. By Friedman [4, Th. 14, p.49],

$$\sum_{i=1}^n b_i^k(x_0, t_0) \varphi_{x_i}(x_0, t_0) > 0. \tag{8}$$

On the other hand, by the boundary conditions concerned and the inequality (4)

$$\sum_{i=1}^{n} b_i^k(x_0, t_0)\varphi_{x_i}(x_0, t_0) = g^k(x_0, t_0, u_0)\cdot n(u_0^k) - b_0^k(x_0, t_0)M \leqslant 0.$$

This contradicts (8) and proves the theorem.

In discussing the existence and uniqueness of solutions we will repeatedly make use of a basic result concerning linear $2b$-parabolic systems due to Solonnikov [5, Th. 10.4, p.621]. For simplicity, we state it only for the following linear parabolic problem

$$\begin{cases} u_t - \sum_{i,j=1}^{n} A_{ij}(x, t)u_{x_ix_j} + \sum_{i=1}^{n} A_i(x, t)u_{x_i} + A(x, t) = F(x, t), \ (x, t)\in Q_T, \\ \sum_{i=1}^{n} B_i(x, t)u_{x_i} + B_0 u = G(x, t), \ (x, t)\in S_T, \\ u(x, 0) = H(x), \quad x\in\overline{\Omega}. \end{cases} \tag{9}$$

Here $u=u(x,t)$, $F(x, t)$, $G(x,t)$ and $H(x)$ are vector functions with N components, $A_{ij}(x,t) = (a_{ij}^1(x,t), \cdots, a_{ij}^N(x,t))_{\text{diag}}$ and $B_i(x,t) = (\underbrace{0, \cdots, 0}_{m}, b_i^{m+1}(x, t), \cdots, b_i^N(x, t))_{\text{diag}}$ are $N\times N$ diagonal matrices, $A_i(x, t)$ and $A(x, t)$ are $N\times N$ matrices, and $B_0(x, t)$ is a $N\times N$ matrix with $(1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(\overbrace{0, \cdots, 0}^{m-1}, 1, 0, \cdots, 0)$ as its first m rows.

Solonnikov's Theorem. *Suppose that $S\in C^2$, $A_{ij}\in C(\overline{Q}_T)$, A_i and $A\in L_\infty(Q_T)$, B_i and $B_0\in H^{1-q^{-1}+\delta}(S_T)$, i, $j=1, \cdots, n$, where $1<q\neq 3$, $\delta>0$ can be arbitrarily small. Then the problem (9) with $F\in L_q(Q_T)$, $G_k\in W_q^{2-q^{-1},(2-q^{-1})/2}(S_T)$ for $k=1, \cdots, m$, $G_k\in W_p^{2-q^{-1},(2-q^{-1})/2}(S_T)$ for $k=m+1, \cdots, N$, adn $H\in W_q^{2-2q^{-1}}(\Omega)$, satisfying necessary compatibility conditions, has unique soluti on u, and the following estimate holds:*

$$\|u\|_{2,1,q} \leqslant C_1\Big(\|F\|_q + \sum_{k=1}^{m} \|G_k\|_{2-q^{-1},(2-q^{-1})/2,q} + \sum_{k=m+1}^{N} \|G_k\|_{1-q^{-1},(1-q^{-1})/2,q} + \|H\|_{2-2q^{-1},q}\Big), \tag{10}$$

where constant C_1 depends onlyon n, N, q, T, Ω and the supremums of the norms in the above spaces of the coefficients.

Now we make use of Theorem 1 to obtain an a priori bound for all the possible solutions of the problem (1).

Theorem 2. *Suppose that $S\in C^2$, $a_{ij}^k\in C(\overline{Q}_T)$, $a_i^k\in L_\infty(Q_T)$, b_i^k and $b_0^k\in H^\gamma(S_T)$. Suppose that f is a measurable function satisfying (5), $g^k\in W_q^{2-q^{-1},(2-q^{-1})/2}(S_T)$ for $k=1, \cdots, s$, g^k locally belongs to $H^{\gamma,\gamma/2,\gamma}(S_T\times R^N)$ for $k=s+1, \cdots, r$* and *$h\in W_q^{2-2q^{-1}}(\Omega)$ and that the necessary compatibility conditions are fulfilled.*

If $u\in W_q^{2,1}(Q_T)$ is a solution to the problem (1) with $|u|_0<M$, $1-q^{-1}<\gamma<1$ and $q\geqslant\frac{n+2}{\varepsilon}$ for ε in (5). Then the following estimate holds:

$$\|u\|_{2,1,q} \leqslant M_1 \tag{11}$$

where constant M_1 depends only on n, N, q, γ, s, T, Ω, M, $\mu(M)$ and the supremums of the norms in the above spaces of f, g, h and all the coefficients of L and B.

Proof Since $q > n+2$, $u \in W_q^{2,1}(Q_T)$ implies $u \in H^{1+\beta}(\bar{Q}_T)$, for any

$$\beta \in \left(1-s,\ 1-\frac{n+2}{q}\right) \quad \text{and} \quad |u|_{1+\beta} \leqslant C_2 \|u\|_{2,1,q}. \tag{12}$$

Define

$$F(x,\ t) = f(x,\ t,\ u(x,\ t),\ \nabla u(x,\ t))$$

and

$$G(x,\ t) = g(x,\ t,\ u(x,\ t)).$$

By HYP. (QG), the interpolation inequalities in $H^{k+\alpha}(\bar{Q}_T)$ (see e. g. [9, Lemma 2]), the inequality (12) and the fact that $|u|_0 \leqslant M$, we have

$$\begin{aligned} \|F\|_q &\leqslant \|\mu(M)(1+|\nabla u|^{2-s})\|_q \leqslant C_3(1+|\nabla u|_0^{2-s}) \\ &\leqslant C_4(1+|u|_{1+\beta}^{(2-s)/(1+\beta)}) \leqslant C_5(1+\|u\|_{2,1,q}^{(2-s)/(1+\beta)}). \end{aligned} \tag{13}$$

Since the embedding operator $H^{\gamma}(\bar{Q}_T) \hookrightarrow W_q^{1-q^{-1},(1-q^{-1})/2}(S_T)$ is bounded, we have

$$\begin{aligned} \|G^k\|_{1-q^{-1},(1-q^{-1})/2,q} &\leqslant C_6 |G^k| \leqslant C_7(1+|\nabla u|_0^{\gamma}) \\ &\leqslant C_8(1+|u|_{1+\beta}^{\gamma/(1+\beta)}) \leqslant C_9(1+\|u\|_{2,1,q}^{\gamma/(1+\beta)}), \\ k &= s+1,\ \cdots,\ r. \end{aligned} \tag{14}$$

Consider u as the solution to the linear boundary value problem

$$\begin{aligned} &Lu = F(x,\ t),\ (x,\ t) \in Q_T, \\ &Bu = G(x,\ t),\ (x,\ t) \in S_T, \\ &u(x,\ 0) = h(x),\ x \in \bar{\Omega}. \end{aligned}$$

and apply the estimates (10), (13) and (14) to it, we find that

$$\|u\|_{2,1,q} \leqslant C_{10}(1+\|u\|_{2,1,q}^{\frac{\gamma}{1+\beta}} + \|u\|_{2,1,q}^{\frac{2-s}{1+\beta}}).$$

Since $\beta > 1-s$ and thus $\dfrac{\gamma}{1+\beta} < 1$, $\dfrac{2-s}{1+\beta} < 1$, (11) follows from the last inequality.

This completes the proof of Theorem 2.

Corollary. *Under the conditions of Theorem 2. there exists $M_2 > 0$ such that*

$$|u|_{1+\beta} \leqslant M_2$$

for any given $\beta \in (0,\ 1-(n+2)/q)$.

Theorem 3. *Suppose that a_{ij}^k, $a_i^k \in H^{\alpha}(\bar{Q}_T)$, b_i^k, $b_0^k \in H^{1+\alpha}(S_T)$, and that $D \subset R^N$ satisfies HYP(D), f, g and h satisfy HYP. (R), (C), (QG), (TG) and (ST) with λg^k, $0 \leqslant \lambda \leqslant 1$, in place of g^k in (3). In addition, assume that*

$$\left.\begin{aligned} &f^k(x,\ 0,\ h(x),\ \nabla h(x)) = 0,\ k = 1,\ \cdots,\ s, \\ &g^k(x,\ 0,\ h(x)) = 0,\ k = s+1,\ \cdots,\ r, \end{aligned}\right\} \quad x \in S.$$

If g^k: $S_T \to D^k$ for $k = 1,\ \cdots,\ s$ and if $h: \bar{\Omega} \subset D$, then the problem (1) has a solution

$u \in H^{2+\alpha}(\bar{Q}_T)$ *with* $u:\bar{Q}_T \to D$.

Proof Firstly, define vector functions $\bar{g}(x, t, u), \bar{\bar{g}}(x, t, u), \bar{h}(x)$ and $\bar{\bar{h}}(x)$ as follows:

for $k=1, \cdots, s$

$$\bar{g}^k(x, t)=g^k(x, t),\ \bar{\bar{g}}^k(x, t, u)=0,\ \bar{h}^k(x)=h^k(x),\ \bar{\bar{h}}^k(x)=0,$$

and for $k=s+1, \cdots, r$

$$\bar{g}^k(x, t)=0,\ \bar{\bar{g}}^k(x, t, u)=g^k(x, t, u),\ \bar{h}^k(x)=0,\ \bar{\bar{h}}^k(x)=h^k(x).$$

Then, define operators F and G as follows: for $w:\bar{Q}_T \to R^N$ define $Fw:\bar{Q}_T \to R^N$ by

$$(Fw)^k(x, t)=\begin{cases} f^k(x, t, w(x, t), \nabla w(x, t))-f^k(x, 0, w(x, 0), \nabla w(x, 0)), \\ \quad k=1, \cdots, s, \\ f^k(x, t, w(x, t), \nabla w(x, t)), \quad k=s+1, \cdots, r; \end{cases}$$

for $w:S_T \to R^N$ define $Gw:S_T \to R^N$ by

$$Gw(x, t)=\bar{\bar{g}}(x, t, w(x, t))-\bar{\bar{g}}(x, 0, w(x, 0)).$$

It is obvious that $F:C^{1,0}(\bar{Q}_T) \to C(\bar{Q}_T)$ is a bounded continuous operator, and that

$$(Fw)^k(x, 0)=0,\ (Gw)^k(x, t)\equiv 0,\ k=1, \cdots, s$$

and

$$Gw(x, 0)=0$$

for $x \in S$ and $t \in [0, T]$.

Next, define operators T_1 and T_2 as follows: for $v \in H^\beta(\bar{Q}_T)$ with $v^k(x, 0)=0$ for $k=1, \cdots, s$ and $x \in S$, let $u=Tv$ be the unique solution of the linear problem

$$\begin{cases} Lu=v(x, t), & (x, t) \in Q_T, \\ Bu=\bar{g}(x, t), & (x, t) \in S_T, \\ u(x, 0)=\bar{h}(x), & x \in \bar{\Omega}. \end{cases}$$

$T_1:H^\beta(\bar{Q}_T) \to H^{2+\beta}(\bar{Q}_T)$ is a bounded continuous operator for any $\beta \in (0, 1)$. By Solonnikov's Theorem stated above, T_1 may be extended to be a bounded continuous operator from $L_q(Q_T)$ to $W_q^{2,1}(Q_T)$ for $1<q\neq 3$. It is easy to see that the embedding operator $C(Q_T) \hookrightarrow L_q(Q_T)$ is bounded. And by the Sobolev embedding results $W_q^{2,1}(Q_T)$ can be compactly embedded in $H^{1+\beta}(\bar{Q}_T)$ for $0<\beta<1-\frac{n+2}{q}$. Therefore, $T_1:C(\bar{Q}_T) \to H^{1+\beta}(\bar{Q}_T)$ is a compact continuous operator.

For $v \in H^{1+\gamma}(S_T)$ with $v(x, 0)=0$ and $v^k(x, t)\equiv 0$ for $x \in S$, $t \in [0, T]$ and $k=1, \cdots, s$, let $u=T_2 v$ be the unique solution of the linear problem

$$\begin{cases} Lu=0, & (x, t) \in Q_T, \\ Bu=v(x, t), & (x, t) \in S_T, \\ u(x, 0)=\bar{\bar{h}}(x), & x \in \bar{\Omega}. \end{cases}$$

$T_2:H^{1+\gamma}(S_T) \to H^{2+\gamma}(\bar{Q}_T)$ is a bounded continuous operator for any $\gamma \in (0, 1)$.

Fix $\beta \in \left(1-\varepsilon,\ 1-\frac{n+2}{q}\right)$ and set $X=C^{1,0}(\bar{Q}_T) \times H^{1+\beta}(\bar{Q}_T)$. For $(u, v) \in X$, define norm$\|(u, v)\| = |u|_{1,0} + |v|_{1+\beta}$. Then X is a Banach space. Let T: $X \to X$ be defined by

$$(U, V) = T(u, v) \equiv (T_1 F(u+v), T_2 G(T_1 F(u+v)+v)).$$

As has been stated above

$$(u, v) \in X \Rightarrow u+v \in C^{1,0}(\bar{Q}_T) \Rightarrow F(u+v) \in C(\bar{Q}_T)$$
$$\Rightarrow U = T_1 F(u+v) \in H^{1+\beta}(\bar{Q}_T) \Rightarrow U+v \in H^{1+\beta}(\bar{Q}_T)$$
$$\Rightarrow G(U+v) \in H^{1+\gamma}(S_T) \Rightarrow V = T_2 G(U+v) \in H^{2+\gamma}(\bar{Q}_T).$$

Since F, G, T_1 and T_2 are all bouned continuous, T is a bounded continuous operator from X to $H^{1+\beta}(\bar{Q}_T) \times H^{2+\beta}(\bar{Q}_T)$, and hence a compact continuous operator from X to itself. If (u, v) is a fixed point of T in X, then u and v satisfy

$$Lu = F(u+v),\ Lv = 0, \qquad (x, t) \in Q_T,$$
$$Bu = \bar{g}(x, t),\ Bv = G(u+v), \qquad (x, t) \in S_T,$$
$$u(x, 0) = \bar{h}(x),\ v(x, 0) = \bar{\bar{h}}(x), \qquad x \in \bar{\Omega},$$

and $w = u+v$ is just a solution of the problem (1) with $w \in H^{1+\beta}(\bar{Q}_T)$. By a bootstrap argument we can show rhat $w \in H^{2+\alpha}(\bar{Q}_T)$. And by Theorem 1, $w(\bar{Q}_T) \subset D$. Thus, the proof of Theorem 3 is reduced to proving that the compact continuous operator T has fixed points.

Let $T_{(\lambda)}: X \times [0, 1] \to X$ be the homotopy defined by

$$T_{(\lambda)}(u, v) = (T_1(\lambda F(u+v)), T_2[\lambda G(T_1(\lambda F(u+v))+v)]).$$

By an argument similar to that for T, we find that $T_{(\lambda)}$ is a compact continuous operator from $X \times [0, 1]$ to X, and that if (u, v) is its fixed point, then $w(\bar{Q}_T) \subset D$ where $w = u+v$.

Let $M = \sup \{|w|; w \in D\}$. As mentioned above for any possible fixed point (u, v) of $T_{(\lambda)}$, $|u+v|_0 \leqslant M$. Imitating the proof of Theorem 2, from this we deduce that

$$\|(u, v)\| \leqslant |u|_{1+\beta} + |v|_{1+\beta} \leqslant M'.$$

Denote

$$\mathcal{O} = \{(u, v) \in X \mid u+v: \bar{Q}_T \to D,\ \|(u, v)\| < M'+1\}.$$

$\mathcal{O}$ is a nonempty open bounded subset of X. It is easy to see that $T_{(1)} = T$ and $I - T_{(0)} = 0$ has unique solution in $\mathcal{O}$. Therefore

$$d(I-T, \mathcal{O}, 0) = d(I-T_{(\lambda)}, \mathcal{O}, 0) = d(I-T_{(0)}, \mathcal{O}, 0) \neq 0$$

and hence T has at least one fixed point. Theorem 3 is thus proved.

Remark. It should be pointed out that Theorem 3 has improved the basic existence results-Theorem 3.3 and Corollary 3, 4 in [7] -of Talaga in several aspects. One is that L_k' s may be different from one another. Another is that the conormal derivative conditions are extended to the general nonlinear regular oblique derivative condtitons. The third one is concerning the restriction on f. The assumption that f is bounded in p is weakened by that f has an almost quadratic growth in p. In addition, it is impossible that the inequality $g(x, 0, h(x)) \cdot n(u_0) \leqslant 0$ which was used to prove Corollary 3.4 in [7] holds for any $u_0\in\partial D$ unless that $g(x, 0, h(x))\equiv 0$.

The example given in [7] shows that, even for the case of one equation with conormal derivative condition, if the strong tangent condition (ST) is changed to the weak tangent condition (WT), then Theorem 1 becomes false, i. e., $h(\bar{\Omega})\subset D$ is not sufficient to ensure $u(\bar{Q}_T)\subset D$ for any solution u. However, Theorem 4.1 in [7] shows that, for linear conormal derivative conditions, $h(\bar{\Omega})\subset D$ ensures that there exists at least one solution that remains in D. Now we generalize this result to more general nonlinear boundary conditions.

Theorem 4. *Let the hypotheses of Theorem 3 hold with (ST) replaced by (WT) Then for any g^k: $S_T\to D^k$, $k=1, \cdots, s$, and any h: $\bar{\Omega}\to D$, there exists a solution $u\in H^{2+\alpha}(\bar{Q}_T)$ to the problem (1) with u: $\bar{Q}_T\to\bar{D}$.*

Proof For $\lambda\in(0, 1]$, consider the perturbed problem

$$\begin{cases} Lu=f(x, t, u, \nabla u)-t\lambda u, & (x, t)\in Q_T,\\ Bu=g(x, t, u), & (x, t)\in S_T,\\ u(x, 0)=h(x), & x\in\bar{\Omega}. \end{cases} \tag{1_λ}$$

It is easy to verify that $f_\lambda=f(x, t, u, \nabla u)-t\lambda u$ is compatible with g and h, and satisfies the strong tangent condition (ST). By Theorem 3 the problem $(1)_\lambda$ has a solution $u_{(\lambda)}\in H^{2+\alpha}(\bar{Q}_T)$ with $u_{(\lambda)}$: $\bar{Q}_T\to D$. By the corollary of Theorem 2, there exists a positive constant M_2 independent of λ such that $|u_{(\lambda)}|_{1+\beta}\leqslant M_2$ for $\beta\in(0, 1)$. Using the Schauder estimate for linear parabolic equations (see e. g. [5, Ths. 5.2 and 5.3, p. 320]) we get $|u_{(\lambda)}|_{2+\alpha}\leqslant M_3$ for some constant M_3. Thus, it follows from Ascoli-Alzera Lemma that there exists a sequence $\{u_{(\lambda_j)}\}$, as $\lambda_j\to 0$, converges in $C^{2,1}(\bar{Q}_T)$ to some function $u\in H^{2+\alpha}(\bar{Q}_T)$. By letting $\lambda_j\to 0$ in (1_{λ_j}) we find that u is a solution to the problem(1). Since $u=\lim\limits_{\lambda_j\to 0} u_{(\lambda_j)}$ and $u_{(\lambda_j)}$:$\bar{Q}_T\to D$, u takes $\bar{Q}_T$ to $\bar{D}$. Theorem 4 is thus proved.

Theorem 5. *Let $D=D^1\times\cdots\times D^r$ and D^k be a nonempty compact convex subset of R^{m_k} containing the origin for $k\in\{1, \cdots, r\}$ with $m_1+\cdots+m_r=N$. Suppose that a_{ij}^k and $a_i^k\in H^\alpha(\bar{Q}_T)$, b_i^k and $b_0^k\in H^{1+\alpha}(S_T)$, and that f, g and h satisfy HYP. (R), (C) and (QG), and f satisfies the weak tangent condition (WT) for any outward normal $n(u_0^k)$ to ∂D^k at u_0^k. In addition, assume that*

$$f^k(x, 0, h(x), \nabla h(x))=0,\ x\in S,\ k=1, \cdots, s$$

and g^k, $k=s+1, \cdots, r$, satisfy either

$$g^k(x, t, u)\equiv 0,\ (x, t)\in S_T,\ u\in\mathbf{R} \tag{15}$$

or

$$\begin{aligned} &g^k(x, 0, h(x))=0,\\ &g^k(x, t, u_0)\cdot n(u_0^k)\leqslant-\delta,\ \delta>0. \end{aligned} \tag{16}$$

Then for any g^k:$S_T\to D^k$, $k=1, \cdots, s$, and any h: $\bar{\Omega}\to D$, there exists a solution $u\in H^{2+\alpha}(\bar{Q}_T)$ to the problem (1) with u: $\bar{Q}_T\to D$.

Proof We consider first the case that g^k, $k=s+1, \cdots, r$, satisfy (15). Since D^k is compact and convex, for each $u^k \in R^{m_k}$, there exists unique $Pu^k \in D^k$ such that $\operatorname{dist}(u^k, Pu^k) = \operatorname{dist}(u^k, D^k)$, $k=1, \cdots, r$. For $u=(u^1, \cdots, u^r)$, define $Pu=(Pu^1, \cdots, Pu^r)$. As is well known, P is uniformly Lipschitz continuous. For $\lambda \in (0, 1]$, consider $D_\lambda = D^1_\lambda \times \cdots \times D^r_\lambda$, where $D^k_\lambda = \{u^k \in R^{m_k}: \operatorname{dist}(u^k, D^k) < \lambda\}$ are open bounded convex subsets of R^{m_k} for $k=1, \cdots, r$. If $u^k \in \partial D^k_\lambda$, then $Pu^k \in \partial D^k$ and the vector $u^k - Pu^k$ is an outer normal to D^k at Pu^k and an outer normal to D^k_λ at u^k. It is easy to verify that, for any open convex set D_λ, the problem

$$\begin{cases} Lu = f(x, t, Pu, \nabla u), & (x, t) \in Q_T, \\ u^k = g^k(x, t), \ k=1, \cdots, s, & (x, t) \in S_T, \\ B^k u^k = 0, \ k=s+1, \cdots, r, & \\ u(x, 0) = h(x), & x \in \overline{\Omega} \end{cases} \tag{17}$$

satisfies all the conditions of Theorem 4. Thus, there exists a solution $u_{(\lambda)} \in H^{2+\alpha}(\overline{Q}_T)$ with $u_{(\lambda)}: \overline{Q}_T \to \overline{D}_\lambda \subset \overline{D}_1$ for each λ. By means of a limit process similar to that used in the proof of Theorem 4. we find a solution $u \in H^{2+\alpha}(\overline{Q}_T)$ to the problem (17) with $u: \overline{Q}_T \to D$. Since u remains in D, $Pu = u$ and so u is the required solution to the problem (1) with g^k satisfying (15) for $k=s+1, \cdots, r$.

Now we assume that (16) holds. For simplicity, we consider only the case that $s=0$, i. e., the boundary conditions do not include the Dirichlet conditions. Then, Theorem 4 does not apply directly to the auxiliary problem

$$\begin{cases} Lu = f(x, t, Pu, \nabla u), & (x, t) \in Q_T, \\ Bu = g(x, t, Pu), & (x, t) \in S_T, \\ u(x, 0) = h(x), & x \in \overline{\Omega}, \end{cases} \tag{18}$$

because the boundary data $g^k(x, t, Pu)$, $k=s+1, \cdots, r$, do not satisfy HYP. (R), but satisfy uniform Lipschitz condition.

Construct a sequence $\{P_{(j)}u\} \subset C^2(\overline{D}_1)$ such that $P_{(j)}u$ converges in $H^\gamma(\overline{D}_1)$ to Pu as $j \to \infty$, with $0<\gamma<1$. Consider the problem

$$\begin{cases} Lu = f(x, t, Pu, \nabla u), & (x, t) \in Q_T, \\ Bu = g_{(j)}(x, t, u), & (x, t) \in S_T, \\ u(x, 0) = h(x), & x \in \overline{\Omega}, \end{cases} \tag{18_j}$$

where $g_{(j)}(x, t, u) = g(x, t, P_{(j)}u) - g(x, 0, P_{(j)}h(x))$. It is easy to see that, for $f=f(x, t, Pu, \nabla u)$, $g=g_{(j)}(x, t, u)$ and $h=h(x)$, HYP. (R) and (C) are fulfilled, and that $g_{(j)}(x, t, u)$, with $D_{1/j}$, satisfies the weak tangent condition $T_{g_{(j)}}$:

$$g^k_{(j)}(x, t, u_0) \cdot (u^k_0 - Pu^k_0) = g^k(x, t, Pu_0) \cdot (u^k_0 - Pu^k_0) + [g^k(x, t, P_{(j)}u_0) \\ - g^k(x, t, Pu_0) - g^k(x, 0, P_{(j)}h(x))] \cdot (u^k_0 - Pu^k_0) \leqslant 0$$

for j large. By Theorem 4, the problem (18_j) has a solution $u_{(j)} \in H^{2+\alpha}(\overline{Q}_T)$ with $u_{(j)}$: $\overline{Q}_T \to \overline{D}_{1/j} \subset \overline{D}_1$ for each j. By Theorem 2 and its Corollary, the estimates $\|u_{(j)}\|_{2,1,q} \leqslant M_1$

and $|u_{(j)}|_{1+\beta}\leqslant M_2$ hold for $q\geqslant\frac{n+2}{s}$ and $\beta\in\left(0,1-\frac{n+2}{q}\right)$, so there is a subsequuence (denoted still by $\{u_{(j)}\}$) of $\{u_{(j)}\}$ strongly converging in $C^{1,0}(\bar{Q}_T)$ and weakly converging in $W_q^{2,1}(Q_T)$ to some function $u\in H^{1+\beta}(\bar{Q}_T)\cap W_q^{2,1}(Q_T)$. Thus, as $j\to\infty$

$$
\begin{aligned}
&f(x,\ t,\ Pu_{(j)},\ \nabla u_{(j)})\to f(x,\ t,\ Pu,\ \nabla u) && \text{in } C(\bar{Q}_T),\\
&g_{(j)}(x,\ t,\ u_{(j)})\to g(x,\ t,\ Pu) && \text{in } C(\bar{Q}_T),\\
&Bu_{(j)}\to Bu && \text{in } C(\bar{Q}_T),\\
&Lu_{(j)}\to Lu \text{ weakly} && \text{in } L_q(Q_T)
\end{aligned}
$$

This shows that u is a solution in $W_q^{2,1}(Q_T)$ to the problem (18). Since $u=\lim_{j\to\infty} u_{(j)}$ and $u_{(j)}:\bar{Q}_T\to\bar{D}_{1/j}$, u takes $\bar{Q}_T$ to D. Therefore, $Pu=u$ and u also solves the problem (1). By a standard reguralily argument, $u\in H^{2+\alpha}(\bar{Q}_T)$. This completes the proof of Theorem 5.

§ 4. The Hukuhara-Kneser Property

In this paragraph we mainly show that the set of solutions of the problem (1) possesses H-K property, i. e., it is compact and connected in $C^{1,0}(\bar{Q}_T)$. A special case of H-K property is the uniqueness of solutions.

Theorem 6. *Let the coefficients of L and B satisfy the same assumption as in Theorem 3, and f, g and h satisfy $HYP(R)$. In addition, assume that $f(x,\ t,\ u,\ p)$ has bounded derivatives with respect to u and p. Then the problem* (1) *has at most one solution in $C^{2,1}(\bar{Q}_T)$*

Proof Let both u and v are solutions in $C^{2,1}(\bar{Q}_T)$ to the problem (1). It is eassy to see that $w=u-v$ satisfies the following linear problem

$$
\begin{cases}
Lw+\sum_{i=1}^{n} A_i(x,\ t)\dfrac{\partial w}{\partial x_i}+A(x,\ t)w=0, & (x,\ t)\in Q_T,\\
Bw+B_0(x,\ t)w=0, & (x,\ t)\in S_T,\\
w(x,\ 0)=0, & x\in\bar{\Omega},
\end{cases}
\tag{19}
$$

where

$$A_i(x,\ t)=\int_0^1\frac{\partial f}{\partial p_i}(x,\ t,\ u(x,\ t)+\tau w(x,\ t),\ \nabla u(x,\ t)+\tau\nabla w(x,\ t))d\tau,$$

$$i=1,\ \cdots,\ n;\ p_i=(p_{i1},\ \cdots,\ p_{iN}),$$

$$A(x,\ t)=\int_0^1\frac{\partial f}{\partial u}(x,\ t,\ u(x,\ t)+\tau w(x,\ t),\ \nabla u(x,\ t)+\tau\nabla w(x,\ t))d\tau,$$

$$B_0(x,\ t)=\int_0^1\frac{\partial g}{\partial u}(x,\ t,\ u(x,\ t)+\tau w(x,\ t))d\tau.$$

Consider (19) as a linear boundary value problem. By using Solonnikov's Theorem with $q\neq3$ and $1<q<\frac{1}{1-\alpha}$, we find that $\|w\|_{2,1,q}=0$, i. e., $u\equiv v$. This thus proves Theorem 6.

Remark. Theorem 6 has sharpened our earlier uniqueness result (see [9]) concerning nonlinear boundary value problem.

The examples given in [7] show that if f does not have enough smoothness then the solutions to the problem (1) may not be unique. Thus, it is interesting to study the H-K property of the set of solutions. Similar to [7], an abstracted version of the result in this aspect (see [2]) is stated without proof here.

B-S Theorem. *Let X be a Banach space, $\mathcal{O}\subset X$ a nonempty, bounded, open set and $T:\bar{\mathcal{O}}\to X$ a compact continuous operator,. Suppose that*

(a) $d(I-T, \mathcal{O}, 0)\neq 0$;

(b) *there exists a sequence of compact continuous operators $T_k:\mathcal{O}\to X$, $k=1, 2, \cdots$ such that $\delta_k=\sup\limits_{w\in\bar{\mathcal{O}}}\{|T_kw-Tw|\}\to 0$ as $k\to+\infty$ and*

(c) *the equation $w=T_kw+w_0-T_kw_0$ has at most one solution in $\mathcal{O}$ for any solution w_0 of $w=Tw$.*

Then the set of fixed points

$$Q=\{w\in\mathcal{O}: w=Tw\}$$

is a compact connected set in X.

Theorem 7. *Let the hypothesis of Theorem 3 hold. For g^k: $S_T\to D^k$, $k=1, \cdots, S$ and h: $\bar{\Omega}\to D$, the set of solutions of the problem (1) is a compact connected set in $C^{1,0}(\bar{Q}_T)$.*

Proof The notation in the proof of Theorem 3 will be used here. Let

$$Q=\{(u, v)\in\mathcal{O}:(u, v)=T(u, v)\}.$$

Since the mapping $(u, v)\to u+v$ from Q to $C^{1,0}(\bar{Q}_T)$ is contininuous, the set Q corresponds to the set of solutions of the problem (1) in an evident way. Therefore, in order to prove the theorem, it is sufficient to show that the set Q is compact and connected in X. It turns out that we need only to verify the conditions of B-S Theorem. By the proof of Theorem 3, (a) is satisfied. The verification of (b) and (c) is similar to that in [7, Theorem 5.6] with the only difference that Theorem 6 should be used to replace Lemma 5.5 in [7].

Theorem 8. *Let the hypothesis of Theorem 4 or 5 hold. For g^k: $S_T\to D^k$, $k=1, \cdots, s$ and h: $\bar{\Omega}\to D$, the set of solutions of the problem (1) that remains in D is a compact connected subset of $C^{1,0}(\bar{Q}_T)$.*

Proof The proof is similar to that of Theorem 5.4 in [7], so it is omitted here.

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