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IDEALS OF TENSOR PRODUCTS OF DIVISION ALGEBRAS AND PRIMITIVE ALGEBRAS

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Abstract

In this paper some lattice isomorphism theorems about one-sided and two-sided ideals in tensor products of primitive algebras and division algebras are proved.

Introduction

In [3], Xu Yonghua showed there is an isomorphism between the lattice of certain left ideals of the tensor product of two primitive algebras and the lattice of submodules of the tensor product of two irreducible modules. His theorem is a generalization of Azumaya-Nakayama theorem.

In this paper, we continue his work. In section I, we give a simple proof of Xu's theorem and deduce an isomorphism theorem about the lattice of certain right ideals of tensor product of two primitive algebras. In section II and section III, we obtain the theorems on the structures of lattices of tensor products of simple primitive algebras and division algebras. In the last section, we show that the lattice of two-sided ideals contained in the tensor product of two primitive simple algebras is isomorphic to the lattice of two-sided ideals contained in the tensor product of their associated division algebras. Throughout this paper, Φ is a field, $\mathfrak{A}$ (or $\mathfrak{A}_i$) is right primitive algebra (not necessary with an identity) over Φ and $\mathfrak{S}$ (or $\mathfrak{S}_i$) is the nonzero socle of $\mathfrak{A}$ (or $\mathfrak{A}_i$).

I. One-sided ideals of $\mathfrak{A}_1\otimes\mathfrak{A}_2$

Lemma 1. *Let $\mathfrak{A}$ be a primitive ring with nonzero socle $\mathfrak{S}$, and L be the r-dim left ideal of $\mathfrak{S}$. Then there is an idempotent E of $\mathfrak{S}$ and a class of orthogonal primitive idempotents*

$$\{E_i\}_{i=1,\cdots,r},\ E_i\in\mathfrak{A},\ \text{such that } L=\mathfrak{A}E,\ E=E_1+\cdots+E_r.$$

Proof It is similar to the proof of a lemma in [1] (p 137) and we omit it.

Lemma 2. *Let $(\mathfrak{M}_i, \Phi)$ $(i=1, 2)$ be the vector space over Φ, $\mathfrak{A}_i$ be the irreducible algebra of linear transformations in $\mathfrak{M}_i$, and $\mathfrak{S}_i$ be the nonzero socle of $\mathfrak{A}_i$. Then we have $N(\mathfrak{S}_1\otimes_\Phi\mathfrak{S}_2)=N$ and $(\mathfrak{S}_1\otimes_\Phi\mathfrak{S}_2)L=L$, where N is any right ideal of $\mathfrak{S}_1\otimes_\Phi\mathfrak{S}_2$ and L is any left ideal of $\mathfrak{S}_1\otimes_\Phi\mathfrak{S}_2$*

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Proof We know that $\mathfrak{S}_1=\sum_{i\in I}\oplus\mathfrak{S}_1E_i$ and $\mathfrak{S}_2=\sum_{j\in J}\oplus\mathfrak{S}_2F_j$, where E_i and F_j are the primitive idempotents of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively. Then

$$\mathfrak{S}_1\otimes\mathfrak{S}_2=\sum_{I,J}\oplus(\mathfrak{S}_1E_i\otimes\mathfrak{S}_2F_j).$$

If u is any element of N, then $u=\sum_{I',J'}x_i\otimes y_j$, $(x_i\in\mathfrak{S}_iE_i,\ y_j\in\mathfrak{S}_2F_j)$ where I' is a finite subset of I, and J' is a finite subset of J. By Lemma 1, without loss of generality, we can assume that $\{E_i\}_{i\in I'}$ and $\{F_j\}_{j\in J'}$ are the orthogonal idempotents. We chose an element s of $\mathfrak{S}_1\otimes\mathfrak{S}_2$: $s=\sum_{I'}\sum_{J'}E_i\otimes F_j$. For every $x_i\otimes y_j$, we have

$$(x_i\otimes y_j)s=(x_iE_i\otimes y_jF_j)s=(x_iE_i\otimes y_jF_j)\left(\sum_{I'}\sum_{J'}E_i\otimes F_j\right)=x_i\otimes y_j.$$

This shows $us=u$, that is $N\subseteq N(\mathfrak{S}_1\otimes\mathfrak{S}_2)$. The converse inclusion is trivial.

Similarly, we can prove $(\mathfrak{S}_1\otimes\mathfrak{S}_2)L=L$, for every left ideal L of $\mathfrak{S}_1\otimes\mathfrak{S}_2$.

Lemma 3. *$\mathfrak{S}_i$, $(\mathfrak{M}_i, \Phi)$ are the same as lemma 2. We regard $\mathfrak{S}_1\otimes\mathfrak{S}_2$ as a left $\mathfrak{S}_1\otimes\mathfrak{S}_2$-module and we assume that Ω is its centralizer. Then, $(\mathfrak{S}_1\otimes\mathfrak{S}_2)_r$ is a subring of Ω, where $(\mathfrak{S}_1\otimes\mathfrak{S}_2)_r$ denotes right multiplication of $\mathfrak{S}_1\otimes\mathfrak{S}_2$. Moreover, Ω-submodule of $\mathfrak{S}_1\otimes\mathfrak{S}_2$ coincides with $(\mathfrak{S}_1\otimes\mathfrak{S}_2)_r$-submodule of $\mathfrak{S}_1\otimes\mathfrak{S}_2$.*

Proof For any $s\in\mathfrak{S}_1\otimes\mathfrak{S}_2$, if $(\mathfrak{S}_1\otimes\mathfrak{S}_2)s=0$, then we have $(\mathfrak{M}_1\otimes\mathfrak{M}_2)(\mathfrak{S}_1\otimes\mathfrak{S}_2)s=0$, that is $(\mathfrak{M}_1\otimes\mathfrak{M}_2)s=0$, so $s=0$. The first assertion is clear. Let N be an Ω-submodule of $\mathfrak{S}_1\otimes\mathfrak{S}_2$, it is easy to see that N is an $\mathfrak{S}_1\otimes\mathfrak{S}_2$-submodule. Conversly, if N is an $\mathfrak{S}_1\otimes\mathfrak{S}_2$-submodule, that is a right ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$. By Lemma 2

$$N\Omega=(N(\mathfrak{S}_1\otimes\mathfrak{S}_2))\Omega=N((\mathfrak{S}_1\otimes\mathfrak{S}_2)\Omega)\subseteq N(\mathfrak{S}_1\otimes\mathfrak{S}_2)=N.$$

Therefore N is also an Ω-submodule.

The next lemma can be proved in a similar way.

Lemma 4. *$\mathfrak{S}_i$, $(i=1, 2)$, $(\mathfrak{M}_i, \Phi)$ are the same as Lemma 2. We regard $\mathfrak{S}_1\otimes\mathfrak{S}_2$ as a right $\mathfrak{S}_1\otimes\mathfrak{S}_2$-module, and $\bar{\Omega}$ is its centralizer. Then $(\mathfrak{S}_1\otimes\mathfrak{S}_2)_l$, the left multiplication of $\mathfrak{S}_1\otimes\mathfrak{S}_2$, can be regarded as a subring of $\bar{\Omega}$. Moreover, $\bar{\Omega}$-submodule and $(\mathfrak{S}_1\otimes\mathfrak{S}_2)_l$-submodule coincide.*

Let $\mathfrak{A}_i(i=1, 2)$ be an irreducible algebra of linear transformations in a vector space $\mathfrak{M}_i$ over a field Φ, and let Δ_i be the centralizer of $\mathfrak{M}_i$ as right $\mathfrak{A}_i$-module. Then $\mathfrak{M}_i$ can be regarded as a left vector space over Δ_i. If $\mathfrak{M}'_i$ is the dual vector space of $(\Delta_i, \mathfrak{M}_i)$ associated with $\mathfrak{A}_i$, then $\mathfrak{M}'_i$ can be regarded as a vector space over Φ (in the natural way). Then we have the tensor product space $\mathfrak{M}'=\mathfrak{M}'_1\otimes_\Phi\mathfrak{M}'_2$. Since $\mathfrak{M}'_i$ is the dual of $\mathfrak{M}_i$ associated with $\mathfrak{A}_i$, $\mathfrak{A}_i$ is also an irreducible algebra of linear transformations in $(\mathfrak{M}'_i, \Delta_i)$ and its centralizer is Δ_i (see [1]). Therefore, $\mathfrak{M}'$ is a left $\mathfrak{A}_1\otimes\mathfrak{A}_2$ and right $\Delta_1\otimes\Delta_2$-bimodule.

Theorem 1. *Let $\mathfrak{A}_i(i=1, 2)$ be an irreducible algebra of linear transformations in a vector space $\mathfrak{M}_i$ over Φ, in which Δ_i is the centralizer of $\mathfrak{M}_i$ as right $\mathfrak{A}_i$-module and $\mathfrak{S}_i$ is the nonzero socle of $\mathfrak{A}_i$. Let $\mathfrak{M}'_i$ be the dual of $(\Delta_i, \mathfrak{M}_i)$ associated with $\mathfrak{A}_i$, and let*

$\mathfrak{M}=\mathfrak{M}_1\otimes_{\varphi}\mathfrak{M}_2$, $\mathfrak{M}'=\mathfrak{M}'_1\otimes_{\varphi}\mathfrak{M}'_2$. *Then, the lattice of right ideals of* $\mathfrak{A}_1\otimes\mathfrak{A}_2$ *which lie in* $\mathfrak{S}_1\otimes\mathfrak{S}_2$ *is isomorphic to the lattice of right* $\Delta_1\otimes\Delta_2$*-submodules of* $\mathfrak{M}'$; *the lattice of left ideals which lie in* $\mathfrak{S}_1\otimes\mathfrak{S}_2$ *is isomorphic to the lattice of* $\Delta_1\otimes\Delta_2$*-submodules of* $\mathfrak{M}$.

Proof By Lemma 2. it is easy to see that N is a right ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$ iff N is a right ideal of $\mathfrak{A}_1\otimes\mathfrak{A}_2$. We know that $\mathfrak{S}_1=\sum_I\oplus\mathfrak{S}_1E_i$, $\mathfrak{S}_2=\sum_J\oplus\mathfrak{S}_2F_j$, and $\mathfrak{S}_1\otimes\mathfrak{S}_2=\sum_{I,J}\oplus(\mathfrak{S}_1E_i\otimes\mathfrak{S}_2F_j)$, where $\mathfrak{S}_1E_i$ and $\mathfrak{S}_2F_j$ and minimal left ideals of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively. As a left $\mathfrak{S}_1$-module, we have $\mathfrak{S}_1E_i\cong\mathfrak{S}_1E_{i'}\cong\mathfrak{M}'_2$ for any i, $i'\in I$. As a left $\mathfrak{S}_2$-module, we also have $\mathfrak{S}_2F_j\cong\mathfrak{S}_2F_{j'}\cong\mathfrak{M}'_2$ for any j, $j'\in J$. It is not difficult to verify that $\mathfrak{S}_1E_i\otimes\mathfrak{S}_2F_j\cong\mathfrak{S}_1E_{i'}\otimes\mathfrak{S}_2F_{j'}\cong\mathfrak{M}'_1\otimes\mathfrak{M}'_2=\mathfrak{M}'$ as a left $\mathfrak{S}_1\otimes\mathfrak{S}_2$-module. Using Lemma 3 and a theorem in [1] (p 111), we conclude that the lattice of right ideals of $\mathfrak{S}_1\otimes\mathfrak{S}_2$ is isomorphic to the lattice of right $\Delta_1\otimes\Delta_2$-submodules of $\mathfrak{M}'$. This is our first statement. The other can be proved in the same way.

II. Representations of one-sided ideals of $\mathfrak{S}_1\otimes\mathfrak{S}_2$

As above, set $\mathfrak{M}=\mathfrak{M}_1\otimes\mathfrak{M}_2$, $\mathfrak{M}'=\mathfrak{M}'_1\otimes\mathfrak{M}'_2$. Let $\mathfrak{M}'\times\mathfrak{M}$ be the product set of $\mathfrak{M}'$ and $\mathfrak{M}$: $\mathfrak{M}'\times\mathfrak{M}=\{(a', b)\mid a'\in\mathfrak{M}', b\in\mathfrak{M}\}$. We define the operation of (a', b) on $\mathfrak{M}$ as follow

$$x(a', b')=(xa')b, \text{ for all } x\in\mathfrak{M}.$$

It is clear that we have defined an endomorphism of $\mathfrak{M}$ as left $\Delta_1\otimes\Delta_2$-module. We denote it by $a'\underline{\otimes}b$. Being endomorphisms, $a'\underline{\otimes}b+c'\underline{\otimes}d$ is the sum of $a'\underline{\otimes}b$ and $c'\underline{\otimes}d$. Let $\mathfrak{M}'\underline{\otimes}\mathfrak{M}$ be an Abelian group generated by $\{a'\underline{\otimes}b\mid a'\in\mathfrak{M}', b\in\mathfrak{M}\}$. Then following equalities are clear:

(i) $(a'_1+a'_2)\underline{\otimes}b=a'_1\underline{\otimes}b+a'_2\underline{\otimes}b$;

(ii) $a'\underline{\otimes}(b_1+b_2)=a'\underline{\otimes}b_1+a'\underline{\otimes}b_2$;

(iii) $a'\alpha\underline{\otimes}b=a'\underline{\otimes}\alpha b$, $\alpha\in\Delta_1\otimes\Delta_2$.

Therefore, $\mathfrak{M}'\underline{\otimes}\mathfrak{M}$ is a product group of $\mathfrak{M}'$ and $\mathfrak{M}$ over $\Delta_1\otimes\Delta_2$. $\mathfrak{M}=\mathfrak{M}_1\otimes\mathfrak{M}_2$ is a free left $\Delta_1\otimes\Delta_2$-module (cf. [1]), so each element of $\mathfrak{M}'\underline{\otimes}\mathfrak{M}$ has the form: $\Sigma a'_i\underline{\otimes}b_i$, where b_i is a base of $\mathfrak{M}$. If $\Sigma a'_i\underline{\otimes}b_i=0$, we shall prove $a_i=0$ for all i. In fact, for any $x\in\mathfrak{M}$, $x(\Sigma a'_i\otimes b_i)=\Sigma(xa'_i)b_i=0$. Since every b_i is a base of $\mathfrak{M}$, xa'_i must be zero. But x is an arbitrary element of $\mathfrak{M}$, so $\mathfrak{M}a'_i=0$, that is $a'_i=0$.

If $\mathfrak{M}'\underline{\otimes}'\mathfrak{M}$ is another product group of $\mathfrak{M}'$ and $\mathfrak{M}$ over $\Delta_1\otimes\Delta_2$ we define a map φ from $\mathfrak{M}'\underline{\otimes}\mathfrak{M}$ to $\mathfrak{M}'\underline{\otimes}'\mathfrak{M}$:

$$\varphi(\Sigma x'_i\underline{\otimes}y_i)=\Sigma x'_i\underline{\otimes}y_i.$$

Without loss of generality, we assume each y_i is a base of $\mathfrak{M}$. If $\Sigma x'_i\underline{\otimes}y_i=0$, then $x'_i=0$, $\varphi(\Sigma x'_i\underline{\otimes}y_i)=\Sigma x'_i\underline{\otimes}'y_i=0$. So φ is well defined. Thus we have proved the following lemma:

Lemma 5. *The product group* $\mathfrak{M}'\underline{\otimes}\mathfrak{M}$ *defined as above is the tensor product of* $\mathfrak{M}'$ *and* $\mathfrak{M}$ *over* $\Delta_1\otimes\Delta_2$. *We denote it by* $\mathfrak{M}'\otimes\mathfrak{M}$.

Now, lets $s_1\otimes s_2$ be an element of $\mathfrak{S}_1\otimes\mathfrak{S}_2$, where $s_1\in\mathfrak{S}_1$, $s_2\in\mathfrak{S}_2$, then $\mathfrak{M}_1s_1$ and

$\mathfrak{M}_2 s_2$ are finite dimensional vector spaces. We assume their bases are $\{v_1, \cdots, v_m\}$ and $\{w_1, \cdots, w_n\}$ respectively. It is known that there are elements $u'_i \in \mathfrak{M}'_1$ $(i=1, \cdots, m)$ and elements $t'_j \in \mathfrak{M}'_2$ $(j=1, \cdots, n)$, such that s_1 can be represented as the mapping $x \to \sum_{i=1}^{n} (xu'_i) v_i$ for each x in $\mathfrak{M}_1$, and s_2 can be represented as the mapping $y \to \sum_{j=1}^{n} (yt'_j) w_j$ for each $y \in \mathfrak{M}_2$. Thus we have

$$(x \otimes y)(s_1 \otimes s_2) = xs_1 \otimes ys_2 = (\sum_i (xu'_i) v_i) \otimes (\sum_j (yt'_j) w_j) = \sum_{i,j} (xu'_i) v_i \otimes (yt'_j) w_j.$$

On the other hand

$$(x \otimes y)(\sum_{i,j} (u'_i \otimes t'_j) \otimes (v_i \otimes w_j)) = \sum_{i,j} (xu'_i \otimes yt'_j)(v_i \otimes w_j) = \sum_{i,j} (xu'_i) v_i \otimes (yt'_j) w_j.$$

This implies

$$(x \otimes y)(s_1 \otimes s_2) = (x \otimes y)(\sum_{i,j} (u'_i \otimes t'_j) \otimes (v_i \otimes w_j)), \text{ for each } x \otimes y \in \mathfrak{M}_1 \otimes \mathfrak{M}_2.$$

Therefore, $s_1 \otimes s_2 = \sum_{i,j} (u'_i \otimes t'_j) \otimes (v_i \otimes w_j) \in \mathfrak{M}' \otimes \mathfrak{M}$, that is $\mathfrak{S}_1 \otimes \mathfrak{S}_2 \subseteq \mathfrak{M}' \otimes \mathfrak{M}$. Conversely, if $(u' \otimes t') \otimes (v \otimes w) \in \mathfrak{M}' \otimes \mathfrak{M}$, where $u' \in \mathfrak{M}'_1$, $v \in \mathfrak{M}_1$, $t' \in \mathfrak{M}'_2$, $w \in \mathfrak{M}_2$, then we can show $(u' \otimes t') \otimes (v \otimes w) \in \mathfrak{S}_1 \otimes \mathfrak{S}_2$. In fact, let s_1 be the linear transformation of $(\Delta_1, \mathfrak{M}_1): x \to (xu')v$, s_2 be the linear transformation of $(\Delta_2, \mathfrak{M}_2): y \to (yt')w$, then $(u' \otimes t') \otimes (v \otimes w) = s_1 \otimes s_2 \in \mathfrak{S}_1 \otimes \mathfrak{S}_2$. This shows $\mathfrak{M}' \otimes \mathfrak{M} \subseteq \mathfrak{S}_1 \otimes \mathfrak{S}_2$, therefore $\mathfrak{M}' \otimes \mathfrak{M} = \mathfrak{S}_1 \otimes \mathfrak{S}_2$. Thus we have proved the following theorem:

Theorem 2. *Let $\mathfrak{A}_i (i=1, 2)$ be an irreducible algebra of linear transformations in a vector space $\mathfrak{M}_i$ over Φ, and Δ_i be its centralizer. Let $\mathfrak{S}_i$ be the nonzero socle of $\mathfrak{A}_i$. $\mathfrak{M}'_i$ is the dual vector space of $(\Delta_i, \mathfrak{M}_i)$ associated with $\mathfrak{A}_i$. $\mathfrak{M}' = \mathfrak{M}'_1 \otimes_\Phi \mathfrak{M}'_2$, $\mathfrak{M} = \mathfrak{M}_1 \otimes_\Phi \mathfrak{M}_2$ and $\mathfrak{M}' \otimes \mathfrak{M}$ is the tensor product of $\mathfrak{M}'$ and $\mathfrak{M}$ over $\Delta_1 \otimes \Delta_2$. Then $\mathfrak{M}' \otimes \mathfrak{M} = \mathfrak{S}_1 \otimes_\Phi \mathfrak{S}_2$.*

Using theorem 2, we are going to obtain a representation of one-sided ideals of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$.

Assume $\mathfrak{N}$ be a $\Delta_1 \otimes \Delta_2$ submodule of $\mathfrak{M}$, consider $\mathfrak{M}' \otimes \mathfrak{N}$. For any element $\sum_i a'_i \otimes b_i$ of $\mathfrak{M}' \otimes \mathfrak{N}$ and any element $\sum_j c'_j \otimes d_j$ of $\mathfrak{M}' \otimes \mathfrak{M}$,

$$(\sum_j c'_j \otimes d_j)(\sum_i a'_i \otimes b_i) = \sum_{i,j} c'_j (d_j a'_i) \otimes b_i \in \mathfrak{M}' \otimes \mathfrak{N}.$$

This shows $\mathfrak{M}' \otimes \mathfrak{N}$ is a left ideal of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$. On the other hand, assume L is a left ideai of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$, by Lemma 2, $L = (\mathfrak{S}_1 \otimes \mathfrak{S}_2) L = (\mathfrak{M}' \otimes \mathfrak{M}) L$. It is easy to verify that $(\mathfrak{M}' \otimes \mathfrak{M}) L = \mathfrak{M}' \otimes \mathfrak{M} L$, Clearly, $\mathfrak{M} L$ is a left $\Delta_1 \otimes \Delta_2$-submodule of $\mathfrak{M}$. Therefore, $L = \mathfrak{M}' \otimes \mathfrak{N}$, where $\mathfrak{N} = \mathfrak{M} L$ is a left $\Delta_1 \otimes \Delta_2$-submodule of $\mathfrak{M}$. Similarly, we can show that every right ideal of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ has the form $\mathfrak{N}' \otimes \mathfrak{M}$, where $\mathfrak{N}'$ is a right $\Delta_1 \otimes \Delta_2$-submodule of $\mathfrak{M}'$. So, the next theorem is true.

Theorem 3. *Let $\mathfrak{M}_i$, $\mathfrak{M}'_i$, $\mathfrak{A}_i$, $\mathfrak{S}_i$, Δ_i and $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ be as in Theorem 2. Then every left (right) ideal of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ has the form $\mathfrak{M}' \otimes \mathfrak{N}$ $(\mathfrak{N}' \otimes \mathfrak{M})$, where $\mathfrak{N}(\mathfrak{N}')$ is a $\Delta_1 \otimes \Delta_2$-submodule of $\mathfrak{M}(\mathfrak{M}')$.*

III. One-sided ideals of $\Delta_1 \otimes \Delta_2$

In this section, we shall study the structures of lattices of one-sided ideals in

$\Delta_1 \otimes \Delta_2$. We have shown that $\mathfrak{M}$ is a right $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-module and $\mathfrak{M}'$ is a left $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-module. Let $(\mathfrak{M}, \mathfrak{M}')$ be the product set of $\mathfrak{M}$ and $\mathfrak{M}'$. We define the left action of (a, b') on $\mathfrak{M}$, where $(a, b') \in (\mathfrak{M}, \mathfrak{M}')$, as follows

$$(a, b')x = (ab')x, \text{ for each } x \in \mathfrak{M}.$$

It is easy to see that we have defined a right $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-module endomorphism of $\mathfrak{M}$. We denote the Abelian group generated by such endomorphisms as $\mathfrak{M} \underline{\otimes} \mathfrak{M}'$. In fact, $\mathfrak{M} \underline{\otimes} \mathfrak{M}'$ is a tensor product of $\mathfrak{M}$ and $\mathfrak{M}'$ over $\mathfrak{S}_1 \otimes \mathfrak{S}_2$. It may be verified directly. We simply denote it by $\mathfrak{M} \odot \mathfrak{M}'$.

We observe that $\Delta_1 \otimes \Delta_2$ is the centralizer of right $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-module $\mathfrak{M}$, then $\mathfrak{M} \otimes \mathfrak{M}' \subseteq \Delta_1 \otimes \Delta_2$. Moreover, if $\sum \alpha_i \otimes \beta_i \in \Delta_1 \otimes \Delta_2$, where $\alpha_i \in \Delta_1$, $\beta_i \in \Delta_2$, for every i, we can find an element $a \otimes b'$ in $\mathfrak{M} \otimes \mathfrak{M}'$ such that $a \otimes b' = \alpha_i \otimes \beta_i$. In fact, since $\mathfrak{M}'_i$ is a dual vector space of $\mathfrak{M}$, we can find $x \in \mathfrak{M}_1$, $x' \in \mathfrak{M}'_1$ such that $xx' = \alpha_i$, and $y \in \mathfrak{M}_2$, $y' \in \mathfrak{M}'_2$, $yy' = \beta_i$, then $\alpha_i \otimes \beta_i = (x \otimes y) \otimes (x' \otimes y') \in \mathfrak{M} \otimes \mathfrak{M}'$. Thus we have proved the following theorem:

Theorem 4. *$\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{M} \otimes \mathfrak{M}'$ and $\Delta_1 \otimes \Delta_2$ being as above, we have* $\mathfrak{M} \otimes \mathfrak{M}' = \Delta_1 \otimes \Delta_2$

Using the same method as we have used, we can prove the next theorem:

Theorem 5. *Let $\mathfrak{A}_i (i=1, 2)$ be an irreducible algebra of linear transformations in a vector space $\mathfrak{M}_i$ over a field Φ, $\mathfrak{S}_i$ be the nonzero socle of $\mathfrak{A}_i$, Δ_i be the centralizer of $\mathfrak{M}_i$ as right $\mathfrak{A}_i$-module, $\mathfrak{M}'_i$ be the dual vector space associated with $\mathfrak{A}_i$, $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$ and $\mathfrak{M}' = \mathfrak{M}'_1 \otimes \mathfrak{M}'_2$. Then every right (left) ideal $T(L)$ of $\Delta_1 \otimes \Delta_2$ has the form $\mathfrak{N} \otimes \mathfrak{M}' (\mathfrak{M} \otimes \mathfrak{N}')$, where $\mathfrak{N} (\mathfrak{N}')$ is the $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-submodule of $\mathfrak{M} (\mathfrak{M}')$, Moreover $\mathfrak{N} = T\mathfrak{M} (\mathfrak{N}' = \mathfrak{M}'L)$.*

The well known Azumaya-Nakayama theorem can be regarded as a corollary of theorem 5.

Corollary. *The lattice of right ideals of $\Delta_1 \otimes \Delta_2$ is isomorphic to the lattice of right $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ (or $\mathfrak{A}_1 \otimes \mathfrak{A}_2$)-submodules of $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$.*

IV. Two-sided ideals

Now, we shall study structure of lattices of two-sided ideals of $\Delta_1 \otimes \Delta_2$ and of $\mathfrak{S}_1 \otimes \mathfrak{S}_2$.

Lemma 6. *If $\mathfrak{N}$ is a right $\mathfrak{S}_1 \otimes \mathfrak{S}_2$-submodule of $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$, then*

$$\mathfrak{N}(\mathfrak{S}_1 \otimes \mathfrak{S}_2) = \mathfrak{N}.$$

Proof It is enough to show $\mathfrak{N} \subseteq \mathfrak{N}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$. Let $n \in \mathfrak{N}$, $n = \sum_1^m x_i \otimes y_i$ where $x_i \in \mathfrak{M}_1$, $y_i \in \mathfrak{M}_2$. Then $\{x_i\}_{i=1,\dots,m}$ span a finite dimensional vector subspace of $(\Delta_1, \mathfrak{M}_1)$ and $\{y_i\}_{i=1,\dots,m}$ span a finite dimensional vector subspace of $(\Delta_2, \mathfrak{M}_2)$. Since $\mathfrak{S}_1$ is a dense ring of linear transformations in $(\Delta_1, \mathfrak{M}_1)$, we can find $s_1 \in \mathfrak{S}_1$ such that $x_i s_1 = x_i$ for $i = 1, \cdots, m$. Similarly, we can find $s_2 \in \mathfrak{S}_2$ such that $y_i s_2 = y_i$. Then $(\sum x_i \otimes y_i)(s_1 \otimes s_2) = \sum x_i \otimes y_i$, This implies $n = n(s_1 \otimes s_2) \in \mathfrak{N}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$.

Lemma 7. *Let $\mathfrak{N}$ be a $\mathfrak{S}_1\otimes\mathfrak{S}_2$-submodule of $\mathfrak{M}=\mathfrak{M}_1\otimes\mathfrak{M}_2$, $\mathfrak{N}'$ be a $(\mathfrak{S}_1\otimes\mathfrak{S}_2, \Delta_1\otimes\Delta_2)$-bisubmodule of $\mathfrak{M}'=\mathfrak{M}_1'\otimes\mathfrak{M}_2'$, and $\mathfrak{K}$ be $\Delta_1\otimes\Delta_2$-submodule of $\mathfrak{M}$. Then we have $(\mathfrak{N}\otimes\mathfrak{N}')\mathfrak{K}=\mathfrak{N}(\mathfrak{N}'\otimes\mathfrak{K})$.*

Proof From the definition, we can verify that $(\mathfrak{N}\otimes\mathfrak{N}')\mathfrak{K}=(\mathfrak{N}\mathfrak{N}')\mathfrak{K}=\mathfrak{N}(\mathfrak{N}'\otimes\mathfrak{K})$.

Theorem 6. *Let $\mathfrak{M}_i$, $\mathfrak{M}_i'$, $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{M}\otimes\mathfrak{M}'$, $\mathfrak{S}_i$, Δ_i be as before. Let I be a two-sided ideal of $\Delta_1\otimes\Delta_2$, then $I=\mathfrak{N}\otimes\mathfrak{M}'$, where $\mathfrak{N}$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodule of $\mathfrak{M}$, Moreover, the lattice of two-sided ideals of $\Delta_1\otimes\Delta_2$ is isomophic to the lattice of $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodules of $\mathfrak{M}$ under the correspondence Λ: $I\to I\mathfrak{M}$.*

Proof First, we observe that if $\mathfrak{N}$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-sub-module of $\mathfrak{M}$, then $\mathfrak{N}\otimes\mathfrak{M}'$ is a two-sided ideal of $\Delta_1\otimes\Delta_2$. Conversely, if I is a two-sided ideal of $\Delta_1\otimes\Delta_2$, then $I\mathfrak{M}$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodule of $\mathfrak{M}$. I, as a right ideal of $\Delta_1\otimes\Delta_2$, has the form $I=\mathfrak{N}\otimes\mathfrak{M}'$, where $\mathfrak{N}$ is a $\mathfrak{S}_1\otimes\mathfrak{S}_2$-submodule of $\mathfrak{M}$. By the Theorem 5, $\mathfrak{N}=I\mathfrak{M}$, so $\mathfrak{N}$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-submodule of $\mathfrak{M}$. Then $I=\mathfrak{N}\otimes\mathfrak{M}'=I\mathfrak{M}\otimes\mathfrak{M}'$. Assume Γ is the map: $\mathfrak{N}\to\mathfrak{N}\otimes\mathfrak{M}'$, from the lattice of bisubmodules of $\mathfrak{M}$ to the lattice of two-sided ideals of $\Delta_1\otimes\Delta_2$, then it is clear that $\Gamma\Lambda(I)=I$ for every two-sided ideal I of $\Delta_1\otimes\Delta_2$. Conversely, $\Lambda\Gamma(\mathfrak{N})=\Lambda(\mathfrak{N}\otimes\mathfrak{M}')=(\mathfrak{N}\otimes\mathfrak{M}')\mathfrak{M}=\mathfrak{N}(\mathfrak{M}'\otimes\mathfrak{M})=\mathfrak{N}(\mathfrak{S}_1\otimes\mathfrak{S}_2)$ $=\mathfrak{N}$. Therefore, we have proved that Λ is a lattice isomophism.

Theorem 7. *Let $\mathfrak{M}_i$, $\mathfrak{M}_i'$, $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{M}'\otimes\mathfrak{M}$, $\mathfrak{S}_i$ and Δ_i be as before. Let I be a two-sided ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$. Then I has the form $I=\mathfrak{M}'\otimes\mathfrak{N}$, where $\mathfrak{N}$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodule of $\mathfrak{M}$. Moreover, the lattice of two-sided ideals of $\mathfrak{S}_1\otimes\mathfrak{S}_2$ is isomophic to the lattice of $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodules of $\mathfrak{M}$ under the correspondence Λ: $I\to\mathfrak{M}I$.*

Proof If I is a two-sided ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$, then $\mathfrak{M}I$ is a bisubmodule of $\mathfrak{M}$. Conversely. if $\mathfrak{N}$ is a bisubmodule of $\mathfrak{M}$, then $\mathfrak{M}'\otimes\mathfrak{N}$ is a two-sided ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$. As a left ideal of $\mathfrak{S}_1\otimes\mathfrak{S}_2$, by Theorem 3, I has the form: $I=\mathfrak{M}'\otimes\mathfrak{N}$, where $\mathfrak{N}=\mathfrak{M}I$ is a $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodule of $\mathfrak{M}$. Define Γ to be the map: $\mathfrak{N}\to\mathfrak{M}'\otimes\mathfrak{N}$, then Γ is a map from the lattice of $(\Delta_1\otimes\Delta_2, \mathfrak{S}_1\otimes\mathfrak{S}_2)$-bisubmodules of $\mathfrak{M}$ to the lattice of two-sided ideals of $\mathfrak{S}_1\otimes\mathfrak{S}_2$. It is easy to see.

$$\Gamma\Lambda(I)=\Gamma(\mathfrak{M}I)=\mathfrak{M}'\otimes\mathfrak{M}I=I,$$

$$\Lambda\Gamma(\mathfrak{N})=\Lambda(\mathfrak{M}'\otimes\mathfrak{N})=\mathfrak{M}(\mathfrak{M}'\otimes\mathfrak{N})=(\mathfrak{M}\otimes\mathfrak{M}')\mathfrak{N}=(\Delta_1\otimes\Delta_2)\mathfrak{N}=\mathfrak{N}.$$

Thus Λ is a lattice isomophism.

As a consequence of Theorem 6 and Theorem 7, we have the following:

Theorem 8. *The lattice of two-sided ideals of $\mathfrak{S}_1\otimes\mathfrak{S}_2$ is isomophic to the lattice of two-sided ideals of $\Delta_1\otimes\Delta_2$.*

Corollary. *$\mathfrak{S}_1\otimes\mathfrak{S}_2$ is a simple algebra iff $\Delta_1\otimes\Delta_2$ is a simple algebra* (see [3]).

Reference

[1] Jacobson, N., Structure of rings, *Amer. Math soci. Colloq*, Pub., 137 (1956).

[2] Xu Yonghua, On the theory of rings which are isomophic to rings of full linear transformations (1), *Acta of Math. Sinica*, **22**: 2 (1980).

[3] Xu Yonghua, On the structure of tensor product of primitive algebras (to appear.).