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CONSTRAINED RATIONAL APPROXIMATION*

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Abstract

Let P and Q be convex sets in $C(X)$, and $q(x)>0$ in X for all $q\in Q$. The approximating family is then the class

$$R=\{p/q: p\in P,\ q\in Q\}.$$

The Chebyshev approximation to $f\in C(X)$ an element in R is investigated and the characterizations of a best approximation, and the necessary and sufficient condition for the unique best approximation are obtained.

1. Introduction

Let X be a compact metric space and $C(X)$ the space of continuous real-valued functions defined on X with the norm

$$\|f\|=\max_{x\in X}|f(x)|.$$

We now suppose that P and Q both are subsets in $C(X)$ and $q(x)>0$ in X for all $q\in Q$. Our approximating family is then the class

$$R=\{p/q: p\in P,\ q\in Q\}$$

and our approximating problem is, of course, given an element $f\in C(X)$ to find $r_0\in R$ such that

$$\|f-r_0\|=\inf_{r\in R}\|f-r\|,$$

such an r_0 (if any) is said to be a best approximation to f in R.

In this paper we present the characterizations of a best approximation and of the unique best approximation when P and Q both are arbitrary convex sets.

2. Characterization and Uniqueness

Write

$$X_r=\{x\in X: |f(x)-r(x)|=\|f-r\|\}.$$

We may state the following lemma.

Lemma. *For any* $r_1,\ r_2\in R$,

$$\|f-r_1\|\leqslant(<)\|f-r_2\|$$

implies that

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$$\max_{x\in X_{r_2}}(r_2(x)-r_1(x))(f(x)-r_2(x))\leqslant(<)0.$$

Proof It is easy to see that

$$\|f-r_1\|\leqslant(<)\|f-r_2\|$$

implies

$$[(f(x)-r_2(x))-(f(x)-r_1(x))](f(x)-r_2(x))\geqslant(>)0,\ \forall x\in X_{r_2}$$

or

$$(r_2(x)-r_1(x))(f(x)-r_2(x))\leqslant(<)0,\ \forall x\in X_{r_2},$$

i. e.

$$\max_{x\in X_{r_2}}(r_2(x)-r_1(x))(f(x)-r_2(x))\leqslant(<)0.$$

We will be able to characterize the best approximations in R.

Theorem 1 (Characterization). *Let P and Q be convex in $C(X)$. An element $r_0\in R$ is a best approximation to $f\in C(X)$ in R if and only if*

$$\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x))\geqslant0,\ \forall r\in R. \tag{1}$$

Proof Sufficiency. Suppose not and let $r\in R$ satisfy

$$\|f-r\|<\|f-r_0\|.$$

Then, by Lemma

$$\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x))<0. \tag{2}$$

This is a contradiction.

Necessity. suppose on the contrary that it is possible to find an element $r\in R$ satisfies (2). Putting $r_0=p_0/q_0$ and $r=p/q$, where p_0, $p\in P$ and q_0, $q\in Q$, write

$$r_t=\frac{(1-t)p_0+tp}{(1-t)q_0+tq}.$$

The remainder of the proof is devoted to showing how to select t, $0<t\leqslant1$, so that $\|f-r_t\|<e\equiv\|f-r_0\|$.

Let $y\in X_{r_0}$. From (2) it follows that

$$\begin{aligned}|f(x)-r_t(x)|&=|(f(x)-r_0(x))+(r_0(x)-r_t(x))|\\&=\left|(f(x)-r_0(x))+\frac{tq(x)}{(1-t)q_0(x)+tq(x)}(r_0(x)-r(x))\right|\\&=|f(x)-r_0(x)|-\frac{tq(x)}{(1-t)q_0(x)+tq(x)}|r_0(x)-r(x)|\end{aligned}$$

if $t>0$ and $|x-y|$ both are small enough. So there exist a number $t_y\in(0,1]$ and a neighborhood N_y of the point y such that

$$|f(x)-r_t(x)|<e,\ \forall t\in(0,t_y],\ \forall x\in N_y. \tag{3}$$

For $y\in X\backslash X_{r_0}$ we have

$$|f(y)-r_0(y)|<e.$$

Then there also exist $t_y>0$ and a neighborhood N_y of y such that (3) is valid, because $\lim\limits_{t\to0+}r_t=r_0$.

Now from the open cover $\{N_y\}$ of the compact metric space X we may select a finite subcover $N_{y_1},\cdots,N_{y_n}$. Taking the minimum of the corresponding positive numbers $t_{y_1}, \cdots, t_{y_n}$, denoted by t, we have $0<t\leqslant 1$ and

$$|f(x)-r_t(x)|<e, \ \forall x\in X.$$

Hence

$$\|f-r_t\|<e.$$

We have reached a contradiction, because

$$r_t=\frac{(1-t)p_0+tp}{(1-t)q_0+tq}\in R.$$

Theorem 2 (Characterization). *Under the assumptions of Theorem 1 if $f\in C(X)$ possesses a best approximation in R, then $r_0\in R$ is a best approximation to f in R if and only if*

$$\max_{x\in X_r}(r(x)-r_0(x))(f(x)-r(x))\leqslant\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x)), \ \forall r\in R. \tag{4}$$

Proof If r_0 is a best approximation to f in R, then

$$\|f-r_0\|\leqslant\|f-r\|, \ \forall r\in R.$$

Hence it follows by Lemma that

$$\max_{x\in X_r}(r(x)-r_0(x))(f(x)-r(x))\leqslant 0, \ \forall r\in R,$$

which and (1) imply (4).

Conversely, assume that r_0 satisfies (4). Suppose on the contrary that r_0 is not a best approximation to f in R but $r\in R\backslash\{r_0\}$ is. Thus by Theorem 1

$$\max_{x\in X_r}(r(x)-r_0(x))(f(x)-r(x))\geqslant 0, \tag{5}$$

and by Lemma

$$\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x))<0.$$

This is a contradiction.

Theorem 3 (*Uniqueness*). *Under the assumptions of Theorem 2 the following statements are equivalent to each other*:

(a) $\|f-r_0\|<\|f-r\|, \ \forall r\in R\backslash\{r_0\}$;

(b) $\max_{x\in X_r}(r(x)-r_0(x))(f(x)-r(x))<0, \ \forall r\in R\backslash\{r_0\}$;

(c) $\max_{x\in X_r}(r(x)-r_0(x))(f(x)-r(x))<\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x)),$
$\forall r\in R\backslash\{r_0\}$.

Proof (a)$\Rightarrow$(b). By Lemma it follows directly.

(a)$\Rightarrow$(c). Since (a) implies (1) by Theorem 1, (c) follows from (1) and (b).

(b)$\Rightarrow$(a) and (c)$\Rightarrow$(a). Suppose not and let $r\in R\backslash\{r_0\}$ be a best approximation to f in R. Thus, by Theorem 1, (5) is valid and by Lemma

$$\max_{x\in X_{r_0}}(r_0(x)-r(x))(f(x)-r_0(x))\leqslant 0. \tag{6}$$

But (5) contradicts (b), and (5) and (6) together contradict (c).

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