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EXISTENCE THEOREMS FOR RANDOM MEASURES

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Abstract

Two existence theorems of random measures on a separable complete metric space are proved. It seems that the theory of random measures in locally compact spaces and that in separable complete metric spaces. are essentially different by noting that the critera for tightness of the locally finite measures is much more tedious than that of the Radom measures.

1° In this paper we shall prove two existence theorems, i. e. the Theorems **5°** and **6°** below, for random measures on a separable complete metric space X with metric ρ. Restricting to the locally compact space X or to the stochastic point processes, our Theorem 6 is just Theorem 5.3 in [2] and Theorem 1.3.5 in [1] respectively. Our Theorem **5°** is More useful than Theorem **6°** becuse the continuity conditions in it are denumerable (i. e. (26) and (27)). In another paper relevant theorems for infinite divisible random measures are obtained mainly by our Theorem 5. It seems that the theory of random measures on locally compact metric spaces and in separable complete metric spaces are essentially different by noting that the criteria for tightness of locally finite measures in [1, Theorem 3.2.5] is much more tedious than the Radom measures in [2, A 7.5].

2° At first, let us recall the definition of random measure. Let $\mathbf{B}$ be the Borel algebra of X, $\mathbf{B}^*$ be the ring consisting of all bounded sets in $\mathbf{B}$, and let $\mathbf{W}$ be Borel algebra in real line. We say that a measure μ on $(X, \mathbf{B})$ is locally finite if $\mu A<\infty$ for all $A\in\mathbf{B}^*$. Let $\mathbf{M}$ be the class of all such measures, and let $\mathbf{B}(\mathbf{M})$ be the σ-ring generated by the sets such as

$$\{\mu;\ \mu A_i\in W_i,\ i=1,\ 2,\ \cdots,\ n\}$$

for $A_i\in\mathbf{B}^*$, $W_i\in\mathbf{W}$ and $n=1, 2, \cdots$. By a random measure we mean any measurable mapping of some fixed probablity space $(\Omega, \mathfrak{A}, P)$ into $(\mathbf{M}, \mathbf{B}(\mathbf{M}))$. For a mapping $\xi:\Omega\to\mathbf{M}$ we have

$$\xi^{-1}\{\mu;\ \mu A_i\in W_i,\ i=1,\ 2,\ \cdots,\ n\}=\{\omega;\ \xi(A_i,\ \omega)\in W_i,\ i=1,\ 2,\ \cdots,\ n\}$$

and ξ is a random measure if and only if

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$$\{\omega;\ \xi(A_i,\ \omega)\in W_i,\ i=1,\ 2,\ \cdots,\ n\}\in\mathfrak{A} \tag{1}$$

for $A_i\in\mathbf{B}^*$, $W_i\in\mathbf{W}$ and $n=1,\ 2,\ \cdots$.

3° Let $X_s=\{x_1,\ x_2,\ \cdots\}$ be the denumerable dense subset of X. Let $\mathscr{D}$ be a denumerable dense subset of the positive numbers $(0,\ \infty)$, and let

$$\mathbf{B}_0=\{\text{open ball } B(x,\ r);\ x\in X_s,\ r\in\mathscr{D}\},$$

$$\mathbf{B}_1=\text{the ring generated by } \mathbf{B}_0.$$

For fixed sequences $r_k,\ r'_k\in\mathscr{D}$ satisfying

$$r_1>r_2>\cdots>r_n\to 0,\ k<r'_k<k+1, \tag{2}$$

let

$$\Gamma_0=\{B(x_1,\ r'_{j+1})-B(x_1,\ r'_j);\ j=1,\ 2,\ \cdots\}\cap\{B(x_1,\ r'_1)\}.$$

Assume that $\Gamma_0,\ \Gamma_1,\ \cdots,\ \Gamma_n$ have been constructed and let

$$\Gamma_{n+1}=\{[B(x_j,\ r_{n+1})-\bigcup_{i<j}B(x_i,\ r_{n+1})]\cap A;\ A\in\Gamma_n,\ j=1,\ 2,\ \cdots\}.$$

Such a family of $\Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \cdots$ is called the family of dissecting systems related to $\mathscr{D}$. It is evident that

$$A\cap B=\emptyset \text{ for } A,\ B\in\Gamma_n \text{ and } A\neq B; \tag{3}$$

$$\bigcup_{A\subset B,\,A\in\Gamma_{n+1}} A=B \text{ for } B\in\Gamma_n; \tag{4}$$

$$\bigcup_{A\in\Gamma_n} A=X; \tag{5}$$

$$\text{each } A\in\Gamma_{n+1} \text{ is contained in a } B\in\Gamma_n; \tag{6}$$

$$\text{each } \Gamma_n \text{ is denumerable and contained in } \mathbf{B}_1; \tag{7}$$

and

$$\dim A\leqslant 2r_n \text{ for } A\in\Gamma_n \text{ and } n\geqslant 1. \tag{8}$$

4° Lemma *Suppose that μ is a finite nonnegative additive set function on $\mathbf{B}_1$ and satisfies*

$$\lim_{r\to t\ r\in\mathscr{D}}\mu B(x,\ r)=\mu B(x,\ t) \text{ for } B(x,\ t)\in\mathbf{B}_0, \tag{9}$$

and

$$\sum_{A\subset B,\,A\in\Gamma_{n+1}}\mu A=\mu B \tag{10}$$

for $B\in\Gamma_n$, $n=0,\ 1,\ 2,\ \cdots$. Then μ is σ additive.

Proof Let $X_i\in\mathbf{B}_1(i=1,\ 2,\ \cdots)$ and $X_0=\bigcup_i X_i\in\mathbf{B}_1$. It is sufficient to prove

$$\mu X_0\leqslant\sum_i\mu X_i. \tag{11}$$

Let A^c, A^0, $\overline{A}$ and ∂A be the complement, the interior, the closure and the boundary of the set A respectively, and $\rho(A_1,\ A_2)$ be the distance between the sets A_1 and A_2. For $A\in\mathbf{B}_1$ and $\varepsilon>0$ there exist $A_1,\ A_2\in\mathbf{B}_1$ satisfying

$$\overline{A}_1\subset A^0\subset\overline{A}\subset A_2^0,\ \ \mu(A_2-A_1)<\varepsilon, \tag{12}$$

$$\rho(A_1,\ A_2^c)>0,$$

since the family $\{A;\ A\in\mathbf{B}_1$, for $\varepsilon>0$ there exist $A_1,\ A_2\in\mathbf{B}_1$ satisfying (12)$\}$ forms a ring and contains $\mathbf{B}_0$ by (9).

For a natural number k, let Y_1, Y_2 be the sets in $\mathbf{B}_1$ satisfying

$$\overline{Y}_1 \subset X_0 \subset \overline{X}_0 \subset Y_2^0, \quad (Y_2 - Y_1) < \frac{1}{2^k} \mu X, \tag{13}$$

$$\rho(Y_1, Y_2^c) > 0,$$

then by (2) there is a natural number n_0 satisfying

$$2r_{n_0} < \rho(Y_1, Y_2^c).$$

Let

$$\Gamma^*_{n_0k} = \{B;\ B \in \Gamma_{n_0},\ B \subset Y_2^0\},$$

then

$$\Gamma^*_{n_0k} \supset \{B;\ B \in \Gamma_{n_0},\ B \cap Y_1 \neq \emptyset\}.$$

Now let us prove that

$$\mu Y_1 \leqslant \sum_{B \in \Gamma^*_{n_0k}} \mu B. \tag{14}$$

Let j be a natural number such that

$$Y_2 \subset B(x_1, r'_j),$$

and suppose (14) is false, i. e.

$$\mu Y_1 > \sum_{B \in \Gamma^*_{n_0k}} \mu B. \tag{15}$$

By (10) (15) and

$$\bigcup_{B \subset B(x_1, r'_j) B \in \Gamma_{n_0} - \Gamma^*_{n_0k}} B \subset B(x_1, r'_j) - Y_1,$$

we have the following contradiction

$$\mu B(x_1, r'_j) = \sum_{B \subset B(x_1, r'_j), B \in \Gamma_{n_0}} \mu B = \sum_{B \in \Gamma^*_{n_0k}} \mu B + \sum_{B \subset B(x_1, r'_j), B \in \Gamma_{n_0} - \Gamma^*_{n_0k}} \mu B$$
$$< \mu Y_1 + \mu(B(x_1, r'_j) - Y_1) = \mu B(x_1, r'_j),$$

and (14) is obtained. By (13) and (14), we have

$$\mu X_0 \leqslant \mu Y_2 < \mu Y_1 + \frac{1}{2^k} \mu X_0 \leqslant \sum_{B \in \Gamma^*_{n_0k}} \mu B + \frac{1}{2^k} \mu X_0,$$

or

$$\left(1 - \frac{1}{2^k}\right) \mu X_0 < \sum_{B \in \Gamma^*_{n_0k}} \mu B < \left(1 + \frac{1}{2^k}\right) \mu X_0.$$

The last inequality is due to (13) and the definition of $\Gamma^*_{n_0k}$, so we can select a finite subset Γ_{n_0k} of $\Gamma^*_{n_0k}$ such that

$$\left(1 - \frac{1}{2^k}\right) \mu X_0 < \mu \bigcup_{B \in \Gamma_{n_0k}} B < \left(1 + \frac{1}{2^k}\right) \mu X_0, \tag{16}$$

$$\Gamma_{n_0k} \subset \Gamma_{n_0}.$$

Now we may select a finite subset Γ_{nk} of Γ_n for each $n > n_0$ by induction such that

$$\bigcup_{A \in \Gamma_{nk}} A \subset \bigcup_{A \in \Gamma_{n-1,k}} A, \tag{17}$$

$$\mu \bigcup_{A \subset B, A \in \Gamma_{nk}} A \geqslant \left(1 - \frac{1}{2^{k+(n-n_0)}}\right) \mu B \text{ for } B \in \Gamma_{n-1,k}. \tag{18}$$

By (10) it is possible. Then by (16) (17) and (18), we have

$$\left(1 - \frac{1}{2^{k-1}}\right) \mu X_0 < \mu \bigcup_{A \in \Gamma_{nk}} A < \left(1 + \frac{1}{2^k}\right) \mu X_0, \quad \text{for } n \geqslant n_0, \tag{19}$$

and for fixed k, the set

$$K_k \triangleq \bigcap_{n=n_0}^{\infty} \bigcup_{A\in\Gamma_{nk}} \bar{A}$$

is totally bounded and closed, so it is compact.

Now we shall prove that for fixed k and fixed $n \geqslant n_0$

$$\bar{A} \cap K_k \neq \emptyset \tag{20}$$

provided that $A\in\Gamma_{nk}$ and $\mu A>0$. By (18), we choose $A_{n+m}\in\Gamma_{n+m,k}$ by induction with $A_{n+0}=A$ for each $m=1, 2, \cdots$ such that

$$A_{n+m}\subset A_{n+m-1},$$

and

$$\mu A_{n+m}>0.$$

By (2), (8) and the property of the completeness of X, the intersection $\bigcap_{m=1}^{\infty} \bar{A}_{n+m}$ consists of a single point belonging to $\bar{A}\cap K_k$. (20) is obtained.

Let B, C be two sets in $\mathbf{B}_1$ such that

$$\bar{C}\cap K_k\subset B^0,$$

equivalently

$$(\bar{C}-B^0)\cap K_k=\emptyset.$$

Then we have

$$(\bar{C}-B^0)\cap \bigcup_{A\in\Gamma_{nk}, \mu A>0} A=\emptyset \tag{21}$$

as soon as $2r_n<\rho(K, \bar{C}-B^0)$. (21) is equivalent to

$$\bar{C}\cap \bigcup_{A\in\Gamma_{nk}, \mu A>0} A\subset B^0 \text{ for large } n \tag{22}$$

provided that $\bar{C}\cap K_k\subset B^0$ and $B, C\in\mathbf{B}_1$. For fixed k and C with $C\in\mathbf{B}_1$ and $C\subset X_0$, let $B\in\mathbf{B}_1$ such that $B^0\supset K_k\cap\bar{C}$. Then for sufficient large n, we have

$$\begin{aligned} \mu C &\leqslant \mu \bigcup_{A\in\Gamma_{nk}, \mu A>0} A\cap C+\mu\Big[C-\bigcup_{A\in\Gamma_{nk}, \mu A>0} A\Big] \\ &\leqslant \mu B+\mu\Big[Y_2-\bigcup_{A\in\Gamma_{nk}, \mu A>0} A\Big]\leqslant \mu B+\mu Y_2-\mu\bigcup_{A\in\Gamma_{nk}} A \\ &\leqslant \mu B+\mu X_0+\frac{1}{2^k}\mu X_0-\Big(\mu X_0-\frac{1}{2^{k-1}}\mu X_0\Big)\leqslant \mu B+\frac{1}{2^{k-2}}\mu X_0. \end{aligned} \tag{23}$$

Here the second inequality is due to (22) and $C\subset X_0\subset Y_2$, and the fourth inequality is due to (13) and (19).

Now let us prove (11). For any given $\varepsilon>0$ let us choose $A\in\mathbf{B}_1$ and $A_i\in\mathbf{B}_1$ such that

$$\begin{aligned} &\bar{A}\subset X^0, \quad \mu(X_0-A)<\varepsilon/2, \\ &A_i^0\supset X_i, \quad \mu(A_i-X_i)<\varepsilon/2^{i+1} \end{aligned} \tag{24}$$

for $i=1, 2, \cdots$. Then for fixed k we have

$$\bar{A}\cap K_k\subset\bigcup_{i=1}^{m} A_i^0$$

for some m because $\bar{A}\cap K_k$ is compact, and moreover we have

$$\mu X_0 - \frac{\varepsilon}{2} - \frac{1}{2^{k-2}}\mu X_0 \leqslant \mu A - \frac{1}{2^{k-2}}\mu X_0 \leqslant \sum_{i=1}^{m} \mu A_i$$
$$\leqslant \sum_{i=1}^{m} (\mu X_i + \varepsilon/2^{i+1}) \leqslant \sum_{i=1}^{\infty} \mu X_i + \varepsilon/2,$$

where the second inequality is due to (23). Therefore

$$\mu X_0 \leqslant \sum_{i=1}^{\infty} \mu X_i,$$

and the Lemma is proved.

5° Theorem *Suppose* $\{\xi(A, \omega);\ A \in \mathbf{B}_1\}$ *is a nonnegative random process on* $(\Omega, \mathfrak{A}, P)$ *such that*

$$\xi A_1 + \xi A_2 = \xi(A_1 \cup A_2) \quad a.s. \text{ for } A_1,\ A_2 \in \mathbf{B}_1 \text{ and } A_1 \cap A_2 = \emptyset, \tag{25}$$

$$\lim_{r \to t_0 r \in \mathscr{D}} \xi B(x, r) = \xi B(x, t) \quad a.s. \text{ for } B(x, t) \in \mathbf{B}_0, \tag{26}$$

$$\sum_{A \subset B,\, A \in \Gamma_{n+1}} \xi A = \xi B \quad a.s. \text{ for } B \in \Gamma_n,\ n = 1, 2, \cdots, \tag{27}$$

$$\xi A < \infty \quad a.s. \text{ for } A \in \mathbf{B}_1, \tag{28}$$

then there exists a random measure η *satisfying*

$$\eta A = \xi A, \quad \eta \partial A = 0 \quad a.s. \text{ for } A \in \mathbf{B}_1.$$

Proof Since $\mathbf{B}_0$, Γ_n and $\mathbf{B}_1$ are all denumerable, we can choose $\Omega_0 \subset \Omega$ such that $P(\Omega - \Omega_0) = 0$ and (25), (26), (27) and (28) are valid for $\omega \in \Omega_0$. Then by Lemma 4°, $\xi(\cdot, \omega)$ is σ-additive for $\omega \in \Omega_0$, and so there exists a measure $\eta(\cdot, \omega)$ on $\mathbf{B}$ such that

$$\eta(A, \omega) = \xi(A, \omega), \quad \forall A \in \mathbf{B}_1. \tag{29}$$

Let $\eta(\cdot, \omega) \equiv 0$ for $\omega \overline{\in} \Omega_0$. We shall prove that η is the desired random measure. For this, let

$$\mathbf{B}_n^* = \{B;\ B \in \mathbf{B},\ B \subset B(x, r_n')\},$$
$$\mathbf{B}_{1n}^* = \{B;\ B \in \mathbf{B}_1,\ B \subset B(x, r_n')\}.$$

Then for fixed n, B_n^* is a σ-ring and $\mathbf{B}_{1n}^*$ is a ring generating $\mathbf{B}_n^*$ and

$$\bigcup_{n=1}^{\infty} \mathbf{B}_n^* = \mathbf{B}^*. \tag{30}$$

Now for fixed n we have

$$\mathbf{B}_n^* = [A;\ \eta A \text{ is a random variable, } A \in \mathbf{B}_n^*\} \tag{31}$$

since the right hand side of (31) is a monotone class by (28) and by the σ-additivity of η, and contains $\mathbf{B}_{1n}^*$ by (29). Hence ηB is a random variable for each $B \in \mathbf{B}^*$ by (31) and (30), and so (1) is valid. The theorem is proved by the equality

$$\mathbf{B}_1 = \{A;\ A \in \mathbf{B}_1,\ \eta \partial A = 0 \quad \text{a.s.}\},$$

because its right hand side is a ring and contains $\mathbf{B}_0$ by (26).

6° Theorem *Let* $\{\xi(A, \omega);\ A \in \mathbf{B}^*\}$ *be a given nonnegative random process on* $(\Omega, \mathfrak{A}, P)$ *such that*

$$\xi(A_1) + \xi(A_2) = \xi(A_1 \cup A_2) \quad a.\,s. \text{ for } A_1 \cap A_2 = \emptyset, \tag{32}$$

$$\lim_{n\to\infty} \xi(A_n)=0 \quad a.s. \ for \ A_n\downarrow\emptyset, \tag{33}$$

$$\xi A<\infty \ a.s. \tag{34}$$

Then there exists a random measure η satisfying

$$\eta A=\xi A \quad a.s. \ for \ A\in\mathbf{B}^*.$$

Proof We prove this theorem by using Theorem **5°** and at first we must have a denumerable dense subset $\mathscr{D}$ of $(0, \infty)$ such that

$$\xi\partial B(x, r)=0, \quad \text{a.s.} \tag{35}$$

is valid for all open ball $B(x, r)$ with centers $x\in X_s$ and radii $r\in\mathscr{D}$. For this, let Z_{mnlx} be the set of positive numbers r satisfying

$$r<l, \ P\left\{\omega; \ \xi\,\partial B(x, r)\geqslant\frac{1}{n}\right\}\geqslant\frac{1}{m},$$

then Z_{mnlx} is a finite set for fixed natural numbers m, n, l and $x\in X_s$. Otherwise we have $r_j(j=1, 2, \cdots)$ satisfying

$$P\left\{\omega; \ \xi\partial B(x, r_j)\geqslant\frac{1}{n}\right\}\geqslant\frac{1}{m}, \quad r_j<l, \tag{36}$$

and we have the following contradiction

$$0=P\{\omega; \ \xi B(x, l)=\infty\}\geqslant P\,\overline{\lim_j}\left\{\omega; \ \xi\partial B(x, r_j)\geqslant\frac{1}{n}\right\}$$

$$\geqslant\overline{\lim_j}\,P\left\{\omega; \ \xi\partial B(x, r_j)\geqslant\frac{1}{n}\right\}\geqslant\frac{1}{m},$$

where the equality is due to (34) and the first and the last inequalities are due to (32) and (36) respectively. Now any dense subset of

$$(0, \infty)-\bigcup_{x\in X_s}\bigcup_n\bigcup_m\bigcup_l Z_{mnlx}$$

may be taken as the desired $\mathscr{D}$. Then we have $\mathbf{B}_0$, $\mathbf{B}_1$, $\Gamma_n(n=1, 2, \cdots)$ as in **3°** and (33) implies (26) and (27). By Theorem **5°** we have a random measure η satisfying

$$\eta(A)=\xi(A) \quad \text{a.s. for } A\in\mathbf{B}_1.$$

Since the right hand sides of (37) below is a monotone class by (34) and contains $\mathbf{B}_{1n}^*$ and so

$$\mathbf{B}_n^*=\{A; \ \eta A=\xi A \ \text{ a.s.}, \ A\in\mathbf{B}_n^*\} \tag{37}$$

is valid, and by (37) and (30) we have

$$\eta(A)=\xi(A) \quad \text{a.s.} \ \forall A\in\mathbf{B}^*,$$

the proof is complete.

Reference

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[2] Kallenberg, O., Random measures, Akadmie-Verlag. Berlin 1974.