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THE BOUNDARY VALUE PROBLEMS FOR QUASILINEAR HIGHER ORDER ELLIPTIC EQUATIONS WITH A SMALL PARAMETER

JIANG FURU (江福汝)
(*Fudan University*)

Abstract

In this paper we deal with the general boundary value problems for quasilinear higher order elliptic equations with a small parameter before higher derivatives. By using the method of multiple scales, we have proved that if the solution of degenerated boundary value problem exists, then under certain assumptions as the small parameter is sufficiently small, the solution of the origional boundary value problem exists as well and it is unique in a certain function space. Besides, the asymptotic expansion of the solution has been constructed.

§ 1. Formal Asymptotic Approximation

Since 1956, there are so many works dealing with the boundary value problems for linear higher order elliptic equations[1-8], but there are few works dealing with the quasilinear ones. In this paper we consider the boundary value problems for the quasilinear elliptic equations subject to some general boundary conditions.

Consider boundary value problem of the following type

$$\begin{cases} N_\varepsilon u_\varepsilon \equiv \varepsilon^{2l_1} N_1 u_\varepsilon + N_0 u_\varepsilon = f(x) \quad (0<\varepsilon\ll 1,\ x\in\Omega\subset R^n), & (1.1)\\ B_j u_\varepsilon|_{\partial\Omega} = h_j(x)|_{\partial\Omega} \quad (j=0,\ 1,\ \cdots,\ l_1+l_0-1), & (1.2)\end{cases}$$

where N_1 and N_0 are strongly elliptic differential operators of order $2(l_1+l_0)$ and $2l_0$ on Ω respectively,

$$N_1 u_\varepsilon \equiv \sum_{|\beta|\le 2(l_1+l_0)} A_\beta(x,\ u_\varepsilon) D^\beta u_\varepsilon, \tag{1.3}$$

$$N_0 u_\varepsilon \equiv \sum_{|\beta|\le 2l_0} a_\beta(x,\ u_\varepsilon) D^\beta u_\varepsilon, \tag{1.4}$$

wheae

$$(-1)^{l_1+l_0} \sum_{|\beta|=2(l_1+l_0)} A_\beta(x,\ w_0)\xi^\beta \geqslant \delta_1 |\xi|^{2(l_1+l_0)}, \tag{1.5}$$

$$(-1)^{l_0} \sum_{|\beta|=2l_0} a_\beta(x,\ w_0)\xi^\beta \geqslant \delta_0 |\xi|^{2l_0}, \tag{1.6}$$

δ_1 and δ_0 are certain positive numbers, Ω is a bounded domain in R^n, $\beta=(\beta_1,\ \cdots,\ \beta_n)$, $\xi=(\xi_1,\ \cdots,\ \xi_n)$, $\xi^\beta=\xi_1^{\beta_1}\cdots\xi_n^{\beta_n}$, $D^\beta=D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}$, $D_{x_i}^{\beta_i}=\dfrac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}$, $|\beta|=\beta_1+\cdots+\beta_n$, and

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$$B_j \equiv \sum_{|\mu|\leqslant m_j} b_\mu^{(j)}(x) D^\mu \quad (j=0,\ 1,\ \cdots,\ l_1+l_0-1),$$

$$(0\leqslant m_0<m_1<\cdots<m_{l_1+l_0-1},\ \leqslant 2(l_1+l_0)-1), \tag{1.7}$$

are boundary operators defined on $\partial\Omega$, where $b_\mu^{(j)}\in D(\partial\Omega)$, $D(\partial\Omega)$ is the set of infinitely differentiable functions defined on $\partial\Omega$, w_0 is the solution of the degenerated problem:

$$\begin{cases} N_0 w_0 = f(x), & (1.8)\\ B_j w_0|_{\partial\Omega} = h_j(x)|_{\partial\Omega} \quad (j=0,\ 1,\ \cdots,\ l_0-1). & (1.9)\end{cases}$$

Assume that

(H1) Coefficients of the differential equation and boundary conditions are infinitely differentiable, and boundary $\partial\Omega$ is sufficiently smooth,

(H2) The solution of the degenerated problem (1.8)—(1.9) exists, and $w_0\in C^{2(l_1+l_0)}(\overline{\Omega})$.

Let the outer solution of the original boundary value problem (1.1)—(1.2) be

$$W\sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x). \tag{1.10}$$

Substituting it into equation (1.1), and expanding all the coefficients $A_\beta\left(x,\ \sum_{n=0}^{\infty}\varepsilon^n w_n\right)$, $\cdots$ with respect to ε

$$A_\beta\left(x,\ \sum_{n=0}^{\infty}\varepsilon^n w_n\right)=\sum_{j=0}^{\infty}\varepsilon^j A_\beta^{(j)}(x)$$

with

$$A_\beta^{(0)}=A_\beta(x,\ w_0),\quad A_\beta^{(1)}=\frac{\partial A_\beta(x,\ w_0)}{\partial u}w_1,$$

$$A_\beta^{(j)}=\frac{\partial A_\beta(x,\ w_0)}{\partial u}w_j+\overline{A}_\beta^{(j)}(x,\ w_0,\ \cdots,\ w_{j-1})\quad (j=2,\ 3,\ \cdots)$$

and

$$\overline{A}_\beta^{(j)}=\sum_{\substack{n_1+2n_2+\cdots+pn_p=j\\(p<j)}}\frac{1}{n_1!\,n_2!\cdots n_p!}\,\frac{\partial^{n_1+n_2+\cdots+n_p}A_\beta(x,\ w_0)}{\partial u^{n_1+n_2+\cdots+n_p}}\,w_1^{n_1}w_2^{n_2}\cdots w_p^{n_p},$$

$\cdots$, and equating all the coefficients of the powers of ε to zero, we obtain the recursive equations for $w_n (n=0,\ 1,\ \cdots)$:

$$N_0 w_0 \equiv \sum_{|\beta|\leqslant 2l_0} a_\beta(x,\ w_0) D^\beta w_0 = f(x), \tag{1.11}$$

$$\sum_{|\beta|\leqslant 2l_0}\left[a_\beta(x,\ w_0)D^\beta+\frac{\partial a_\beta(x,\ w_0)}{\partial u}D^\beta w_0\right]w_n$$

$$=-\sum_{|\beta|\leqslant 2l_0}\overline{a}_\beta^{(n)}(x,\ w_0,\ \cdots,\ w_{n-1})\,D^\beta w_0-\sum_{j=1}^{n-1}\sum_{|\beta|\leqslant 2l_0}a_\beta^{(j)}D^\beta w_{n-j}$$

$$-\sum_{j=0}^{n-2l_1}\sum_{|\beta|\leqslant 2(l_1+l_0)}A_\beta^{(j)}D^\beta w_{n-2l_1-j},\quad (n=1,\ 2,\ \cdots), \tag{1.12}$$

in which the letters with negative subscript are interpreted as zero. Since all of these equations are the elliptic equations of $2l_0$ *th* order, we can not make each of them to satisfy $2(l_1+l_0)$ boundary conditions. So the expansion constructed above can only be the outer solution of boundary value problem (1.1)—(1.2).

In the following, we shall construct the boundary layer correction to replenish W to satisfy the whole boundary condition.

In the neighborhood of $\partial\Omega$, we introduce local coordinates as follows: at each point of $\partial\Omega$ construct an inner formal to $\partial\Omega$, and take length η of the inner normal so small that none of them intersect. Let P be a point in the η neighborhood of $\partial\Omega$, we take the distance from P to $\partial\Omega$ as its ρ_1 coordinate, and take the curvilinear coordinates of the foot point of the inner normal past P as its $\rho_2, \cdots, \rho_n$ coordinates.

In terms of the local coordinates, equation (1.1) and the boundary conditions in (1.2) take the forms

$$\tilde{N}_\varepsilon \tilde{u}_\varepsilon \equiv \varepsilon^{2l_1} \sum_{|\beta|\leq 2(l_1+l_0)} \tilde{A}_\beta(\rho, \tilde{u}_\varepsilon)\tilde{D}^\beta \tilde{u}_\varepsilon + \sum_{|\beta|\leq 2l_0} \tilde{a}_\beta(\rho, \tilde{u}_\varepsilon)\tilde{D}^\beta \tilde{u}_\varepsilon = \tilde{f}(\rho), \tag{1.13}$$

$$\tilde{B}_j \tilde{u}_\varepsilon|_{\rho_1=0} = \tilde{h}_j(\rho)|_{\rho_1=0} (j=0, 1, \cdots, l_1+l_0-1), \tag{1.14}$$

where $\tilde{D}^\beta = \tilde{D}^{\beta_1}_{\rho_1} \cdots \tilde{D}^{\beta_n}_{\rho_n}$, $\tilde{D}^{\beta_i}_{\rho_i} = \dfrac{\partial^{\beta_i}}{\partial \rho_i^{\beta_i}}$.

In the η neighborhood of $\partial\Omega$, we introduce $n+1$ variables with multiple scales

$$f = \frac{g(\rho)}{\varepsilon}, \quad \xi_1 = \rho_1, \cdots, \xi_n = \rho_n. \tag{1.15}$$

In terms of these variables, the partial derivatives with respect to $\rho_i (i=1, 2, \cdots, n)$ can be expanded as (cf. [8])

$$\tilde{D}^{\beta_i}_{\rho_i} = \varepsilon^{-\beta_i}(\tilde{\delta}^{(0)}_{\rho_i^{\beta_i}} + \varepsilon \tilde{\delta}^{(1)}_{\rho_i^{\beta_i}} + \cdots + \varepsilon^{\beta_i} \tilde{\delta}^{(\beta_i)}_{\rho_i^{\beta_i}}) \quad (i=1, 2, \cdots, n), \tag{1.16}$$

where

$$\tilde{\delta}^{(0)}_{\rho_i^{\beta_i}} = g^{\beta_i}_{\rho_i} \frac{\partial^{\beta_i}}{\partial f^{\beta_i}},$$

$$\tilde{\delta}^{(1)}_{\rho_i^{\beta_i}} = \beta_i g^{\beta_i-1}_{\rho_i} \frac{\partial^{\beta_i}}{\partial f^{\beta_i-1}\partial \xi_i} + \frac{(\beta_i-1)\beta_i}{2!} g^{\beta_i-2}_{\rho_i} g_{\rho_i\rho_i} \frac{\partial^{\beta_i-1}}{\partial f^{\beta_i-1}},$$

$$\tilde{\delta}^{(s)}_{\rho_i^{\beta_i}} = \frac{\partial^{\beta_i-s}}{\partial f^{\beta_i-s}}\left(\sum_{j=0}^{s} H_j(\xi)\frac{\partial^j}{\partial \xi^j}\right) \quad (s=2, 3, \cdots, \beta_i).$$

So we have

$$\tilde{D}^\beta = \varepsilon^{-|\beta|}(\tilde{\delta}^{(0)}_\beta + \varepsilon \tilde{\delta}^{(1)}_\beta + \cdots + \varepsilon^{|\beta|}\tilde{\delta}^{(|\beta|)}_\beta), \tag{1.17}$$

where

$$\tilde{\delta}^{(0)}_\beta \equiv \tilde{\delta}^{(0)}_{\rho_1^{\beta_1}} \tilde{\delta}^{(0)}_{\rho_2^{\beta_2}} \cdots \tilde{\delta}^{(0)}_{\rho_n^{\beta_n}}, \tag{1.18}$$

$$\tilde{\delta}^{(j)}_\beta \equiv \sum_{|q|=j} \tilde{\delta}^{(q_1)}_{\rho_1^{\beta_1}} \tilde{\delta}^{(q_2)}_{\rho_2^{\beta_2}} \cdots \tilde{\delta}^{(q_n)}_{\rho_n^{\beta_n}} \quad (j=1, 2, \cdots, |\beta|). \tag{1.19}$$

Suppose that the boundary layer correction to W has the expansion of the form

$$\tilde{V} \sim \varepsilon^p \sum_{n=0}^{\infty} \varepsilon^n \tilde{v}_n(f, \xi), \tag{1.20}$$

and let the expansion of solution of boundary value problem (1.1)—(1.2), in the neighborhood of $\partial\Omega$, takes the form

$$\tilde{u}_\varepsilon = \tilde{W} + \tilde{V} \sim \sum_{n=0}^{\infty} \varepsilon^n \tilde{w}_n(\xi) + \varepsilon^p \sum_{n=0}^{\infty} \varepsilon^n \tilde{v}_n(f, \xi), \tag{1.21}$$

where $\tilde{w}_n(\xi) = \tilde{w}_n(\rho)$ is the expression of $w_n(x)$ in local coordinates, and p is an undeterminate constant.

Substituting $\tilde{u}_\varepsilon$ into Eq. (1.13), and expanding all coefficients $\tilde{A}_\beta(\xi, \tilde{u}_\varepsilon)$, $\cdots$ into the power series of ε

$$\tilde{A}_\beta(\xi, \tilde{u}_\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j\tilde{A}_\beta^{(j)}(f, \xi),$$

where

$$\tilde{A}_\beta^{(0)}=\tilde{A}_\beta(\xi, \tilde{w}_0), \quad \tilde{A}_\beta^{(1)}=\frac{\partial\tilde{A}_\beta(\xi, \tilde{w}_0)}{\partial u}(\tilde{w}_1+\tilde{v}_{1-p}),$$

$$\tilde{A}_\beta^{(j)}=\frac{\partial\tilde{A}_\beta(\xi, \tilde{w}_0)}{\partial u}(\tilde{w}_j+\tilde{v}_{j-p})+\bar{\tilde{A}}_\beta^{(j)} \quad (j=2, 3, \cdots)$$

and

$$\bar{\tilde{A}}_\beta^{(j)}=\sum_{\substack{n_1+2n_2+\cdots+qn_q=j\\(q<j)}}\frac{1}{n_1!\,n_2!\cdots n_q!}\,\frac{\partial^{n_1+n_2+\cdots+n_q}\tilde{A}_\beta(\xi, \tilde{w}_0)}{\partial u^{n_1+n_2+\cdots+n_q}}\,\tilde{w}_1^{n_1}\cdots(\tilde{w}_p+\tilde{v}_0)^{n_p}\cdots(\tilde{w}_q+\tilde{v}_{q-p})^{n_q},$$

$\cdots$, and equating all the coefficients of the powers of ε to zero, we obtain the recursive equations for $\tilde{v}_i$ $(i=0, 1, \cdots)$

$$\sum_{|\beta|=2(l_1+l_0)}\tilde{A}_\beta^{(0)}\tilde{\delta}_\beta^{(0)}\tilde{v}_0+\sum_{|\beta|=2l_0}\tilde{a}_\beta^{(0)}\tilde{\delta}_\beta^{(0)}\tilde{v}_0=0, \tag{1.22}$$

$$\begin{aligned}&\sum_{|\beta|=2(l_1+l_0)}\tilde{A}_\beta^{(0)}\tilde{\delta}_\beta^{(0)}\tilde{v}_i+\sum_{|\beta|=2l_0}\tilde{a}_\beta^{(0)}\tilde{\delta}_\beta^{(0)}\tilde{v}_i\\&=-\Big[\sum_{p_1+p_2+p_3=i,\,(p_3\neq i)}\Big(\sum_{|\beta|=2(l_1+l_0)}\tilde{A}_\beta^{(p_1)}\tilde{\delta}_\beta^{(p_2)}\tilde{v}_{p_3}-\sum_{|\beta|=2l_0}\tilde{a}_\beta^{(p_1)}\tilde{\delta}_\beta^{(p_2)}\tilde{v}_{p_3}\Big)\\&\quad+\sum_{t=1}^{i}\sum_{p_1+p_2+p_3=i-t}\Big(\sum_{|\beta|=2(l_1+l_0)-t}\tilde{A}_\beta^{(p_1)}\tilde{\delta}_\beta^{(p_2)}\tilde{v}_{p_3}+\sum_{|\beta|=2l_0-t}\tilde{a}_\beta^{(p_1)}\tilde{\delta}_\beta^{(p_2)}\tilde{v}_{p_3}\Big)\Big]\\&\quad-\lambda_i\sum_{q=0}^{i-2l_0}(\tilde{a}_\beta^{(p+q)}-\mathring{\tilde{a}}_\beta^{(p+q)})\tilde{D}^\beta\tilde{w}_{i-2l_0-q}-\mu_i\sum_{q=0}^{i-2(l_1+l_0)}(\tilde{A}_\beta^{(p+q)}-\mathring{\tilde{A}}_\beta^{(p+q)})\tilde{D}^\beta\tilde{w}_{i-2(l_1+l_0)-q},\end{aligned} \tag{1.23}$$

where

$$\lambda_i=\begin{cases}0, & \text{for } i<2l_0\\ 1, & \text{for } i\geqslant 2l_0,\end{cases}\qquad \mu_i=\begin{cases}0, & \text{for } i<2(l_1+l_0)\\ 1, & \text{for } i\geqslant 2(l_1+l_0),\end{cases}$$

and $\mathring{\tilde{a}}_\beta^{(p+q)}$ is $\tilde{a}_\beta^{(p+q)}$ with $\tilde{v}_j\equiv 0$, $(j\geqslant 0)$, etc.

From equation (1.22), we see that if we take $g(\rho)$, in (1.15), to be the solution of the following Cauchy problem for nonlinear first order differential equation

$$\begin{cases}(-1)^{l_1}\displaystyle\sum_{|\beta|=2(l_1+l_0)}\tilde{A}_\beta(\xi, \tilde{w}_0)g_\rho^\beta=\sum_{|\beta|=2l_0}\tilde{a}_\beta(\xi, \tilde{w}_0)g_\rho^\beta, & (1.24)\\ g(\rho)\,|_{\rho_1=0}=0, & (1.25)\end{cases}$$

then it reduces to the equation with constant coefficients

$$(-1)^{l_1}\frac{\partial^{2(l_1+l_0)}\tilde{v}_0}{\partial f^{2(l_1+l_0)}}+\frac{\partial^{2l_0}\tilde{v}_0}{\partial f^{2l_0}}=0. \tag{1.26}$$

We can prove that the solution of Cauchy problem (1.24)—(1.25) exists, and is positive in the neighborhood of $\partial\Omega$. From [6] we know that the solution of equation (1.26) of boundary layer type is

$$\tilde{v}_0(f, \xi)=\sum_{k=1}^{l_1}\alpha_k^{(0)}(\xi)e^{-\lambda_k f}, \tag{1.27}$$

where $\alpha_k^{(0)}(k=0, 1, \cdots, l_1)$ are the functions determined by the boundary conditions given below, and λ_k $(k=0, 1, \cdots, l_1)$ are the roots of the characteristic equation corresponding to (1.26)

$$(-1)^{l_1}\lambda^{2(l_1+l_0)}+\lambda^{2l_0}=0 \tag{1.28}$$

with positive real part.

As $\tilde{v}_0$ is determined, substituting it into (1.23) (with $i=1$), we obtain the differential equation governing $\tilde{v}_1$

$$\sum_{|\beta|=2l_0}\tilde{a}_{\beta}g_{\rho}^{\beta}\left[(-1)^{l_1}\frac{\partial^{2(l_1+l_0)}\tilde{v}_1}{\partial f^{2(l_1+l_0)}}+\frac{\partial^{2l_0}\tilde{v}_1}{\partial f^{2l_0}}\right]$$
$$=-\Big[\sum_{|\beta|=2(l_1+l_0)}(\tilde{A}_{\beta}^{(0)}\tilde{\delta}_{\beta}^{(1)}+\tilde{A}_{\beta}^{(1)}\tilde{\delta}_{\beta}^{(0)})\tilde{v}_0+\sum_{|\beta|=2l_0}(\tilde{a}_{\beta}^{(0)}\tilde{\delta}_{\beta}^{(1)}+\tilde{a}_{\beta}^{(1)}\tilde{\delta}_{\beta}^{(0)})\tilde{v}_0$$
$$+\sum_{|\beta|=2(l_1+l_0)-1}\tilde{A}_{\beta}^{(0)}\tilde{\delta}_{\beta}^{(0)}\tilde{v}_0+\sum_{|\beta|=2l_0-1}\tilde{a}_{\beta}^{(0)}\tilde{\delta}_{\beta}^{(0)}\tilde{v}_0\Big]. \tag{1.29}$$

Since its right hand side decreases to zero exponentially, we can prove that it also has the solution of boundary layer type. Similarly, we can prove that all of the remainder equations of (1.23) have the solution of boundary layer type. Later, we shall find out the boundary conditions for $\tilde{w}_i$ and $\tilde{v}_i$ $(i=0,1,\cdots)$.

Substituting $\tilde{u}_\varepsilon$ into boundary condition (1.14), we obtain

$$\tilde{B}_j\sum_{n=0}^{\infty}\varepsilon^n\tilde{w}_n+\sum_{|\mu|\leqslant m_j}\tilde{b}_{\mu}^{(j)}\varepsilon^{p-|\mu|}(\tilde{\delta}_{\mu}^{(0)}+\varepsilon\tilde{\delta}_{\mu}^{(1)}+\cdots+\varepsilon^{|\mu|}\tilde{\delta}_{\mu}^{(|\mu|)})$$
$$\times\sum_{n=0}^{\infty}\varepsilon^n\tilde{v}_n=\tilde{h}_j \quad (j=0,1,\cdots,l_1+l_0-1). \tag{1.30}$$

Here, we omit the sign of taking $\rho_1=0$. In order to obtain the recursive boundary conditions for $\tilde{w}_i$ and $\tilde{v}_i$ $(i=0,1,\cdots)$, we should take

$$p=m_{l_0} \tag{1.31}$$

in (1.30). By equating all the coefficients of the powers of ε to zero in (1.30), we obtain

$$\tilde{B}_j\tilde{w}_0=\tilde{h}_j \quad (j=0,1,\cdots), \tag{1.32}$$

$$\begin{cases}\tilde{B}_{l_0}\tilde{w}_0+\sum\limits_{|\mu|=m_{l_0}}\tilde{b}_{\mu}^{(l_0)}\tilde{\delta}_{\mu}^{(0)}\tilde{v}_0=\tilde{h}_{l_0}, & (1.33a)\\ \sum\limits_{|\mu|=m_j}\tilde{b}_{\mu}^{(j)}\tilde{\delta}_{\mu}^{(0)}\tilde{v}_0=0 \quad (j=l_0+1,\cdots,l_1+l_0-1). & (1.33b)\end{cases}$$

....

In order to write out their general recursive relations, denote

$$r_s=m_s-m_{s-1} \quad (s=1,2,\cdots,l_1+l_0-1),$$

then we have

$$\begin{cases}\tilde{B}_j\tilde{w}_\alpha=0 \quad (j=0,1,\cdots,l_0-2), & (1.34a)\\ \tilde{B}_{l_0-1}\tilde{w}_\alpha+\delta_{\alpha,r_{l_0}}\sum\limits_{|\mu|=m_{l_0}-1}\tilde{b}_{\mu}^{(l_0-1)}\tilde{\delta}_{\mu}^{(0)}\tilde{v}_0=0, & (1.34b)\end{cases}$$

$(\delta_{\alpha,r_{l_0}}=0$, for $1\leqslant\alpha<r_{l_0}$; $\delta_{r_{l_0},r_{l_0}}=1$, for $\alpha=1,2,\cdots,r_{l_0})$,

$$\begin{cases}\tilde{B}_{l_0}\tilde{w}_\alpha+\sum\limits_{k=0}^{\alpha}\sum\limits_{|\mu|=m_{l_0}-k}\tilde{b}_{\mu}^{(l_0)}(\tilde{\delta}_{\mu}^{(0)}\tilde{v}_{\alpha-k}+\cdots+\tilde{\delta}_{\mu}^{(\alpha-k)}\tilde{v}_0)=0, & (1.35a)\\ \sum\limits_{|\mu|=r_{l_0+1}}\tilde{b}_{\mu}^{(l_0+1)}(\tilde{\delta}_{\mu}^{(0)}\tilde{v}_\alpha+\cdots+\tilde{\delta}_{\mu}^{(\alpha)}\tilde{v}_0)+\delta_{\alpha,r_{l_0+1}}\tilde{B}_{l_0+1}\tilde{w}_0=\tilde{h}_{l_0+1}, & (1.35b)\\ \sum\limits_{|\mu|=m_j}\tilde{b}_{\mu}^{(j)}(\tilde{\delta}_{\mu}^{(0)}\tilde{v}_\alpha+\cdots+\tilde{\delta}_{\mu}^{(\alpha)}\tilde{v}_0)=0 \quad (j=l_0+2,\cdots,l_1+l_0-1), & (1.35c)\end{cases}$$

$(\delta_{\alpha,r_{l_0+1}}=0$, for $1\leqslant\alpha<r_{l_0+1}$, $\delta_{r_{l_0+1},r_{l_0+1}}=1$, $\alpha=1,2,\cdots,r_{l_0+1})$,

$$\begin{cases}\widetilde{B}_j\widetilde{w}_{r_{l_0}+\alpha}=0 \quad (j=0, 1, \cdots, l_0-3), & (1.36a)\\ \widetilde{B}_{l_0-2}\widetilde{w}_{r_{l_0}+\alpha}+\delta_{\alpha, r_{l_0-1}}\sum\limits_{|\mu|=m_{l_0-2}}\widetilde{b}_\mu^{(l_0-2)}\widetilde{\delta}_\mu^{(0)}\widetilde{v}_0=0, & (1.36b)\\ \widetilde{B}_{l_0-1}\widetilde{w}_{r_{l_0}+\alpha}+\sum\limits_{k=0}^{\alpha}\sum\limits_{|\mu|=m_{l_0-1}-k}\widetilde{b}_\mu^{(l_0-1)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{\alpha-k}+\cdots+\widetilde{\delta}_\mu^{(\alpha-k)}\widetilde{v}_0)=0 & (1.36c)\end{cases}$$

$(\delta_{\alpha, r_{l_0-1}}=0$, for $1\leqslant\alpha<r_{l_0-1}$, $\delta_{r_{l_0-1}, r_{l_0-1}}=1$, for $\alpha=1, 2, \cdots, r_{l_0-1})$,

$$\begin{cases}\widetilde{B}_{l_0}\widetilde{w}_{r_{l_0+1}+\alpha}+\sum\limits_{k=0}^{r_{l_0+1}+\alpha}\sum\limits_{|\mu|=m_{l_0}-k}\widetilde{b}_\mu^{(l_0)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_{l_0+1}+\alpha-k}+\cdots+\widetilde{\delta}_\mu^{(r_{l_0+1}+\alpha-k)}\widetilde{v}_0)=0, & (1.37a)\\ \widetilde{B}_{l_0+1}\widetilde{w}_{r_{l_0+1}+\alpha}+\sum\limits_{k=0}^{r_{l_0+1}+\alpha}\sum\limits_{|\mu|=m_{l_0+1}-k}\widetilde{b}_\mu^{(l_0+1)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_{l_0+1}+\alpha-k}+\cdots+\widetilde{\delta}_\mu^{(r_{l_0+1}+\alpha-k)}\widetilde{v}_0)=0, & (1.37b)\\ \sum\limits_{|\mu|=m_{l_0+2}}\widetilde{b}_\mu^{(l_0+2)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_{l_0+1}+\alpha}+\cdots+\widetilde{\delta}_\mu^{(r_{l_0+1}+\alpha)}\widetilde{v}_0)+\delta_{\alpha, r_{l_0+2}}\widetilde{B}_{l_0+2}\widetilde{w}_0=\delta_{\alpha, r_{l_0+2}}\widetilde{h}_{l_0+2}, & (1.37c)\\ \sum\limits_{|\mu|=m_j}\widetilde{b}_\mu^{(j)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_{l_0+1}+\alpha}+\cdots+\widetilde{\delta}_\mu^{(r_{l_0+1}+\alpha)}\widetilde{v}_0)=0 \quad (j=l_0+3, \cdots, l_1+l_0-1), & (1.37d)\end{cases}$$

$(\delta_{\alpha, r_{l_0+2}}=0$, for $1\leqslant\alpha<r_{l_0+2}$, $\delta_{r_{l_0+2}, r_{l_0+2}}=1$, for $\alpha=1, 2, \cdots, r_{l_0+2})$

....

and

$$\widetilde{B}_j\widetilde{w}_{r_{l_0}+\cdots+r_1+\alpha}+\sum_{k=0}^{r_1+\cdots+r_j+\alpha}\sum_{|\mu|=m_j-k}\widetilde{b}_\mu^{(j)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_1+\cdots+r_j+\alpha-k}+\cdots$$
$$+\delta_\mu^{(r_1+\cdots+r_j+\alpha-k)}\widetilde{v}_0)=0 \quad (j=0, 1, \cdots, l_0-1;\ \alpha=0, 1, \cdots), \tag{1.38}$$

$$\widetilde{B}_j\widetilde{w}_{r_{l_0+1}+\cdots+r_{l_1+l_0-1}+\alpha}+\sum_{k=0}^{r_{l_0+1}+\cdots+r_{l_1+l_0-1}+\alpha}\sum_{|\mu|=m-k_j}\widetilde{b}_\mu^{(j)}(\widetilde{\delta}_\mu^{(0)}\widetilde{v}_{r_{l_0+1}+\cdots+r_{l_1+l_0-1}+\alpha-k}+\cdots$$
$$+\delta_\mu^{(r_{l_0+1}+\cdots+r_{l_1+l_0-1}+\alpha-k)}\widetilde{v}_0)=0 \quad (j=l_0, l_0+1, \cdots, l_1+l_0-1;\ \alpha=0, 1, \cdots). \tag{1.39}$$

Let $\lambda_k(k=1, 2, \cdots, l_1)$ be the different roots of characteristic equation (1.28). Then all of the systems of equations governing $\widetilde{v}_n$ $(n=0, 1, \cdots)$ can be solved.

So the main term $w_0(x)$ of the expansion of u_ε is the solution of the degenerated problem (1.8)—(1.9). After $w_0(x)$ is determined, substituting it into (1.33), we obtain the boundary conditions for $\widetilde{v}_0$, so $\widetilde{v}_0$ can be determined. Substituting $\widetilde{v}_0$ into (1.35) (with $\alpha=1$), we obtain the boundary condition for $w_1(x)$. Combining this with Eq. (1.12) (with $n=1$), we can solve $w_1(x)$. In the same way, we can determine all of the terms w_n, $\widetilde{v}_n(n=0, 1, \cdots)$.

Define

$$V=\psi(x)\widetilde{V}, \tag{1.40}$$

where $\psi(x)$ is an infinitely differentiable truncated function

$$\psi(x)=\begin{cases}1, & \text{for } x\in\Omega_{\eta/3}=\left\{\rho\in\overline{\Omega}\,|\,0\leqslant\rho_1\leqslant\dfrac{\eta}{3}\right\},\\ 0, & \text{for } x\in\overline{\Omega}/\Omega_{2\eta/3}.\end{cases}$$

We can easily show (cf. [6])

Theorem 1. *Under assumptions (H1) and (H2), the function*

$$U_m=W_m+V_m=\sum_{n=0}^{m}\varepsilon^n w_n(x)+\psi(x)\varepsilon^{m_{l_0}}\sum_{n=0}^{M}\varepsilon^n\widetilde{v}_n(f, \xi) \tag{1.41}$$

is the formal approximation of order m to the solution of the boundary value problem

(1.1)—(1.2), *where* $M=\max\{m+2l_0-m_{l_0},\ m+m_{l_1+l_0-1}-m_{l_0}\}$ *has been constructed above.*

§ 2. Estimate for the Remainder Term and Existence of Solution u_ε

Let L_ε be the linearized differential operator of N_ε

$$\begin{aligned} L_\varepsilon \equiv{}& \varepsilon^{2l_1}\sum_{|\beta|\le 2(l_1+l_0)} A_\beta(x, U_m)D^\beta + \sum_{|\beta|\le 2l_0} a_\beta(x, U_m)D^\beta \\ &+\varepsilon^{2l_1}\sum_{|\beta|\le 2(l_1+l_0)} \frac{\partial A_\beta(x, U_m)}{\partial u} D^\beta U_m + \sum_{|\beta|\le 2l_0} \frac{\partial a_\beta(x, U_m)}{\partial u} D^\beta U_m \end{aligned} \tag{2.1}$$

and let Z_m be the remainder of U_m

$$Z_m = u_\varepsilon - U_m. \tag{2.2}$$

It is easily shown that (cf. [10]).

$$L_\varepsilon Z_m = \varepsilon^{m+1}\Phi_m(x, \varepsilon) + R_\varepsilon(Z_m), \tag{2.3}$$

$$B_j Z_m|_{\partial\Omega} = \varepsilon^{m+1}\Psi_m^{(j)}(x, \varepsilon) \quad (j=0, 1, \cdots, l_1+l_0-1), \tag{2.4}$$

where $\Phi_m = O(1)$, $\Psi_{(u)}^{(j)} = O(1)$, and

$$\begin{aligned} R_\varepsilon(Z_m) \equiv{}& L_\varepsilon Z_m - N_\varepsilon u_\varepsilon + N_\varepsilon U_m \\ ={}& \varepsilon^{2l_1}\sum_{|\beta|\le 2(l_1+l_0)} [A_\beta(x, U_m) - A_\beta(x, U_m+Z_m)]D^\beta(U_m+Z_m) \\ &+ \sum_{|\beta|\le 2l_0} [a_\beta(x, U_m) - a_\beta(x, U_m+Z_m)]D^\beta(U_m+Z_m) \\ &+ \varepsilon^{2l_1}\sum_{|\beta|\le 2(l_1+l_0)} \frac{\partial A_\beta(x, U_m)}{\partial u}(D^\beta U_m)Z_m \\ &+ \sum_{|\beta|\le 2l_0} \frac{\partial a_\beta(x, U_m)}{\partial u}(D^\beta U_m)Z_m. \end{aligned} \tag{2.5}$$

Setting

$$\tilde Z_m = Z_m - \varepsilon^{m+1}\Psi_m,$$

where Ψ_m is any differentiable function which satisfies the boundary conditions

$$B_j\Psi_m|_{\partial\Omega} = \Psi_m^{(j)} \quad (j=0, 1, \cdots, l_1+l_0-1),$$

we obtain the boundary value problem for $\tilde Z_m$

$$L_\varepsilon \tilde Z_m = \varepsilon^{m+1}\Phi_m^* + R(\tilde Z_m + \varepsilon^{m+1}\Psi_m), \tag{2.6}$$

$$B_j\tilde Z_m|_{\partial\Omega} = 0, \quad (j=0, 1, \cdots, l_1+l_0-1). \tag{2.7}$$

Banach space $C^{2(l_1+l_0)+\alpha}(\Omega)$, $(0<\alpha<1)$ is constituted by the functions of $C^{2(l_1+l_0)}(\bar\Omega)$ with the norm

$$|u|_{2(l_1+l_0)+\alpha} \equiv |u|_{2(l_1+l_0)} + [u]_{2(l_1+l_0)+\alpha} \tag{2.8}$$

finite, where

$$|u|_l \equiv \sum_{j=0}^{l}[u]_j, \quad [u]_j \equiv \sup_{x\in\bar\Omega}|D^j u(x)|,$$

$$[u]_{2(l_1+l_0)+\alpha} \equiv \sup_{x, y\in\bar\Omega, x\ne y} \frac{|D^{2(l_1+l_0)}u(x) - D^{2(l_1+l_0)}u(y)|}{|x-y|^\alpha}.$$

We have

Lemma 1. (cf. [6] Theorem 3) *Consider the boundary value problem*

$$L_\varepsilon u_1 \equiv (\varepsilon_1 L_1 + L_0) u_1 = f(x), \quad x \in \Omega, \ 0 < \varepsilon_1 \ll 1, \tag{2.9}$$

$$B_j u_1 |_{\partial\Omega} = 0 \quad (j = 0, 1, \cdots, l_1 + l_0 - 1), \tag{2.10}$$

where L_0 and L_1 are strongly elliptic operators of order $2l_0$ and $2(l_1+l_0)$ in $\overline{\Omega}$ respectively, and satisfy the conditions

$$(L_0 u, u) \geqslant \delta_0 \|u\|^2_{L_2}, \quad \forall u \in \hat{C}^{2(l_1+l_0)}(\overline{\Omega}),$$

$$(L_1 u, u) \geqslant -k_0 \|u\|^2_{L_2}, \quad \forall u \in \hat{C}^{2(l_1+l_0)}(\overline{\Omega}),$$

where $\hat{C}^{2(l_1+l_0)}(\overline{\Omega})$ is the set of functions of $C^{2(l_1+l_0)}(\overline{\Omega})$ satisfying all the boundary conditions in (2.10). Assume that the coefficients of L_0 and L_1 belong to $C^{l-2(l_1+l_0)+\alpha}(\Omega)$. where $l \geqslant 2(l_1+l_0)$, and $\partial\Omega \in C^2$, and the system of boundary operators $\{B\}_0^{l_1+l_0-1}$ is regular and covers L_ε on $\partial\Omega$[11], then

$$[u_1]_j \leqslant k [\varepsilon_1^{(l-2l_1+1-j)/(2l_1-1)} |f|_{l-2(l_1+l_0)+\alpha} + \varepsilon_1^{-\left(j+\frac{n}{2}\right)/(2l_1-1)} \|f\|_{L_2}, \tag{2.11}$$

Suppose that the linearized operator L_ε of N_ε satisfies the conditions given in Lemma 1, and L_ε^{-1} exists in $\hat{C}^{2(l_1+l_0)}(\overline{\Omega})$. Define an operator equation

$$u = T_\varepsilon u \tag{2.12}$$

in $\hat{C}^{2(l_1+l_0)}(\overline{\Omega})$, where

$$T_\varepsilon u \equiv L_\varepsilon^{-1} [\varepsilon^{m+1} \Phi_m^* + R(u + \varepsilon^{m+1} \Psi_m)]. \tag{2.13}$$

Let S^{m+1} be a sphere in $\hat{C}^{2(l_1+l_0)}(\overline{\Omega})$

$$S^{m+1} = \{u \in \hat{C}^{2(l_1+l_0)}(\overline{\Omega}) \mid |u|_{2(l_1+l_0)} \leqslant \varepsilon^{m+1}\}.$$

Then, we have

Lemma 2. *If $u \in S^{m+1}$, then $T_\varepsilon u \in S^{m+1}$ for $m > \gamma + 2l_0 - 1$, where*

$$\gamma = {}^{2l\left(2l_1+2l_0+\frac{1}{2}\right)/(2l_1-1)}.$$

Proof From Lemma 1, we have

$$|T_\varepsilon u|_{2(l_1+l_0)} \leqslant \varepsilon^{-\gamma} k (\varepsilon^{m+1} |\Phi_m^*|_0 + |R(u + \varepsilon^{m+1} \Psi_m)|_0).$$

Since

$$\begin{aligned} |R(u+\varepsilon^{m+1}\Psi_m)|_0 \leqslant \varepsilon^{2l_1} \sum_{|\beta| \leqslant 2(l_1+l_0)} \Bigg\{ & \Bigg| \frac{\partial^2 A_\beta(x, U_m + \theta_2(U_m + \theta_1(u + \varepsilon^{m+1}\Psi)))}{\partial u^2} \\ & \times (u+\varepsilon^{m+1}\Psi_m)^2 D^\beta U_m \Bigg|_0 \\ & + \Bigg| \frac{\partial A_\beta(x, U_m + \theta_1(u+\varepsilon^{m+1}\Psi_m))}{\partial u} (u+\varepsilon^{m+1}\Psi_m) D^\beta (u+\varepsilon^{m+1}\Psi_m) \Bigg|_0 \Bigg\} \\ & + \sum_{|\beta| \leqslant 2l_0} \Bigg\{ \Bigg| \frac{\partial^2 a_\beta(x, U_m + \theta_2(\cdots))}{\partial u^2} (u+\varepsilon^{m+1}\Psi_m)^2 D^\beta U_m \Bigg|_0 \\ & + \Bigg| \frac{\partial a_\beta(x, U_m + \theta_1(u+\varepsilon^{m+1}\Psi_m))}{\partial u} (u+\varepsilon^{m+1}\Psi_m) D^\beta (u+\varepsilon^{m+1}\Psi_m) \Bigg|_0 \Bigg\} \\ & \leqslant k_1 \varepsilon^{2(m+1)-2l_0}, \end{aligned}$$

where $0 \leqslant \theta_1, \ \theta_2 \leqslant 1$, we have

$$|T_\varepsilon u|_{2(l_1+l_0)} \leqslant k_2 \varepsilon^{m+1-\gamma-2l_0} \varepsilon^{m+1}.$$

In case $m > \gamma + 2l_0 - 1$, we can limit ε so small that $k_2 \varepsilon^{m+1-\gamma-2l_0} < 1$, thus the lemma is proved.

Lemma 3. *If* $u_1\in S^{m+1}$, $u_2\in S^{m+2}$, *then*

$$|T_\varepsilon u_1-T_\varepsilon u_2|_{2(l_1+l_0)}\leqslant k|u_1-u_2|_{2(l_1+l_0)}$$

for $m>\gamma+2l_0-1$, *where* $0<k<1$.

Proof From Lemma 1, we have

$$|T_\varepsilon u_1-T_\varepsilon u_2|_{2(l_1+l_0)}\leqslant \varepsilon^{-\gamma}k_1|R(u_1+\varepsilon^{m+1}\Psi_m)-R(u_2+\varepsilon^{m+1}\Psi_m)|_0.$$

Since

$$\begin{aligned}&|R(u_1+\varepsilon^{m+1}\Psi_m)-R(u_2+\varepsilon^{m+1}\Psi_m)|_0\\ &\leqslant \varepsilon^{2l_1}\sum_{|\beta|\leqslant 2(l_1+l_0)}\left\{\left|\frac{\partial^2 A_\beta}{\partial u^2}[\varepsilon^{m+1}\Psi_m+\theta(u_1-u_2)](D^\beta U_m)(u_1-u_2)\right|_0\right.\\ &+\left|\frac{\partial A_\beta}{\partial u}(D^\beta\varepsilon^{m+1}\Psi_m)(u_1-u_2)\right|_0\\ &\left.+\left|\frac{\partial A_\beta}{\partial u}(\varepsilon^{m+1}\Psi_m+u_1)D^\beta(u_1-u_2)\right|_0+\left|\frac{\partial A_\beta}{\partial u}(D^\beta u_2)(u_1-u_2)\right|_0\right\}\\ &+\sum_{|\beta|\leqslant 2l_0}\left\{\left|\frac{\partial^2 a_\beta}{\partial u^2}[\varepsilon^{m+1}\Psi_m+\theta(u_1-u_2)](D^\beta U_m)(u_1-u_2)\right|_0\right.\\ &+\left|\frac{\partial a_\beta}{\partial u}(D^\beta\varepsilon^{m+1}\Psi_m)(u_1-u_2)\right|_0+\left|\frac{\partial a_\beta}{\partial u}(\varepsilon^{m+1}\Psi_m+u_1)D^\beta(u_1-u_2)\right|_0\\ &\left.+\left|\frac{\partial a_\beta}{\partial u}(D^\beta u_2)(u_1-u_2)\right|_0\right\}\leqslant k_2\varepsilon^{m+1}|u_1-u_2|_{2(l_1+l_0)},\end{aligned}$$

we have

$$|T_\varepsilon u_1-T_\varepsilon u_2|_{2(l_1+l_0)}\leqslant k_3\varepsilon^{m+1-\gamma}|u_1-u_2|_{2(l_1+l_0)}.$$

Owing to $m>\gamma-1$, we can limit ε so small that $k_3\varepsilon^{m+1-\gamma}\leqslant k<1$, thus the lemma is proved.

From Lemma 2 and Lemma 3 we know that T_ε is a contraction mapping which maps a convex subset S^{m+1} in a Banach space into itself, hence there exists a unique element $u_0\in S^{m+1}$ such that

$$u_0=T_\varepsilon u_0.$$

So we have the theorem

Theorem 2. Under the assumptions (H1), (H2), and all of the conditions mentioned above, there exists a unique solution $u_\varepsilon\in C^{2(l_1+l_0)}(\bar{\Omega})$ as ε is sufficiently small, and it possesses the asymptotic expansion

$$u_\varepsilon=U_m+Z_m, \tag{2.14}$$

where U_m is defined by (1.41), and $|Z_m|_{2(l_1+l_0)}\leqslant k\varepsilon^{m+1}$ as $m>\gamma+2l_0-1$, where k is a constant independent of ε.

By the same process as that given in [10], we have

Theorem 2'. *Under the same assumptions as in Theorem 2, there exists a solution* $u_\varepsilon\in C^{2(l_1+l_0)}(\bar{\Omega})$, *and it is unique in* $C^0(\bar{\Omega})$ *as* ε *is sufficiently small, and has the same asymptotic expansion of the form* (2.14), *where* $Z_m=O(\varepsilon^{m+1})$ *as* $m\geqslant 0$.

It is the extension of Fife's[12] work.

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