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ON THE EQUIVALENCE PRINCIPLE FOR CONTRACTIVE MAPPINGS

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Abstract

The main result is:

Theorem 1. Let T be a continuous selfmapping of a complete metric space (X, d) and have the unique fixed point property. If there exists $n: X \to N$ (the set of all positive integers) which is locally bounded such that for each $x \in X$ and for all $r \in N$, $r \geqslant n(x)$,

$$D(O_T(T^{n(x)}x, 0, r)) \leqslant \varphi(D(O_T(x, 0, r))), \text{ or}$$

$$D(O_T(T^{n(x)}x, 0, r)) \leqslant \psi(D(O_T(x, 0, r))),$$

where φ and ψ are contractive gauge functions, then

(a) T has a unique fixed point x^*;

(b) For each $x \in X$, $T^n x \to x^*$ as $n \to \infty$;

(c) There exists a neighborhood $U(x^*)$ of x^* such that $\lim\limits_{n\to\infty} T^n(U(x^*)) = \{x^*\}$;

(d) x^* is stable;

(e) For any given $C \in (0, 1)$ there exists a metric d^* topologically equivalent to d such that T is a Banach contraction under d^* with Lipschitz constant C.

By Theorem 1 it is shown that many contractive type mappings defined in [1—26] are topologically equivalent to each other.

§ 1. Introduction

In recent ten years, a number of important generalizations of the well-known Banach contraction principle are obtained in various directions. In [1], Rhoades has discussed the comparison and classification of various definitions for contraction mappings and established some relations between them. The author of [2, 24] has also proved that under a suitable metric d^* which is topologically equivalent to d for the given metric space (X, d), a number of contraction mapping principles are topologically equivalent to each other.

Recently, Fisher[3], Browder [4], Walter[5] and the writer[6-8] establish some new fixed point theorems for contraction type mappings. In this paper we shall show that these new contraction principles are also topologically equivalent to Banach contraction principle. Our equivalent principles unify and generalize a number of recent

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results obtained by many authors.

§ 2. Notations and Lemmas

We denote by ω the set of all nonnegative integers and by R^+ the set of all nonnegative real numbers.

Let T be a mapping of a metric space (X, d) into itself. For each $x\in X$, $O_T(x, 0, \infty)$ denotes the orbit of T at x and for all $i, j\in\omega, j>i$, write $O_T(x, i, j)=\{T^ix, T^{i+1}x, \cdots, T^jx\}$. For any $A\subset X$, $D(A)=\sup\{d(x, y):x, y\in A\}$ denotes the diameter of A.

In the following we introduce two families Φ and Ψ of contractive gauge functions.

We say that $\varphi\in\Phi$, if $\varphi:R^+\to R^+$ satisfies the following properties:

(Φ_1) φ is nondecreasing,

(Φ_2) $\lim\limits_{n\to\infty}\varphi^n(t)=0, \forall t>0$, where φ^n is the n-th iteration of φ,

(Φ_3) $\lim\limits_{t\to\infty}(t-\varphi(t))=\infty$.

We say that $\psi\in\Psi$, if $\psi:R^+\to R^+$ satisfies the following properties:

(Ψ_1) ψ is nondecreasing and continuous from the right,

(Ψ_2) For each $q\in R^+$ there exists the maximal solution $u(q)$ of the equation $u=\psi(u)+q$ and $u(0)=0$. The class is defined by Kwapisz[9].

Lemma 1.[9] *Let $\psi\in\Psi$, $q\in R^+$ and $u(q)$ is the maximal solution of the equation $u=\psi(u)+q$. If $p\in R^+$ satisfies $p\leqslant\psi(p)+q$ then $p\leqslant u(q)$.*

Lemma 2.[10, 11] *Let T be a continuous selfmapping of a complete metric space (X, d) with the following properties:*

(i) T has a unique fixed point x^*,

(ii) For each $x\in X$ the sequence $\{T^nx\}_{n\in\omega}$ converges to x^*,

(iii) There exists an open neighborhood U of x^* with the property that for any given open set V including x^* there is $n_0\in N$ such that $n\geqslant n_0$ implies $T^nU\subset V$.

Then for each $C\in(0, 1)$ there exists a metric d^* topologically equivalent to d such that T is a Banach contraction under d^* with Lipschitz constant C.

Lemma 3. *Let T be a continuous selfmapping of a complete metric space (X, d). Suppose for each $x\in X$, $D(O_T(x, 0, \infty))<\infty$. If there exists a function $n:X\to N$ such that for all $x\in X$ any one of the following conditions holds:*

$$D(O_T(T^{n(x)}x, 0, \infty))\leqslant\varphi(D(O_T(x, 0, \infty))), \tag{1}$$

$$D(O_T(T^{n(x)}x, 0, \infty))\leqslant\psi(D(O_T(x, 0, \infty))), \tag{2}$$

where $\varphi\in\Phi$ and $\psi\in\Psi$, then for each $x\in X$ $\{T^nx\}_{n\in\omega}$ converges to a fixed point x^* of T.

Proof If (1) is true, then the conclusion of Lemma 3 follows from Theorem 1 of [7]. If (2) is true, then noting that $D(O_T(x, 0, \infty)<\infty$, $\forall x\in X$, and using the same argument as in the proof of Theorem 1 of [24], the conclusion of Lemma 3 also holds.

Remark 1. The lemma improves and generalizes the main results of [12] and [6].

§ 3. Main results

A mapping $T: X\to X$ is said to have the unique fixed point property in the following sense: if the fixed point of T exists, then it is unique.

We say that the function $n: X\to N$ is locally bounded, if for each $x\in X$ there exist a neighborhood $U(x)$ of x and $p\in N$ such that $n(y)\leqslant p$, $\forall y\in U(x)$.

Theorem 1. *Let T be a continuous selfmapping of a complete metric space (X, d) and have the unique fixed point property. If there exists a function $n: X\to N$ which is locally bounded such that for all $x\in X$ and for all $r\in N$, $r\geqslant n(x)$, any one of the following conditions holds:*

$$D(O_T(T^{n(x)}x, 0, r))\leqslant\varphi(D(O_T(x, 0, r))), \tag{3}$$

$$D(O_T(T^{n(x)}x, 0, r))\leqslant\psi(D(O_T(x, 0, r))), \tag{4}$$

where $\varphi\in\Phi$ and $\psi\in\Psi$, then

(a) *T has a unique fixed point x^*,*

(b) *For each $x\in X$, $\{T^n x\}_{n\in\omega}$ converges to x^*,*

(c) *$U(x^*)$-uniform convergence: there exists a neighborhood $U(x^*)$ of x^* such that $\lim\limits_{n\to\infty} T^n(U(x^*))=\{x^*\}$, this means that for any open set V including x^* there exists $n_0\in N$ such that $T^n(U(x^*))\subset V$, $\forall n\geqslant n_0$,*

(d) *Stability of the fixed point x^*: for any neighborhood $W(x^*)$ of x^* there exists some neighborhood $V(x^*)$ of x^* such that $T^n(x)\in W(x^*)$, $\forall x\in V(x^*)$ and $n\in\omega$,*

(e) *For any given $C\in(0, 1)$ there exists a metric d^* topological equivalent to d such that T is a Banach contraction under d^* with Lipschitz constant C*, i. e. $d^*(Tx, Ty)\leqslant Cd^*(x, y)$, $\forall x, y\in X$.

Proof If (3) holds, then for all $r\geqslant n(x)$,

$$\begin{aligned} D(O_T(x, 0, r))&\leqslant D(O_T(x, 0, n(x)))+D(O_T(T^{n(x)}x, 0, r)) \\ &\leqslant D(O_T(x, 0, n(x)))+\varphi(D(O_T(x, 0, r))). \end{aligned} \tag{5}$$

If $D(O_T(x, 0, \infty))=\infty$, then $\lim\limits_{r\to\infty} D(O_T(x, 0, r))=\infty$. By (5) and (Φ_3) we have

$$\infty=\lim_{r\to\infty}[D(O_T(x, 0, r))-\varphi(D(O_T(x, 0, r)))]\leqslant D(O_T(x, 0, n(x))).$$

This is a contradiction and hence

$$D(O_T(x, 0, \infty))<\infty, \ \forall x\in X. \tag{6}$$

If (4) holds, then for all $r\geqslant n(x)$,

$$D(O_T(x, 0, r)) \leqslant D(O_T(x, 0, n(x))) + D(O_T(T^{n(x)}x, 0, r))$$
$$\leqslant D(O_T(x, 0, n(x))) + \psi(D(O_T(x, 0, r))). \tag{7}$$

Let $M = D(O_T(x, 0, n(x))$ and $u(M)$ be the maximal solution of the equation $u = \psi(u) + M$. It follows from (7) and Lemma 1 that $D(O_T(x, 0, r)) \leqslant u(M)$, $\forall r \geqslant n(x)$ and hence

$$D(O_T(x, 0, \infty)) < \infty, \ \forall x \in X. \tag{8}$$

On the other hand, obviously (3) and (4) imply (1) and (2) respectively. It follows from Lemma 3 that for each $x \in X$ $\{T^n x\}_{n \in \omega}$ converges to a fixed point x^* of T. Since T has the unique fixed point property, the conclusions (a) and (b) hold.

Now we prove the conclusion (c). Since $n: X \to N$ is locally bounded, there exist a neighborhood $U_1(x^*)$ of x^* and $p \in N$ such that $n(y) \leqslant p$, $\forall y \in U_1(x^*)$. It follows from the continuous property of T that $T^k (k = 2, 3, \cdots, p)$ is also continuous and so for some given $(M/2) \in R^+$, $(M/2) > 0$, there exists $\eta \in R^+$, $0 < \eta < M/2$ such that

$$d(T^k x, x^*) \leqslant \frac{M}{2}, \ \forall k \in \{0, 1, \cdots, p\}, \ x \in U_2(x^*) = \{x: d(x, x^*) < \eta\}.$$

Hence we have

$$d(T^i x, T^j x) \leqslant d(T^i x, x^*) + d(T^j x, x^*) \leqslant M, \ x \in U_2(x^*), \ i, j \in \{0, 1, \cdots, p\}$$

and so

$$\sup_{x \in U_2(x^*)} D(O_T(x, 0, p)) \leqslant M.$$

Putting $U(x^*) = U_1(x^*) \cap U_2(x^*)$, we obtain

$$\sup_{x \in U(x^*)} D(O_T(x, 0, n(x))) \leqslant \sup_{x \in U(x^*)} D(O_T(x, 0, p)) \leqslant \sup_{x \in U_2(x^*)} D(O_T(x, 0, p)) \leqslant M. \tag{9}$$

An analysis for the proofs of (5)—(6) and (7)—(8) shows that when in (5), (6), (7) and (8) we take the supremum for $x \in U(x^*)$, they are still true. Then we easily get

$$\lim_{m \to \infty} \sup_{x \in U(x^*)} D(O_T(T^{mp}x, 0, \infty)) \leqslant \lim_{m \to \infty} \sup_{x \in U(x^*)} D(O_T(x_m, 0, \infty)) = 0, \tag{10}$$

where $x_m = T^{n(x) + \sum_{k=1}^{m-1} n(x_k)} x$, $mp \geqslant n(x) + \sum_{k=1}^{m-1} n(x_k)$, $\forall x \in U(x^*)$.

Now we take the fixed open neighborhood $U(x^*)$ of x^*. For any given $\varepsilon > 0$, it follows from (10) that there exists $m_0 \in N$ such that

$$d(T^i x, T^j x) \leqslant \frac{\varepsilon}{2}, \ \forall x \in U(x^*), \ j \geqslant i \geqslant m_0 p.$$

Putting $j \to \infty$ we obtain

$$d(T^i x, x^*) \leqslant \frac{\varepsilon}{2}, \ \forall x \in U(x^*), \ i \geqslant m_0 p.$$

Therefore $D(T^i(U(x^*)))$ satisfies

$$D(T^i(U(x^*))) = \sup_{x, y \in U(x^*)} d(T^i x, T^i y) \leqslant \sup_{x \in U(x^*)} d(T^i x, x^*) + \sup_{y \in U(x^*)} d(T^i y, x^*)$$

for all $i \geqslant m_0 p$, which implies $D(T^i(U(x^*))) \to 0$ as $i \to \infty$. Thus for large $i \in N$, $T^i(U(x^*))$ squeezes into any neighborhood $V(x^*)$ of x^* and so the conclusion (c) holds.

According to Lemma 2 the conclusion (e) is also true. By Lemma 2.1 of олойцев[11] the conclusion (d) also holds. This completes the proof of Theorem 1.

Theorem 2. *Let T be a continuous selfmapping of a complete metric space (X, d). If there exist functions $n, m: X \to N$ which are locally bounded such that for all $x, y \in X$ and for all $r \geqslant \max\{n(x), m(y)\}$, any one of the following conditions holds:*

$$D(O_T(T^{n(x)}x, 0, r) \cup O_T(T^{m(y)}y, 0, r)) \leqslant \varphi(D(O_T(x, 0, r) \cup O_T(y, 0, r))), \tag{11}$$

$$D(O_T(T^{n(x)}x, 0, r) \cup O_T(T^{m(y)}y, 0, r)) \leqslant \psi(D(O_T(x, 0, r) \cup O_T(y, 0, r))), \tag{12}$$

where $\varphi \in \Phi$ and $\psi \in \Psi$, then the conclusions (a), (b), (c), (d) and (e) of Theorem 1 still hold.

Proof By (11) and (12) it is easy to check that T has the unique fixed point property. Obviously, (11) and (12) imply (3) and (4) respectively. Therefore the conclusions of Theorem 2 follow from Theorem 1.

Theorem 3. *Let T be a continuous selfmapping of a complete metric space (X, d). If there exists a function $n: X \to N$ which is locally bounded such that for each $x \in X$ and for all $n \geqslant n(x)$, $y \in X$, any one of the following conditions holds:*

$$d(T^n x, T^n y) \leqslant \varphi(D(O_T(x, 0, n) \cup O_T(y, 0, n))), \tag{13}$$

$$d(T^n x, T^n y) \leqslant \psi(D(O_T(x, 0, n) \cup O_T(y, 0, n))), \tag{14}$$

then the conclusions of Theorem 1 hold.

Proof By (13) and (14), it is easy to prove that T has the unique fixed point property. For any $x \in X$, $r \geqslant n \geqslant n(x)$, $i, j \in \omega$ and $0 \leqslant i+j \leqslant r-n$, letting $n = n(x)+i$, $y = T^j x$ in (13) we have

$$d(T^{n(x)+i}x, T^{n(x)+i+j}x) \leqslant \varphi(D(O_T(x, 0, n(x)+i) \cup O_T(T^j x, 0, n(x)+i))) \leqslant \varphi(D(O_T(x, 0, r)).$$

From the arbitrariness of i and j it follows that for all $x \in X$ and for all $r \geqslant n(x)$

$$D(O_T(T^{n(x)}x, 0, r)) \leqslant \varphi(D(O_T(x, 0, r))).$$

Similarly, by (14) we can prove that for all $x \in X$ and for all $r \geqslant n(x)$

$$D(O_T(T^{n(x)}x, 0, r)) \leqslant \psi(D(O_T(x, 0, r))).$$

Therefore the conclusion of Theorem 3 follows from Theorem 1.

Remark 2. Theorem 3 improves and generalizes the main results of Browder[4] and Walter[5].

Remark 3. By Theorem 1 of [2] and Theorem 7 of [24], it is easy to prove that many known contraction conditions imply (13), (14), (3) and (4) respectively, so Theorems 1 and 3 unify, improve and generalize the many known results. Moreover, it is easy to prove that under a suitable metric d^* topologically equivalent to d these contraction type mappings are topologically equivalent to each other.

Theorem 4. *Let T be a continuous selfmapping of a complete metric space (X, d). If there exist a decreasing function $\alpha: (0, \infty) \to [0, 1)$ and $p \in N$ such that for all $x, y \in X$, $x \neq y$*

$$d(T^p x, T^p y) \leqslant \alpha(d(x, y)) \max \{d(x, y), d(x, T^p x), d(y, T^p y), \frac{1}{2}[d(x, T^p y) + d(y, T^p x)]\}, \tag{15}$$

then for each $x \in X$, the sequence $\{T^{mp}x\}_{m \in \omega}$ converges to a unique fixed point x^* of T.

Proof For any fixed $x \in X$ we consider the sequence

$$\{x_m\}_{m \in \omega} = \{T^{mp}x; m \in \omega\}.$$

By (15) we have

$$d(x_{m+1}, x_m) = d(T^p x_m, T^p x_{m-1}) \leqslant \alpha(d(x_m, x_{m-1})) \max \{d(x_m, x_{m-1}), d(x_m, x_{m+1}), d(x_{m-1}, x_m), \frac{1}{2}[0 + d(x_{m-1}, x_{m+1})]\}.$$

Since $\alpha(d(x_m, x_{m-1})) < 1$, it follows that

$$d(x_{m+1}, x_m) \leqslant \alpha(d(x_m, x_{m-1})) \cdot d(x_m, x_{m-1}) < d(x_m, x_{m-1}) \tag{16}$$

and so $\{d(x_{m+1}, x_m)\}_{m \in \omega}$ is a decreasing sequence. Let $b_m = d(x_{m+1}, x_m) \to b$, and suppose $b > 0$. Then $b_m \geqslant b$, $\forall m \in \omega$, which yields $\alpha(b_m) \leqslant \alpha(b)$, $\forall m \in \omega$. By (16)

$$b_m = d(x_{m+1}, x_m) \leqslant \alpha(b) d(x_m, x_{m-1}) \leqslant \cdots \leqslant [\alpha(b)]^m d(x_1, x_0) \to 0,$$

as $m \to \infty$.

Now we prove that $\{x_m\}_{m \in \omega}$ is a Cauchy sequence. If it is not, then there exist $\varepsilon > 0$ and the sequences $\{p(m)\}$ and $\{q(m)\}$ such that

$$p(m) > q(m) > m, \tag{17}$$

$$d(x_{p(m)}, x_{q(m)}) \geqslant \varepsilon, \tag{18}$$

and (by the well-ordered principle)

$$d(x_{p(m)-i}, x_{q(m)-j}) < \varepsilon, \quad \forall 1 \leqslant i \leqslant p(m), \ 1 \leqslant j \leqslant q(m). \tag{19}$$

Let $c_m = d(x_{p(m)}, x_{q(m)})$. By (18) and (19) we have

$$\varepsilon \leqslant c_m \leqslant d(x_{p(m)}, x_{p(m)-1}) + d(x_{p(m)-1}, x_{q(m)}) < b_{p(m)-1} + \varepsilon,$$

which implies that $\{c_m\}$ converges to ε from the right. Let $v_m = d(x_{p(m)-1}, x_{q(m)-1})$, $d_m = d(x_{p(m)-1}, x_{q(m)})$ and $w_m = d(x_{p(m)}, x_{q(m)-1})$. Since $c_m \to \varepsilon$ and $b_m \to 0$, it is easy to show that v_m, d_m and w_m converge to ε from the left. Using (15) we have

$$c_m = d(x_{p(m)}, x_{q(m)}) = d(T^p x_{p(m)-1}, T^p x_{q(m)-1}) \leqslant \alpha(v_m) \max \{v_m, b_{p(m)-1}, b_{q(m)-1}, \frac{1}{2}(d_m + w_m)\}. \tag{20}$$

By the assumption of α, without loss of generality, we may regard α as continuous function from the left. Letting $m \to \infty$ in (20) we obtain $\varepsilon \leqslant \alpha(\varepsilon)\varepsilon < \varepsilon$. This is a contradiction and hence $\{x_m\}_{m \in \omega}$ is a Cauchy sequence. Let $x_m \to x^*$. Since T is continuous, T^p is also continuous. Thus $x_{m+1} = T^p x_m \to T^p x^*$ and so $x^* = T^p x^*$. If y^* is also a fixed point of T^p and $x^* \neq y^*$, then by (15)

$$d(x^*, y^*)=d(T^px^*, T^py^*)\leq\alpha(d(x^*, y^*))d(x^*, y^*)<d(x^*, y^*),$$

which yields a contradiction. Hence $x^*=y^*$. Then x^* is a unique fixed point of T^p. But $x^*=T^px^*$ implies $Tx^*=T^pTx^*$. So Tx^* is also a fixed point of T^p. By the uniqueness of x^* we obtain $x^*=Tx^*$. Hence x^* is a unique fixed point of T and $\{x_m\}_{m\in\omega}=\{T^{mp}x\}_{m\in\omega}$ converges to x^* for each $x\in X$.

Theorem 5. *Let T be a continuous selfmapping of a complete metric space (X, d). If there exists a decreasing function $\alpha: (0, \infty)\to[0, 1)$ such that for all $x, y\in X$, $x\neq y$,*

$$d(Tx, Ty)\leq\alpha(d(x, y))\max\{d(x, y), \frac{1}{2}[d(x, Tx)+d(y, Ty)], \frac{1}{2}[d(x, Ty)+d(y, Tx)]\}, \tag{21}$$

then the conclusions of Theorem 1 hold.

Proof By Theorem 4 with $p=1$, we see that T has a unique fixed point x^* and $\{T^nx\}_{n\in\omega}$ converges to x^* for each $x\in X$. Hence the conclusions (a) and (b) of Theorem 1 hold. By (21) we have

$$d(T^nx, x^*)=d(T^nx, Tx^*)\leq\alpha(d(T^{n-1}x, x^*))\max\{d(T^{n-1}x, x^*), \frac{1}{2}[d(T^{n-1}x, x^*)+d(T^nx, x^*)]\}.$$

It follows that

$$d(T^nx, x^*)\leq\alpha(d(T^{n-1}x, x^*))d(T^{n-1}x, x^*)<d(T^{n-1}x, x^*)$$

and so $\{d(T^nx, x^*)\}_{n\in\omega}$ is a decreasing sequence. Since α is a decreasing function, we have

$$d(T^nx, x^*)\leq[\alpha(d(T^nx, x^*))]^nd(x, x^*).$$

Now take the open neighborhood $U(x^*)=\{x\in X:d(x, x^*)<1\}$ of x^*. Assume $V(x^*)$ is any open neighborhood including x^*. Take $1>\varepsilon>0$ such that $\tilde{V}(x^*)=\{x\in X:d(x, x^*)<\varepsilon\}\subset V(x^*)$. Since $\alpha\left(\frac{\varepsilon}{4}\right)<1$, there exists $n_0\in N$ such that $\left[\alpha\left(\frac{\varepsilon}{4}\right)\right]^n<\frac{\varepsilon}{2}$, $\forall n\geq n_0$. For $n\geq n_0$, let $U_1(x^*)=\left\{x\in U(x^*):d(T^nx, x^*)<\frac{\varepsilon}{4}\right\}$ and $U_2(x^*)=\left\{x\in U(x^*):d(T^nx, x^*)\geq\frac{\varepsilon}{4}\right\}$. $D(T^n(U(x^*)))$ satisfies

$$\begin{aligned}D(T^n(U(x^*)))&=\sup_{x,y\in U(x^*)}d(T^nx, T^ny)\leq\sup_{x,y\in U(x^*)}[d(T^nx, x^*)+d(T^ny, x^*)]\\&\leq2\sup_{x\in U(x^*)}d(T^nx, x^*)\\&\leq2\max\{\sup_{x\in U_1(x^*)}d(T^nx, x^*), \sup_{x\in U_2(x^*)}d(T^nx, x^*)\}\\&\leq2\max\left\{\frac{\varepsilon}{4}, \sup_{x\in U_2(x^*)}[\alpha(d(T^nx, x^*))]^nd(x, x^*)\right\}\\&\leq2\max\left\{\frac{\varepsilon}{4}, \left[\alpha\left(\frac{\varepsilon}{4}\right)\right]^n\right\}<2\max\left\{\frac{\varepsilon}{4}, \frac{\varepsilon}{2}\right\}=\varepsilon, \forall n\geq n_0.\end{aligned}$$

Hence $T^n(U(x^*))\subset\tilde{V}(x^*)\subset V(x^*)$, $\forall n\geq n_0$. This shows that the conclusion (c) of Theorem 1 holds. From Lemma 2.1 of [11] and Lemma 2 it follows that the conclusions (d) and (e) of Theorem 1 are also true. This completes the proof of Theorem 5.

Remark 4. Theorems 4 and 5 improve and generalize the corresponding results in [25, 26] and [1]. Using Theorems 1, 3 and 5, we can prove that many contraction type mappings defined in [1—26] are topologically equivalent to each other and each of them is also topologically equivalent to Banach contraction.

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