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# APPROXIMATION THEOREMS BASED ON RANDOM PARTITIONS FOR STOCHASTIC DIFFERENTIAL EQUATION AND THEIR APPLICATIONS

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**Abstract**

This kind of problems is discussed: When we use certain smooth approximations of the Brownian motion $W$ as substitutes for it in stochastic line integral and stochastic differential equation, do these resultant integrals and solutions converge to the original one? The corresponding approximation theoroms for two kinds of apprximations are proved, which are wider than those discussed in [1]. Some limit theorems about stochastic line integral and solutions of stochastic differential equations with respect to random walks are obtained by using the idea of "embeding a random walk into the Brownian motion" first proposed by A. V. Skorohod[11]. It seems to be remarkable that the method used here is not only effective for the one dimensional case, but also for the multi-dimensional case.

**Introduction.**

Considering the irregular character of Brownian motion sample functions $w$, we hope to use a certain smooth approximation of $W$ as a substitute for $W$ in the integral and in the equation. However, do these resultant solutions converge to original one? This meaningful problem both in theory and in application has been discussed by several authors (cf. [1—8]), in particular, recently Prof. N. Ikeda and Prof. S. Watanabe in their excellent book[1] discussed the problem for a wider kind of approximation of $W$ by a unified way.

On account of all approximations for W used in [1] were based on deterministic sequence of partitions for the time space $[0, \infty)$, the purpose of this paper is to extend the kind of approximations discussed in [1] (§ 1), and to prove corresponding approximation theorems (§ 2), then to obtain some limit theorems about stochastic line integrals and solutions of stochastic differential equations with respect to random walks by use of an idea proposed by A. V. Skorohod in [11]. It seems to be remarkable that the method used here is not only effective for the one dimentional case but also for the muiti-dimentional case.

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## 1. Assumptions and Definitions.

**Assumption 1.1** $\{\sigma_n^{(l)}\}$ $(l=1, 2, \cdots, n=0, 1, 2, \cdots)$ *be a sequence of stopping times defined on* $(W_0^r, \mathscr{B}(W_0^r), (\mathscr{B}_t(W_0^r))_{t\geq 0}, P)^*$, *such that*

*for each* $l=1, 2, \cdots$, $\sigma_0^{(l)}\equiv 0$, $\sigma_1^{(l)}>0$ *and* $\sigma_n^{(l)}(w)=\sigma_{n-1}^{(l)}(w)+\sigma_1^{(l)}(\theta_{\sigma_{n-1}^{(l)}}w)$, (c.1)

$$\lim_{l\uparrow\infty} E(\sigma_1^{(l)})=0. \tag{c.2}$$

$$E[(\sigma_1^{(l)3}]\leqslant c[E(\sigma_1^{(l)})]^{3**}, \tag{c.3}$$

$$E\left[\left(\int_0^{\sigma_1^{(l)}}|w(s)|ds\right)^2\right]\leqslant c[E(\sigma_1^{(l)})]^3 \text{ and } E\left[\int_0^{\sigma_1^{(l)}}|w(s)|^2ds\right]\leqslant c[E(\sigma_1^{(l)})]^2, \tag{c.4}$$

$$E[(\sigma_1^{(l)})^4\leqslant c[E(\sigma_1^{(1)})]^4; \tag{c.5}$$

**Assumption1.2.** *Suppose that on* $(W_0^r, \mathscr{B}(W_0^r), (\mathscr{B}_t(W_0^r)), P)$

*for every* $l=1, 2, \cdots$, $\{\sigma_n^{l,i}\}_{n\geq 0}$ $(i=1, 2, \cdots, r)$ *are r sequences of stopping times such that* $\{\sigma_n^{l,i}\}_{n\geq 0}$ *only depend on* $w^i$ *and* $\sigma_0^{l,i}\equiv 0$,

$$\sigma_n^{l,i}(w)=\sigma_{n-1}^{l,i}(w)+\sigma_1^{l,i}(\theta_{\sigma_{n-1}^{l,i}}w^i) \text{ for } n>1. \tag{c.1)'$$

$$\sigma_1^{l,i}>0 \text{ and } E(\sigma_1^{l,1})=E(\sigma_1^{l,2})=\cdots=E(\sigma_1^{l,r})=\delta(l)\to 0 \text{ as } l\uparrow\infty. \tag{c.2)'$$

$$E[(\sigma_1^{l,i})^4]\leqslant c[E(\sigma_1^{l,i})]^{4**}. \tag{c.3)'$$

$$E\left[\left(\int_0^{\sigma_1^{l,i}}|w^i(s)|ds\right)^2\right]\leqslant c[E(\sigma_1^{l,i})]^3. \tag{c.4)'$$

$$E\left[\int_0^{\sigma_1^{l,i}}|w(s)|^2ds\right]\leqslant c[E(\sigma_1^{l,i})]^2, \tag{c.5)'$$

*and hence by writing* $\tau_n^l=(\sigma_n^{l,1}, \cdots, \sigma_n^{l,r})$ *we have*

$$\tau_n^l(w)=\tau_{n-1}^l(w)+\tau_1^l(\theta_{\tau_{n-1}^l}w) \text{ for } n>1 \tag{c.6)'$$

*where* $(\theta_{(t_1,\cdots,t_r)}w)(s)=((\theta_{t_1}w^1)(s), \cdots, (\theta_{t_r}w^r)(s))$.

**Definition 1.1** *A family* $\{B_l(t, w)=(B_l^1(t, w), \cdots, B_l^r(t, w))\}_{l\geq 1}$ *of r-dimensional continuous processes defined on* $(W_0^r, P)$ *is said to be an approximation of I-class of the Wiener process*$(W(t)=(W^1(t), \cdots, W^r(t))$, *if*

(I) *for every* $w\in W_0^r$, $t\mapsto B_l(t, w)$ *is piecewise continuious differentiable.*

(II) $B_l(0, w)$ *is* $\mathscr{B}_{\sigma_1^{(l)}}$*-measurable,*

(III) $B_l(t+\sigma_k^{(l)}, w)=B_l(t, \theta_{\sigma_k^{(l)}}w)+w(\sigma_k^{(l)})$ *for every* $k=1, 2, \cdots$, *and* $t\geqslant 0$, $w\in W_0^r$.

(IV) $E[B_l^i(0)]=0$, *for* $i=1, 2, \cdots, r$,

(V) $E[|B_l^i(0, w)|^6]\leqslant c[E(\sigma_1^{(l)})]^3$ *for* $i=1, 2, \cdots, r$;

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* Except special explanation, the notations of this paper are the same as in [1].

** All positive constants $c, c', c_1, c_2, \cdots, d_1\cdots$ in this paper are independent of $l$.

(VI) $E\left[\left(\int_0^{\sigma_1^{(l)}} |\dot{B}_l^i(s)|ds\right)^6\right] \leqslant c[E(\sigma_1^{(l)})]^3$ *for* $i=1, 2, \cdots, r$

*where*

$$\dot{B}_l^i(t, w) = \frac{d}{dt} B_l^i(t, w) \text{ for } i=1, 2, \cdots, r. \tag{1.1}$$

As an example, it is easy to see that the piecewise linear approximation satisfies all conditions (I)—(VI).

**Definition 1.2.** *A family* $\{B_l(t, w) = (B_l^1(t, w), \cdots, B_l^r(t, w))\}_{l\geqslant 1}$ *of* $r$*-dimensional continuous processes defined on* $(W_0^r, P)$ *is said to be an approximation of D-class of the Wiener process* $(W(t) = (W^1(t), \cdots, W^r(t))$, *if*

(I) *for every* $w \in W_0^r$, $t \mapsto B_l(t, w)$ *is piecewise continuious differentiable.*

(II) $B_l(0, w)$ *is* $\mathscr{B}_{\tau_1^l}$*-measurable, where*

$$\mathscr{B}_{\tau_k^l} = \mathscr{B}_{\sigma_k^{l,1}}(W^1) \otimes \mathscr{B}_{\sigma_k^{l,2}}(W^2) \otimes \cdots \otimes \mathscr{B}_{\sigma_k^{l,r}}(W^r),$$

(III) $B(B_l(t, w)) = B_l\left(t - \frac{k}{l}, \theta_{\tau_k^l} w\right) + B(\tau_k^l)$ *for every* $k=1, 2, \cdots$, $t \geqslant \frac{k}{l}$ *and* $w \in W_0^r$.

(IV) $E(B_l^i(0)) = 0 \ (i=1, 2, \cdots, r)$.

(V) $E(|B_l^i(0)|^6) \leqslant c[\delta(l)^3] \ (i=1, 2, \cdots, r)$,

(VI) $E\left[\left(\int_0^{\frac{1}{l}} |\dot{B}_l^i(s)|ds\right)^6\right] \leqslant c[\delta(l)^3] \ (i=1, 2, \cdots, r)$, *where*

$$\dot{B}_l^i(t) = \frac{d}{dt} B_l^i(t) \text{ for } i=1, 2, \cdots, r.$$

**Definition 1.3.** *Let* $\alpha$ *be a differential 1-form on* $R^r$ *given by*

$$\alpha = \sum_{i=1}^r \alpha_i(x) dx^i \tag{1.2}$$

*where* $\alpha_i(x) \in C^2(R^r)$ *and* $\alpha_i^{(1)}(x) \in C_b^1(R^r)$ *for* $i=1, 2, \cdots, r$, *and* let $\{B_l\}_{l\geqslant 1}$ *be an approximation of I-class of D-class of* $W$. *We define stochastic line integral* $A(t, \alpha; w)$ *and* $A(t, \alpha; B_1)$ *respectively by*

$$A(t, \alpha; w) = \int_{w[0,t]} \alpha = \sum_{i=1}^r \int_0^t \alpha_i(w(s)) \circ dw^i(s) \tag{1.3}$$

and

$$A(t, \alpha; B_1) = \int_{B_l[0,t]} \alpha = \sum_{i=1}^r \int_0^t \alpha_i(B_l(s)) \dot{B}_l^i(s) ds. \tag{1.4}$$

**Assumption 1.3.** (i) *For the approximation* $B_l$ *of I-class, there is a skew-symmetric* $r \times r$*-matrix* $(S_{ij})$ *such that*

$$\lim_{l \uparrow \infty} S_{ij}^{(l)}(l) = S_{ij} \text{ for } i, j=1, 2, \cdots, r, \tag{1.5}$$

*where*

$$S_{ij}^{(l)}(k) = \frac{1}{E(\sigma_k^{(l)})} E\left[\frac{1}{2}\int_0^{\sigma_k^{(l)}} [B_l^i(s)\dot{B}_l^j(s) - \dot{B}_l^i(s) B_l^j(s)] ds\right],$$

$$C_{ij}^l(k) = \frac{1}{E(\sigma_k^{(l)})} E\left[\int_0^{\sigma_k^{(l)}} \dot{B}_l^i(s) [B_l^j(\sigma_k^{(l)}, w) - B_l^j(s, w)\right] ds \tag{1.6}$$

*for* $i, j=1, 2, \cdots, r$, $l=1, 2, \cdots$, $k=1, 2, \cdots$; *and* (ii) *For the approximation* $\overline{B}_l$ *of*

*D-class, there is a skew-symmetric $r\times r$-matrix $(\bar{S}_{ij})$ such that*

$$\lim_{l\uparrow\infty}\bar{S}^{(l)}_{ij}(l)=\bar{S}_{ij}\ \textit{for } i,\ j=1,\ 2,\ \cdots,\ r, \tag{1.5)'$$

*where*

$$\left.\begin{aligned}\bar{S}^{(l)}_{ij}(k)&=(k\delta(l))^{-1}E\Big[\frac{1}{2}\int_0^{k\delta(l)}(\bar{B}^i_l(s)\dot{\bar{B}}^j_l(s)-\dot{\bar{B}}^i_l(s)\bar{B}^j_l(s))ds\Big]\\ \bar{C}^{(l)}_{ij}(k)&=(k\delta(l))^{-1}E\Big[\int_0^{k\delta(l)}\Big(\dot{\bar{B}}^i_l(s)\Big(\bar{B}^j_l\Big(\frac{k}{l}\Big)-\bar{B}^j_l(s)\Big)ds\Big]\end{aligned}\right\}\ \textit{for } k>1 \tag{1.6)'$$

## 2. Approximation for stochrstic line integral.

**Lemma 1.** *Let $\{K(l)\}_{l\geq1}\subset Z^+$ be a sequence such that $K(l)\uparrow\infty$ as $l\uparrow\infty$. Then under Assumption 1.3*

$$\lim_{\uparrow\infty}C^l_{ij}(k(l))=C_{ij}(\lim_{l\uparrow\infty}\bar{C}^l_{ij}(K(l))=\bar{C}_{ij})\ \text{for } i,\ j=1,\ 2,\ \cdots,\ r \tag{2.1}$$

*where*

$$C_{ij}=S_{ij}+\frac{1}{2}\delta_{ij}\Big(\bar{C}_{ij}=\bar{S}_{ij}+\frac{1}{2}\delta_{ij}\Big)\ \text{for } i,\ j=1,\ 2,\ \cdots,\ r. \tag{2.2}$$

*Proof* It is similar to that for lamma VI-7.1 of [1].

**Theorem 1.** *Let $\{B_l(t,\ w)\}(l=1,\ 2,\ \cdots)$ be an approximation of I-class of $W$ satisfing Assumption 1.3, and $\{n(l)\}_{l\geq1}\subset Z^+$ be a increasing sequence with*

$$\lim_{l\uparrow\infty}n(l)^4E(\sigma^{(l)}_1)=0.$$

*Then for any increasing sequence $\{N(l)\}_{l\geq1}\subset Z^+$ which satisfies the following conditions*

$$\lim_{l\uparrow\infty}N(l)=\infty,\quad N(l)\equiv0(\operatorname{mod} n(l)),\quad 0<\overline{\lim_{l\uparrow\infty}}N(l)E(\sigma^{(l)}_1)<\infty, \tag{2.3}$$

*the equality*

$$\lim_{l\uparrow\infty}E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}}|A(t,\ \alpha,\ B_l)-A(t,\ \alpha;\ w)-\int_0^t\sum_{i,j=1}^r S_{ij}\partial_i\alpha_j(w(s))ds|^2]=0 \tag{2.4}$$

*holds, where $\partial_i\alpha_j=\dfrac{\partial}{\partial x^i}\alpha_j\ (i,\ j=1,\ 2,\ \cdots,\ r)$.*

*Proof* Just the same as the proof of theorem VI-7.1 of [1], it is enough to show that

$$\begin{aligned}\lim_{l\uparrow\infty}E\Big[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}}\Big|\int_0^t u(B_l(s,\ w))\dot{B}^j_l(s,\ w)ds-\int_0^t u(w(s))\circ dw^j(s)\\ -\int_0^t\Big(\sum_{i=1}^r S_{ij}u_i(w(s))\Big)ds\Big|^2\Big]=0\end{aligned} \tag{2.5}$$

for any $T>0$, $j=1,\ 2,\ \cdots,\ r$. Here $u\in C^2(R^r)$. and all of its partial derivatives $u_i(x)=\dfrac{\partial}{\partial x^i}u(x)$ are bounded. Now set

$$\left.\begin{aligned}&\hat{\sigma}^{(l)}_k=\sigma^{(l)}_{kn(l)}\ \text{for } k=0,\ 1,\ 2,\ \cdots,\ l=1,\ 2,\ \cdots,\\ &\left.\begin{aligned}&[s]^+_l(w)=\hat{\sigma}^{(l)}_{k+1}(w)\\ &[s]^-_l(w)=\hat{\sigma}^{(l)}_k(w)\end{aligned}\right\}\ \text{if } \hat{\sigma}^{(l)}_k(w)\leq s<\sigma^{(l)}_{k+1}(w)\ \text{for } k=0,\ 1,\ \cdots,\\ &m_l(t,\ w)=k;\ [t]^-_l(w)=\hat{\sigma}^{(l)}_k,\ \text{for } l=1,\ 2,\ \cdots.\end{aligned}\right. \tag{2.6}$$

Then, similar to (7.21) of [1], we have the following for $1\leq j\leq r$

$$\int_0^t u(B_l(s,w))dB_l^j(s,w)-\int_0^t u(w(s))\circ dw^j(s)-\int_0^t\sum_{i=1}^r S_{ij}u_i(w(s))ds$$
$$=-[u(B_l(t,w))-u(B_l([t]_l^-,w))][B_l^j([t]_l^+,w)-B_l^j(t,w)]$$
$$-[u(B_l([t]_l^-,w))-u(w([t]_l^-))][B_l^j([t]_l^+,w)-B_l^j(t;w)]$$
$$-u(w([t]_l^-))[B_l^j([t]_l^+,w)-B_l^j(t,w)]$$
$$+[u(B_l([t]_l^-,w))-u(w([t]_l^-))][B_l^j([t]_l^+,w)-B_l^j([t]_l^-,w)]$$
$$+u(w([t]_l^-))([B_l^j([t]_l^+,w)-B_l^j([t]_l^-,w)]$$
$$+\sum_{i=1}^r\int_{[t]_l^-}^t u_i(B_l(B_l(s)))\dot{B}_l^i(s)[B_l^j([t]_l^+)-B_l^j(s)]ds$$
$$-u(w([t]_l^-))[w^j(t)-w^j([t]_l^-)]$$
$$+u(w([t]_l^-))[w^j(t)-w^j([t]_l^-)]-\int_{[t]_l^-}^t u(w(s))dw^j(s)$$
$$-\int_{[t]_l^-}^t\sum_{i=1}^r C_{ij}u_i(w(s))ds$$
$$+\sum_{k=0}^{m_l(t)-1}[u(B_l(\hat{\sigma}_k^{(l)}))-u(w(\hat{\sigma}_k^{(l)}))][B_l^j(\hat{\sigma}_{k+1}^{(l)})-B_l^j(\hat{\sigma}_k^{(l)})]$$
$$+\sum_{k=0}^{m_l(t)-1}u(w(\hat{\sigma}_k^{(l)}))[B_l^j(\hat{\sigma}_{k+1}^{(l)})-w^j(\hat{\sigma}_{k+1}^{(l)})]$$
$$-\sum_{k=0}^{m_l(t)-1}u(w(\hat{\sigma}_k^{(l)}))[B_l^j(\hat{\sigma}_k^{(l)})-w^j(\hat{\sigma}_k^{(l)})]$$
$$+\sum_{k=0}^{m_l(t)-1}u((w(\hat{\sigma}_k^{(l)}))[w^j(\hat{\sigma}_{k+1}^{(l)})-w^j(\hat{\sigma}_k^{(l)})]-\int_0^{[t]_l^-}u(w(s))dw^j(s)$$
$$+\sum_{i=1}^r\sum_{k=0}^{m_l(t)-1}\int_{\hat{\sigma}_k^{(l)}}^{\hat{\sigma}_{k+1}^{(l)}}[u_i(B_l(s))-u_i(w(\hat{\sigma}_k^{(l)}))]\dot{B}_l^i(s)[B_l^j(\hat{\sigma}_{k+1}^{(l)})-B_l^j(s)]ds$$
$$+\sum_{i=1}^r\sum_{k=0}^{m_l(t)-1}\int_{\hat{\sigma}_k^{(l)}}^{\hat{\sigma}_{k+1}^{(l)}}u_i(w(\sigma_k^{(l)}))[\dot{B}_l^i(s)(B_l^j(\hat{\sigma}_{k+1}^{(l)})-B_l^j(s))-C_{ij}^l(n)]ds$$
$$+\sum_{i=1}^r\sum_{k=0}^{m_l(t)-1}u_i(w(\hat{\sigma}_k^{(l)}))[C_{ij}^{(l)}(n)-C_{ij}](\hat{\sigma}_{k+1}^{(l)}-\hat{\sigma}_k^{(l)})$$
$$+\sum_{i=1}^r\sum_{k=0}^{m_l(t)-1}\int_{\hat{\sigma}_k^{(l)}}^{\sigma_{k+1}^{(l)}}[u_i(w(\hat{\sigma}_k^{(l)}))-u_i(w(s))]ds\cdot C_{ij}$$
$$=I_1(t)+\cdots+I_5+\sum_{i=1}^r I_6^i(t)+I_7(t)+\cdots+I_{13}(t)+\sum_{i=1}^r I_{14}^i(t)+\cdots+\sum_{i=1}^r I_{17}^i(t). \tag{2.7}$$

In estimation of these terms, of course, we can make full use of those steps used in theorem VI-7.1 of [1]. In addition, if we use following estimates respectively

$$E\Big[\Big(\int_{\sigma_{k_1}^{(l)}}^{\sigma_{k_1'}^{(l)}}|\dot{B}_l^{i_1}(s)|ds\Big)^{p_1}\cdots\Big(\int_{\sigma_{k_m}^{(l)}}^{\sigma_{k_m'}^{(l)}}|\dot{B}^{i_m}(s)|ds\Big)^{p_m}\leqslant d_1\prod_{q=1}^m(k_q'-k_q)^{p_q}(E(\sigma_1^{(l)}))^{\frac{1}{2}p_q} \tag{2.8}$$

$$E[\sup_{0\leqslant t\leqslant\sigma_{N(l)}^{(l)}}(1+w([t]_l^-)^4]\leqslant d_2(1+N(l)^2E(\sigma_1^{(l)})^2) \tag{2.9}$$

$$E\Big[\int_{\hat{\sigma}_k^{(l)}}^{\hat{\sigma}_{k+1}^{(l)}}|u(w(\hat{\sigma}_k^{(l)}))-u(w(s))|^2ds\Big]\leqslant d_3n(l)^2(E(\sigma_1^{(l)}))^2 \tag{2.10}$$

$$E[(\sigma_{N(l)}^{(l)})^2]\leqslant d_4, \tag{2.11}$$

then it is easy to verify that

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I_j(t)|^2]\to 0 \text{ as } l\uparrow\infty \text{ for } j=1, 2, 4, 6, 10; \tag{2.8)'}$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I^i_{14}(t)|^2]\to 0 \text{ as } l\uparrow\infty \text{ for } i=1, 2, \cdots, r;$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I_j(t)|^2]\to 0 \text{ as } l\uparrow\infty \text{ for } j=3, 5, 7; \tag{2.9)'}$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I_8(t)+I_{13}(t)|^2]\to 0 \text{ as } l\uparrow\infty; \tag{2.10}$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I^i_{16}(t)|^2]\to 0 \text{ as } l\to\infty \text{ for } i=1, 2, \cdots, r \tag{2.11}$$

hold, and similarly as in [1], we have

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I_j(t)|^2]\to 0 \text{ as } l\uparrow\infty \text{ for } j=9, 11, 12. \tag{2.12}$$

To prove $E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |I^i_{15}(t)|^2]\to 0$ as $l\uparrow\infty$ for $i=1, 2, \cdots, r$, we first note that

$$\begin{aligned}
I_{15}(t) &= \sum_{k=0}^{m_l(t)-1} u_i(w(\hat\sigma^{(l)}_k))\int_{\sigma^{(l)}_{kn(l)+1}}^{\hat\sigma^{(l)}_{k+1}} [\dot B^i_l(s)(B^j_l(\hat\sigma^{(l)}_{k+1})-B^j_l(s))-C^{(l)}_{ij}(n(l)-1)]ds \\
&+\sum_{k=0}^{m_l(t)-1} u_i(w(\hat\sigma^{(l)}_k))\int_{\hat\sigma^{(l)}_{k+1}}^{\sigma^{(l)}_{kn(l)+1}} [\dot B^i_l(s)(B^j_l(\hat\sigma^{(l)}_{k+1})-B^j_l(s))-\widetilde C^{(l)}_{ij}(n(l))]ds \\
&+\sum_{k=0}^{m_j(t)-1} u_i(w(\sigma^{(l)}_k))\Big[\frac{1}{n(l)}(\hat\sigma^{(l)}_{k+1}-\sigma^{(l)}_{kn(l)}) \\
&+\frac{n(l)-1}{n(l)}(\sigma^{(l)}_{kn(l)}-\hat\sigma^{(l)}_k)\Big]\cdot[\widetilde C^{(l)}_{ij}(n)-\widetilde C^{(l)}_{ij}(n-1)] \\
&=J_1(t)+J_2(t)+J_3(t).
\end{aligned} \tag{2.13}$$

where $\widetilde C^{(l)}_{ij}(k)=\dfrac{1}{E(\sigma^{(l)}_1)}\Big[E\int_0^{\sigma^{(l)}_1} \dot B^i_l(s)(B^j_l(\sigma^{(l)}_k)-B^j_l(s))ds\Big]$. Because of

$$\frac{1}{n(l)}\widetilde C^{(l)}_{ij}(n(l))\to 0 \text{ as } l\uparrow\infty.$$

it is easy to show that

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |J_1(t)|^2]\leq c_1 n(l)^3 E(\sigma^{(l)}_1)\to 0 \text{ as } l\uparrow\infty.$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |J_2(t)|^2]\leq c_2\Big[\frac{1}{n(l)}+\frac{1}{n(l)}C^l_{ij}(n)\Big]\to 0 \text{ as } l\uparrow\infty. \tag{2.14}$$

$$E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |J_3(t)|^2]\leq c_3[C^l_{ij}(n)-C^l_{ij}(n(l)-1)]^2 n(l)^{-2}\to 0 \text{ as } l\uparrow\infty.$$

Thus we conclude the proof of Theorem 1. As an immediate consequence of the theorem, we have

**Corollary 1.1.** *Under The conditions of Theorem 1, we have*

$$\lim_{l\uparrow\infty} E[\sup_{0\leq t\leq\sigma^{(l)}_{N(l)}} |w(t)-B_l(t, w)|^2]=0. \tag{2.15}$$

**Corollary 1.2.** *If a sequence $\{N(l)\}_{l\geq1}$ satisfies the conditions of Theorem 1, and for any $T>0$*

$$\lim_{l\uparrow\infty} P\{\sigma^{(l)}_{N(l)}>T\}=1. \tag{2.16}$$

*then for any* $\varepsilon>0$

$$\lim_{l\uparrow\infty} P\Big\{\sup_{0\leqslant t\leqslant T\vee\sigma_{N(l)}^{\mathscr{D}}}\Big|A(t,\ \alpha;\ B_l)-A(t,\ \alpha;\ w)-\int_0^t\sum_{i=1}^r S_{ij}\partial_i\alpha_i(w(s))ds\Big|^2\geqslant\varepsilon\Big\}=0. \tag{2.17}$$

*Proof* From Theorem 1 and Corollary 1.1 this is obvious.

**Theorem 2.** *Let* $\overline{B}_l(t,\ w)$ $(l=1,\ 2,\ \cdots)$ *be an approximation of D-class of W satisfying Assmption* 1.3, *and* $\{n(l)\}_{l\geqslant1}\subset Z^+$ *be an increasing sequence with* $\lim\limits_{l\uparrow\infty} n(l)^4\delta(l)=0$. *Then for any increasing sequence* $\{N(l)\}_{l\geqslant1}\subset Z^+$ *which satisfies following conditions*

$$\lim_{l\uparrow\infty} N(l)=\infty,\quad N(l)\equiv0\pmod{n(l)},\ 0<\varlimsup_{l\uparrow\infty} N(l)\delta(l)<\infty. \tag{2.18}$$

*the equality*

$$\lim_{l\uparrow\infty} E\Big[\sup_{0\leqslant t\leqslant T}\Big|A(t,\ \alpha;\ \overline{B}_l)-A(t,\ \alpha;\ w)-\int_0^t\sum_{ij=1}^r \overline{S}_{ij}\partial_i\alpha_j(w(s))ds\Big|^2\Big]=0 \tag{2.19}$$

*holds, where* $\partial_i\alpha_j=\dfrac{\partial}{\partial x^i}\alpha_j(i,\ j=1,\ 2,\ \cdots,\ r)$.

*Proof* Just as in the proof of Theorem 1, we may achieve the proof of Thearem 2 provided we can show that for any given $T>0$ and $1\leqslant j\leqslant r$

$$\lim_{l\uparrow\infty} E\Big[\sup_{0\leqslant t\leqslant T}\Big|\int_0^t u(\overline{B}_l(s))\dot{\overline{B}}{}_l^j(s)ds-\int_0^t u(w(s))\circ dw^j(s)-\int_0^t\sum_{i=1}^r \overline{S}_{ij}u_i(w(s))ds\Big|^2\Big]=0 \tag{2.20}$$

where $u\in C^2(R^r)$ and all of its partial derivatives $u_i\in C_b^1(R^r)$. Let

$$\tilde{\delta}=n(l)\delta(l)\qquad \left.\begin{array}{l}[s]^+(\tilde{\delta})=(k+1)\tilde{\delta}\\ [s]^-(\tilde{\delta})=k\tilde{\delta}\end{array}\right\}\text{ if } k\tilde{\delta}\leqslant s<(k+1)\tilde{\delta} \tag{2.21}$$

$$m(t)=[t]^-(\tilde{\delta})/\tilde{\delta}\qquad \left.\begin{array}{l}[s]_l^+=\sigma_{(k+1)n(l)}^{l,j}\\ [s]_l^-=\sigma_{kn(l)}^{l,j}\end{array}\right\}\text{ if } \sigma_{kn(l)}^{l,j}\leqslant s<\sigma_{(k+1)n(l)}^{l,j}.$$

we have

$$\begin{aligned}
&\int_0^t u(\overline{B}_l(s))d\overline{B}_l^j(s)-\int_0^t u(w(s))\circ dw^j(s)-\int_0^t\sum_{i=1}^r \overline{S}_{ij}u_i(w(s))ds\\
&=-[u(\overline{B}_l(t))-u(\overline{B}_l([t]^-))][\overline{B}_l^j([t]^+)-\overline{B}_l^j(t)]\\
&\quad-[u(\overline{B}_l([t]^-))-u(w(\hat{\tau}_{m(t)}^l))][\overline{B}_l^j([t]^+)-\overline{B}_l^j(t)]\\
&\quad-u(w(\hat{\tau}_{m(t)}^l))[\overline{B}_l^j([t]^+)-\overline{B}_l^j(t)]\\
&\quad+[u(\overline{B}_l([t]^-))-u(w(\hat{\tau}_{m(t)}^l))][\overline{B}_l^j([t]^+)-\overline{B}_l^j([t]^-)]\\
&\quad+u(w(\hat{\tau}_{m(t)}^l))[\overline{B}_l^j[t]^+)-\overline{B}_l^j([t]^-)]\\
&\quad+\sum_{i=1}^r\int_{[t]^-}^t u_i(\overline{B}_l(s))\dot{\overline{B}}{}_l^i(s)[\overline{B}_l^j([t]^+)-\overline{B}_l^j(s)]ds\\
&\quad-u(w(\hat{\tau}_{m(t)}^l))[w^j(t)-w^j(\hat{\sigma}_{m(t)}^{l,j})]\\
&\quad+u(w\hat{\tau}_{m(t)}^j))[w^j(t)-w^j(\hat{\sigma}_{m(t)}^{l,j})]-\int_{[t]^-}^t u(w(s))dw^j(s)\\
&\quad-\int_{[t]^-}^t\sum_{i=1}^r \overline{C}_{ij}u_i(w(s))ds
\end{aligned}$$

$$+\sum_{k=0}^{m(t)-1}[u(B_l(k\tilde{\delta}))-u(w(\hat{\tau}_k^l))][B_l^j((k+1)\tilde{\delta})-B_l^j(k\tilde{\delta})]$$

$$+\sum_{k=0}^{m(t)-1}u(w(\hat{\tau}_k^l))[\bar{B}_l^j((k+1)\tilde{\delta})-w^j(\hat{\sigma}_{k+1}^{l,j})]$$

$$+\sum_{k=0}^{m(t)-1}u(w(\hat{\tau}_k^l))[\bar{B}_l^j(k\tilde{\delta})-w^j(\hat{\sigma}_k^{l,j})]$$

$$+\sum_{k=0}^{m(t)-1}u(w(\hat{\tau}_k^l))[w^j(\hat{\sigma}_{k+1}^{l,j})-w^j(\hat{\sigma}_k^{l,j})]-\int_0^{[t]^-}u(w(s))dw^j(s)$$

$$+\sum_{i=1}^{r}\sum_{k=0}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}[u_i(\bar{B}_l(s)-u_i(\bar{B}_l(k\tilde{\delta}))]\dot{\bar{B}}_l^i(s)[\bar{B}_l^j((k+1)\tilde{\delta})-\bar{B}_l^j(s)]ds$$

$$+\sum_{i=1}^{r}\sum_{k=0}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}u_i(\bar{B}_l(k\tilde{\delta}))[\dot{\bar{B}}_l^i(s)(\bar{B}_l^j((k+1)\tilde{\delta})-\bar{B}_l^j(s))-\bar{C}_{ij}^l(n)]ds$$

$$+\sum_{i=1}^{r}\sum_{k=0}^{m(t)-1}u_i(\bar{B}_l(k\tilde{\delta}))[\bar{C}_{ij}^l(n(l))-\bar{C}_{ij}]\tilde{\delta}$$

$$+\sum_{i=1}^{r}\sum_{k=0}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}[u_i(\bar{B}_l(k\tilde{\delta}))-u_i(w(s))]ds\cdot\bar{C}_{ij}$$

$$=I_1(t)+\cdots+I_5(t)+\sum_{i=1}^{r}I_6^i(t)+I_7(t)+\cdots+I_{13}(t)$$

$$+\sum_{i=1}^{r}I_{14}^i(t)+\sum_{i=1}^{r}I_{15}^i(t)+\sum_{i=1}^{r}I_{16}^i(t)+\sum_{i=1}^{r}I_{17}^i(t). \tag{2.22}$$

where $\hat{\tau}_k^l=\tau_{kn(l)}^l$. Among the estimates of these terms, the places where the estimates are different from those of theorem VI-7.1 of [1] are only in $I_7$, $I_8+I_{13}$ and $I_{17}$, but by use of the estimaties

$$E[\sup_{0\le k\le m(T)}(1+|w(\hat{\tau}_k^l)|^4)]\le d_5 \tag{2.23}$$

$$E\left[\int_{\hat{\sigma}_k^{l,j}}^{\hat{\sigma}_{k+1}^{l,j}}|w^i(\hat{\sigma}_k^{l,j})-w^i(s)|^2ds\right]\le d_6\delta^2 \tag{2.24}$$

$$E\left[\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|w^i(\hat{\sigma}_k^{l,j})-w^i(s)|^2ds\right]\le d_7k^{\frac{1}{2}}n(l)^{\frac{3}{2}}\delta^2 \tag{2.25}$$

we have that

$$E[\sup_{0\le t\le T}|I_7(t)|^2]\le c_4\tilde{\delta}^{\frac{1}{2}}\to 0 \text{ as } l\uparrow\infty. \tag{2.23}$$

$$E[\sup_{0\le t\le T}|I_8(t)+I_{13}(t)|^2]\le c_5\to 0 \text{ as } l\uparrow\infty. \tag{2.24)'$$

and

$$E[\sup_{0\le t\le T}|I_{17}^i(t)|^2]\le c_6\tilde{\delta}^{\frac{1}{2}}\to 0 \text{ as } l\uparrow\infty\ (i=1,2,\cdots,r) \tag{2.25)'$$

and thus, we conclude the proof of Theorem 2.

## 3. Approximation for the solution of stochastic differential equation.

Let $\{B_l(t)\}_{l\ge 1}$ be an approximation of $I$-class of $W$ and $\{\bar{B}_l\}_{l\ge 1}$ be an approximation of $D$-class of $W$ defined by the Defininitions 1.1 and 1.2 respectively. In this section we will consider the approximations of solutions of $SDE$ (3.1) and (3.1)′, $X_l(t)$ and $\bar{X}_l(t)$, to the solution $X$ of $SDE$ (3.2), where

$$\begin{cases} \dot{X}_l^i(t,w)=\sum_{p=1}^{r} a_p^i(X_l(t,w))\dot{B}_l^p(t,w)+b^i(X_l(t,w)) \\ X_l(0)=x_0^i \quad i=1,2,\cdots,d, \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{\overline{X}}_l^i(t,w)=\sum_{p=1}^{r} a_p^i(\overline{X}_l(t,w))\dot{\overline{B}}_l^p(t,w)+b^i(\overline{X}_l(t,w)) \\ X_l^i(0)=x_0^i \quad i=1,2,\cdots,d, \end{cases} \tag{3.1'}$$

$$\begin{cases} dX^i(t,w)=\sum_{p=1}^{r} a_p^i(X(t,w))dw^p(t)+b^i(X(t,w))dt \\ \qquad\qquad +\sum_{p,q=1}^{r}\sum_{\alpha=1}^{d} C_{pq}a_p^\alpha\partial_\alpha a_q^i(X(t,w))dt \\ X^i(0)=x_0^i \quad i=1,2,\cdots,d, \end{cases} \tag{3.2}$$

and $x_0=(x_0^1,\cdots,x_0^d)\in R^d$, $a_p^i(x)\in C_b^2(R^d)$, $b^i(x)\in C_b^1(R^d)$.

**Theorem 3.** *If $\{|n(l)\}_{l\geq 1}$ and $\{N(l)\}_{l\geq 1}$ satisfy the conditions of Theorem 1, then for any given $T>0$ and $x_0\in R^d$,*

$$\lim_{l\uparrow\infty} E[\sup_{0\leq t\leq T\wedge\sigma_{N(l)}^{(l)}} |X(t,w)-X_l(t,w)|^2]=0. \tag{3.3}$$

*Proof* We will complete the proof by using the same way as for theorem VI-7.2 of [1]. First, we have a decomposition as follows

$$X_l^i(t)-X^i(t)=\sum_{p=1}^{4} H_p(t) \quad \text{for } i=1,2,\cdots,r \tag{3.4}$$

where

$$\begin{aligned} H_1(t)&=-\sum_{p=1}^{r}\int_t^{[t]_l^\ddagger} a_p^i(X_l(s))\dot{B}_l^p(s)ds+\sum_{p=1}^{r}\int_t^{[t]_l^\ddagger} a_p^i(X(s))dw^p(s) \\ &\quad+\sum_{p,q=1}^{r}\sum_{\alpha=1}^{d} C_{pq}\int_t^{[t]_l^\ddagger}(a_p^\alpha\partial_\alpha a_q^i)(X(s))ds \\ H_2(t)&=\sum_{p=1}^{r}\Big\{\int_{\hat\sigma_1^{(l)}}^{[t]_l^\ddagger} a_p^i(X_l(s))\dot{B}_l^p(s)ds-\int_{\hat\sigma_1^{(l)}}^{[t]_l^\ddagger} a_p^i(X(s))dw^p(s) \\ &\quad-\sum_{pq=1}^{r}\sum_{\alpha=1}^{d} C_{qp}\int_{\hat\sigma_1^{(l)}}^{[t]_l^\ddagger}(a_q^\alpha\partial_\alpha a_p^i)X(s))ds\Big\}=\sum_{p=1}^{r} H_2^p(t) \\ H_3&=\sum_{p=1}^{r}\int_0^{\hat\sigma_1^{(l)}} a_p^i(X_l(s))\dot{B}_l^p(s)ds-\sum_{p=1}^{r}\int_0^{\hat\sigma_1^{(l)}} a_p^i(X(s))dw^p(s) \\ &\quad-\sum_{pq=1}^{r}\sum_{\alpha=1}^{d} C_{pq}\int_0^{\hat\sigma_1^{(l)}}(a_p^\alpha\partial_\alpha a_q^i)(X(s))ds \\ H_4(t)&=\int_0^t b^i(X_l(s))ds-\int_0^t b^i(X(s))ds \end{aligned} \tag{3.5}$$

and

$$H_2^p(t)=\sum_{q=1}^{r} I_q^p(t)$$

$$I_1^p(t)=\sum_{k=1}^{m_l(t)} a_p^i(X_l(\sigma_{kn(l)-1}^{(l)}))[w^p(\hat\sigma_{k+1}^{(l)})-w^p(\hat\sigma_k^{(l)})-\int_{\alpha_1^{(l)}}^{[t]_l^\ddagger} a_p^i(X(s))dw^p(s)$$

$$I_2^p(t)=\sum_{k=1}^{m_l(t)}[a_p^i(X_l(\hat\sigma_k^{(l)}))-a_p^i(X_l(\sigma_{kn(l)-1}^{(l)}))][B_l^p(\hat\sigma_{k+1}^{(l)})-B_l^p(\hat\sigma_k^{(l)})]$$

$$I_3^p(t)=\sum_{k=1}^{m_l(t)} a_p^i(X_l(\sigma_{kn(l)-1}^{(l)}))\,[B_l^p(\hat\sigma_{k+1}^{(l)})-w^p(\hat\sigma_{k+1}^{(l)})]$$

$$I_4^p(t)=\sum_{k=1}^{m_l(t)} a_p^i(X_l(\sigma_{kn(l)-1}^{(l)}))\,[B_l^p(\hat\sigma_k^{(l)})-w^p(\hat\sigma_k^{(l)})]$$

$$I_5^p(t)=\sum_{\beta=1}^{d}\sum_{k=1}^{m_l(t)}\int_{\hat\sigma_k^{(l)}}^{\hat\sigma_{k+1}^{(l)}} \partial_\beta a_p^i(X_l(s))\Big[\sum_{q=1}^{r} a_q(X_l(s))B_l^q(s)+b^\beta(X_l(s))\Big]$$

$$\cdot\,[B_l^p(\hat\sigma_{p+1}^{(l)})-B_l^p(s)]ds-\int_{\hat\sigma_1^{(l)}}^{[t]_l^+}\sum_{q=1}^{r} C_{qp}(a_q^\beta\partial_\beta a_p^i)(X(s))ds. \tag{3.6}$$

Among these terms, only the estimates for $I_1^p$ and $J_{54}^{p*}$ of $I_5^p$ are different from those we used in theorem 7.2 of [1]. therefore we may obtain the proof of this theorem provided that we can achieve necessary estimates for $I_1^p$ and $J_{54}^p$. However, by

$$P\{\max_{0\le t\le T}([t]_e^+\wedge\sigma_{N(l)}^{(l)}-[t]_l^-\wedge\sigma_{N(l)}^{(l)})\geqslant s\}\leqslant c_7 n(l)E(\sigma_1^{(l)})/s^2 \tag{3.7}$$

and hence also

$$E[\max_{0\le t\le T}([t]_l^+\wedge\sigma_{N(l)}^{(l)}-[t]_l^-\wedge\sigma_{N(l)}^{(l)})]\leqslant c_8 n(l)^{-3}\to 0 \text{ as } l\uparrow\infty.$$

we have that for any $t_1\in[0, T]$.

$$E\{\sup_{0\le t\le t_1\wedge\sigma_{N(l)}^{(l)}}|I_1^p(t)|^2]\leqslant c_9\int_0^{t_1}E[\sup_{0\le s'\le s\wedge\sigma_{N(l)}^{(l)}}|X_l(s')-X(s')|^2]ds+o(1) \text{ as } l\uparrow\infty. \tag{3.8}$$

As for $J_{54}^q(t)=\sum_{k=1}^{m_l(t)}\int_{\hat\sigma_k^{(l)}}^{\hat\sigma_{k+1}^{(l)}} a_q^\beta\partial_\beta a_p^i(X_l(\hat\sigma_k^{(l)}))-a_q^\beta\partial_\beta a_p^i(X(s))]C_{qp}\,ds$ $(\beta,\ i=1,\cdots,d)$, we have

$$E[\sup_{0\le t\le t_1\wedge\sigma_{N(l)}^{(l)}}|J_{54}^q(t)|^2]\leqslant c_{10}\Big\{\int_0^{t_1}E(\sup_{0\le s'\le s\wedge\sigma_{N(l)}^{(l)}}|X_l(s')-X(s')|^2]ds$$

$$+\int_0^T E[\sup_{0\le s\le\sigma_{N(l)}^{(l)}}|X_l(s)-X_l([s]_l^-)|^2]ds$$

$$+E[\sup_{0\le t\le T}([t]_l^+\wedge\sigma_{N(l)}^{(l)}-[t]_l^-\wedge\sigma_{N(l)}^{(l)})^2]$$

$$=L_1+L_2+L_3 \text{ for } t_1\in[0, T] \text{ and } q=1, 2, \cdots, r. \tag{3.9}$$

Now it is not difficult to obtain that

$$L_2\leqslant c_{11}[n(l)^2(E(\sigma_1^{(l)}))^{\frac{1}{2}}+n(l)E(\sigma_1^{(l)})]\to 0 \text{ as } l\uparrow\infty$$

$$L_3\leqslant c_{12}[n(l)^{-9}+(n(l)^4E(\sigma_1^{(l)}))^3]\to 0 \text{ as } l\uparrow\infty. \tag{3.10}$$

so that for any $t_1\in[0, T]$

$$E[\sup_{0\le t\le t_1\wedge\sigma_{N(l)}^{(l)}}|H_2(t)|^2]\leqslant c_{13}\int_0^{t_1}E[\sup_{0\le s'\le s\wedge\sigma_{N(l)}^{(l)}}|X_l(s')-X(s')|^2]ds+o(1). \tag{3.11}$$

The rest is easy, hence we conclude the proof here.

**Corollary 2.1** *Under the assumptions of Corollary 1.2, for any given $T>0$ and $\varepsilon>0$*

$$\lim_{l\uparrow\infty}P[\sup_{0\le t\le T\vee\sigma_{N(l)}^{(l)}}|X(t)-X_l(t)|\geqslant\varepsilon\}=0. \tag{3.12}$$

Its proof is similar to Corollary 1.2.

* Cf. (7.68) of [1].

**Theorem 4.** *Let the matrix* $(C_{pq})$, $(p, q=1, 2, \cdots, r)$ *in* (3.2) *be* $(\bar{C}_{qq})$ *of Lemma* 1 *corresponding to the case of D-approximation* $\{\bar{B}_l\}$, *and* $\{n(l)\}$, $\{N(l)\}$ *satisfy the conditions of Theorem* 1, *then for any given* $T>0$ *and* $x_0\in R^d$.

$$\lim_{l\uparrow\infty} E[\sup_{0\leqslant t\leqslant T} |\bar{X}_l(t, w)-X(t, w)|^2]=0. \tag{3.13}$$

*Proof* The proof is the same as for theorem VI-7.2 of [1] provided we substitute respectively $w^n(\hat{\sigma}_k^{l,n})$ and $w^n(\hat{\sigma}_{k+1}^{l,n}$ $(n=1, 2, \cdots, r)$ for $w^n((k\tilde{\delta})$ and $w^n((k+1)\tilde{\delta})$ in the representation of $J_1(k)$ (Cf. [1], p. 413), and the substitutions do not lead to any important change of the proof there, thus we obtained this theorem.

## 4. Some limit theorems for stochastic line integrals and solutions of SDE with respect to random walks.

In [11], to study limit theory of stochastic processes, A. V. Skorohod proposed a probabilistic method: assume that $\{\xi_l\}$ $(l=1, 2, \cdots)$ is a sequence sf continuous stochastic processes, which converges to Brownian motion $\xi(t)$ in certain sense (for example, in mean, in probability. etc.). to find the limit distribution in law for certain specific classes of functionals $F(\xi_l)$, we can construct a sequence of processes whose distributivs laws coincide with thore of corrosponding functionals but the study of the asymptotic behavior of the later is easier than the original. By use of this idea, in this setion we will show that, the limit distributions which are what we want for *SLI* and solutions of *SDE* with respect to a sequence of certain random walks are just the distributions of those integrals and solutions of *SDE* which are obtained by using Brownian motion as a substitute for random walks in corresponding line integrals and SDE. To do this, first, we establish the limit theorems for random walks which can be imbeded into Brownian motion by theorems 1~4, then, as an immediate consequence of them, we obtain the expectant theorems.

Now we define $\{\sigma_k^{(l)}\}$, $\{B_l\}_{l\geqslant1}$ and a family of processes $\{W_l\}_{l\geqslant1}$ on $(W_0^r, \mathscr{B}(W_0^r), P)$ by

$$\sigma_0^{(l)}\equiv0,\quad \sigma_1^{(l)}(w)=\inf\left\{t\colon |w(t)|=\sqrt{\frac{r}{l}}\right\} \text{ and}$$

$$\sigma_k^{(l)}(w)=\sigma_{k-1}^{(l)}(w)+\sigma_1^{(l)}(\theta_{\sigma_{k-1}^{(l)}}w) \text{ for } k>1,\ l=1, 2, \cdots, \tag{4.1}$$

$$B_l^i(t, w)=\begin{cases} w^i(\sigma_k^{(l)}) \text{ if } t=\sigma_k^{(l)}, \\ \text{linear if } \sigma_k^{(l)}\leqslant t\leqslant\sigma_{k+1}^{(l)}, \end{cases} \tag{4.2}$$

$$W_l^i(t, w)=\begin{cases} w^i(\sigma_k^{(l)}) \text{ if } t=\dfrac{k}{l}, \\ \text{linear if } \dfrac{k}{l}\leqslant t\leqslant\dfrac{k+1}{l}, \end{cases} \tag{4.3}$$

for $k=0, 1, 2, \cdots$, $l=1, 2, \cdots$ and $i=1, 2, \cdots, r$.

Owing to $E[(\sigma_1^{(l)})]=\frac{1}{l}$ and $E[(\sigma_1^{(l)})^m]\leqslant d_8\left(\frac{1}{l}\right)^m$ for $m>1$*, it is easy to verify that

* Cf. K.. Itô[9], pp. 38—39 for $r=1$ and C. Port & C. J. Stone [10], pp. 28—29 for $r>1$.

$\{B_l\}_{l\geq 1}$ is an approximation of I-class of W and $\{\sigma_k^{(l)}\}$ satisfies the conditions (c.1)—(c.5).

**Theorem 5** *Under the assumptions* (4.1)—(4.3), *for any given* $T>0$ *and* $\varepsilon>0$, *we have*

$$\lim_{l\uparrow\infty} P\{\sup_{0\leq t\leq T} |A(t, \alpha, W_l) - A(t, \alpha; w)| \geq \varepsilon\} = 0. \tag{4.4}$$

*where* $\alpha_i(x)\in C^2(R^d)$ *with bounded partial derivatives* $\alpha_i^{(1)}\in C_b^1(R^d)$.

*Proof* For simplicity and without loss of generality we assume $r=d=1$ first. Then we introduce a lemma below:

**Lemma 2.** *For* $\{\sigma_k^{(l)}\}$, $\{B_l\}$ *and* $\{W_l\}$ *defined by* (4.1)—(4.3) *and any given* $T>0$, *we have*

$$P\Big(\lim_{l\uparrow\infty} \max_{1\leq k\leq [lT]} \Big|\sigma_k^{(l)} - \frac{k}{l}\Big| = 0\Big) = 1 \tag{4.5}$$

$$P(\lim_{l\uparrow\infty} \sup_{0\leq t\leq T} |\sigma_{[lt]}^{(l)} - t| = 0) = 1 \tag{4.6}$$

$$P(\lim_{l\uparrow\infty} \operatorname{snp}_{0\leq t\leq T} |W_l(t) - B_l(t)| = 0) = 1. \tag{4.7}$$

*Proof* (4.5) was proved in [9] (cf. pp. 38—39). Next, the estimate

$$\sup_{0\leq t\leq T} |\sigma_{[lt]}^{(l)} - t| \leq \sup_{0\leq k\leq [lT]} \Big|\sigma_k^{(l)} - \frac{k}{l}\Big| + \frac{1}{l}$$

and (4.5) immediatily lead to (4.6). Finally, from

$$\sup_{0\leq t\leq T} |W_l(t, w) - B_l(t, w)|$$
$$\leq \frac{1}{\sqrt{l}} + \sup_{0\leq t\leq T} |w(\sigma_{[lt]}^{(l)}) - w(t)| + \sup_{0\leq t\leq T} |w(t) - B_l(t, w)| \to 0 \text{ as } l\uparrow\infty,$$

(4.7) becomes obvious.

If for any given sequence $\{n(l)\}$, which has been chosen to satisfy the conditions of Theorem 1 we set

$$N(l) = [l(T+1)] + m(l) \tag{4.8}$$

where the non-negative integer $m(l)$ be so chosen that $N(l)\equiv 0 \pmod{n(l)}$ and $0\leq m(l)<l$, then evidently this $N(l)$ also satisfies conditions of Theorem 1. Moreover, by $\lim_{l\uparrow\infty} \frac{N(l)}{l} > T$ the condition (2.16) is satisfied. Therefore, due to corollary 1.2, it is sufficient to show

$$\lim_{l\uparrow\infty} P\{\sup_{0\leq t\leq T} |A(t, \alpha; W_l) - A(t, \alpha; B_l)| \geq \varepsilon\} = 0 \tag{4.9}$$

For any given $\varepsilon>0$. To prove this, let's consider that

$$\sup_{0\leq t\leq T} |A(t, \alpha; W_l) - A(t, \alpha; B_l)|$$
$$\leq \sup_{t\in[0,T]} \Big|\int_{\frac{[lt]}{l}}^{t} \alpha(W_l(s))\dot{W}_l(s)ds\Big| + \sup_{0\leq t\leq T} \Big|\int_{\sigma_{[lt]}^{(l)}\wedge t}^{\sigma_{[lt]}^{(l)}\vee t} \alpha(B_l(s))B_l(s)\,ds\Big|$$
$$= K_1(l) + K_2(l). \tag{4.10}$$

Evidently, $K_1(t)\to 0$ a.s. as $l\uparrow\infty$, Besides, we have

$$\sup_{0<t<T}\left|\int_{\sigma^{(l)}_{[lt]}\wedge t}^{\sigma^{(l)}_{[lt]}\vee t}\alpha(w(s))dw(s)\right|\to 0 \text{ a.s. as } l\uparrow\infty$$

and

$$\sup_{0<t<T}\left|\int_{\sigma^{(l)}_{[lt]}\wedge t}^{\sigma^{(l)}_{[lt]}\vee t}\alpha'(w(s))ds\right|\to 0 \text{ a. s. as } l\uparrow\infty$$

by noting $Y(t)=\int_0^t \alpha(w(s))dw(s)\in\mathscr{M}_2^c$ and (4.6). Hence, we know $K_2(l)$ converges to 0 in probalility, and we proved (4.10). and also (4.4).

**Theorem 6.** *Let the processes $X_l$ and $X$ be the solutions of SDE* (3.1) *and* (3.2) *respectively, and $\mathring{X}_l=(\mathring{X}_l(t, w))$ be the solution of SDE* (3.1) *which is obtained by using $W_l$ as a substitute for approximation of I-class $B_l$, then for any given $x_0$, $T>0$ and $\varepsilon>0$ we have*

$$\lim_{l\uparrow\infty}P\{\sup_{0\leqslant t\leqslant T}|\mathring{X}_l(t)-X(t)|\geqslant\varepsilon\}=0. \tag{4.11}$$

*Proof* Withont loss of generality, we only sketch the proof for the case of $r=d=1$ here. In the first place, under transformations $\mathring{t}_l$ and $\hat{t}_l$ which are defined by ($l=1, 2, \cdots$)

$$\begin{aligned}\mathring{t}_l(t)&=\frac{t-\sigma_k^{(l)}}{l\cdot\Delta\sigma_k^{(l)}}+\frac{k}{l} \text{ if } \sigma_k^{(l)}\leqslant t\leqslant\sigma_{k+1}^{(l)}\\ \hat{t}_l(t)&=\left(t-\frac{k}{l}\right)l\Delta\sigma_k^{(l)}+\sigma_k^{(l)} \text{ if } \frac{k}{l}\leqslant t\leqslant\frac{k+1}{l}\end{aligned}\quad \text{for } k=0, 1, \cdots \tag{4.12}$$

The process $\hat{X}_l$ defined by $\hat{X}_l(\hat{t})=\mathring{X}_l(\mathring{t}_l(\hat{t}))$ for $\hat{t}\in[0, \infty)$ is a solution of

$$\hat{X}_l(\hat{t})-x_0=\int_0^{\hat{t}}a(\hat{X}_l(s))\dot{B}_l(s)ds+\int_0^{\hat{t}}\hat{b}_l(\hat{X}_l(s))ds \tag{4.13}$$

where $\hat{b}_l(\hat{X}_l(t))=(l\cdot\Delta\sigma_k^{(l)})^{-1}b(\hat{X}_l(t))$ for $\sigma_k^{(l)}\leqslant t\leqslant\sigma_{k+1}^{(l)}$ ($k=1, 2, \cdots$).

Now we are going to prove the following assertion:

$$\lim_{l\uparrow\infty}E[\sup_{0\leqslant t\leqslant T\wedge\sigma^{(l)}_{N(l)}}(|\hat{X}_l(t)-X(t)|^2+|\mathring{X}_l(t)-X(t)|^2)]=0. \tag{4.14}$$

In a similar way as in the proof of Theorem 3, we may estimate, for $t_1\in[0, T]$, $E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma^{(l)}_{N(l)}}|\hat{H}_p(t)|^2]$ ($p=1, 2, 3, 4$) and $E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma^{(l)}_{N(l)}}|\mathring{H}_p(t)|^2]$ ($p=1, 2, 3, 4$) corresponding $\hat{X}_l(t)-X(t)$ and $\mathring{X}_l(t)-X(t)$ respectively, and so long as we use

$$E|\sup_{0\leqslant t\leqslant\sigma^{(l)}_{N(l)}}\left(\int_{\sigma^{(l)}_{m_l(t)n-1}}^{t}\varphi_l(s)ds^2\right]\to 0 \text{ as } l\uparrow\infty$$

$$\sum_{k=1}^{m(l)}E\left[\left(\int_{\hat{\sigma}_k^{(l)}}^{\hat{\sigma}_{k+1}^{(l)}}\varphi_l(s)ds\right)^4\right]\to 0 \text{ as } l\uparrow\infty$$

$$\sum_{k=1}^{m(l)-1}E\left[\left(\int_{\hat{\sigma}_k^{(l)}}^{\hat{\sigma}_{k+1}^{(l)}}\varphi_l(s)ds\right)^2\right]\to 0 \text{ as } l\to\infty$$

$$E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma^{(l)}_{N(l)}}|t-\mathring{t}_l(t)|^2]\to 0 \text{ as } l\uparrow\infty$$

$$E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma^{(l)}_{N(l)}}|t-\hat{t}_l(t)|^2]\to 0 \text{ as } l\uparrow\infty. \tag{4.15}$$

where $\varphi_l(t)=(l\cdot\Delta\sigma_k^{(l)})^{-1}$ for $\sigma_k^{(l)}\leqslant t\leqslant\sigma_{k+1}^{(l)}$ $(k=1, 2, \cdots)$, we have

$$E[\sup_{0\leqslant t\leqslant\sigma_{N(l)}^{(l)}}|\hat{H}_1(t)|^2]\to 0 \text{ and } E[|\hat{H}_3|^2]\to 0 \text{ as } l\uparrow\infty$$

$$E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma_{N(l)}^{(l)}}|\hat{H}_2(t)|^2]\leqslant c_{14}\int_0^{t_1}E[\sup_{0\leqslant s'\leqslant s\wedge\sigma_{N(l)}^{(l)}}|\hat{X}_l(s')-X(s')|^2]ds+o(1)$$

$$E(\sup_{0\leqslant t\leqslant t_1\wedge\sigma_{N(l)}^{(l)}}|\hat{H}_4(t)|^2]\leqslant c_{15}\int_0^{t_1}E[\sup_{0\leqslant s'\leqslant s\wedge\sigma_{N(l)}^{(l)}}|\mathring{X}_l(s')-X(s')|^2]ds+o(1)$$

$$E[\sup_{0\leqslant t\leqslant\sigma_{N(l)}^{(l)}}|\mathring{H}_1(\hat{t}_l(t))|^2]\to 0 \text{ and } E[\mathring{H}_3|^2]\to 0 \text{ as } l\uparrow\infty$$

$$E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma_{N(l)}^{(l)}}|\mathring{H}_2(t)|^2]\leqslant c_{16}\int_0^{t_1}E[\sup_{0\leqslant s'\leqslant s\wedge\sigma_{N(l)}^{(l)}}|\hat{X}_l(s')-X(s')|^2]ds+o(1)$$

$$E[\sup_{0\leqslant t\leqslant t_1\wedge\sigma_{N(l)}^{(l)}}|\mathring{H}_4(t)|^2]\leqslant c_{17}\int_0^{t_1}E[\sup_{0\leqslant s'\leqslant s\wedge\sigma_{N(l)}^{(l)}}|\mathring{X}_l(s')-X(s')|^2]ds+o(1) \quad (4.16)$$

where $t_1\in[0, T]$ and $o(1)$ converge to 0 uniformly on $[0, T]$ as $l\uparrow\infty$. Thus, by Gronwall's inequality, (4.14) becomes obvious, and also (4.11).

As an immediate application we get the following limit theorem:

**Theorem 7.** *If for every $l=1, 2, \cdots$ on any $(\Omega_l, \mathscr{B}_l, P_l)$ the i. i. d. random variables $\{\xi_k^l\}$ $(k=1, 2, \cdots)$ are given such that*

*(i) $\xi_k^l=(\xi_k^{l,1}, \cdots, \xi_k^{l,r})$ and*

*(ii) $P_l\left\{|\xi_k^l|=\sqrt{\frac{r}{l}}\right\}=1$ and on the surface $S^{r-1}=\left\{x\in R^r: |x|=\sqrt{\frac{r}{l}}\right\}$ the distribution of $\xi_k^l$ is uniform distribution,*

*then random walk process $\{W_l(t)=(W_l^1(t), \cdots, W_l^r(t))\}$ defined by*

$$W_l(t)=\begin{cases}0 & \text{if } t=0\\ \xi_1^l+\xi_2^l+\cdots+\xi_k^l & \text{if } t=\frac{k}{l}, k=1, 2, \cdots\\ \text{linear} & \text{if } \frac{k}{l}\leqslant t\leqslant\frac{k+1}{l}, k=1, 2, \cdots\end{cases} \quad (4.17)$$

*have the same distribution as $\{W_l(t)\}$which defined by(4.3). Moreover,*

*(i) $\{A(t, \alpha; W_l)\}_{l\geqslant 1}$ converges to $A(t, \alpha; w)$ in law;*

*(ii) the solutions $\{X_l\}$* of SDE

$$\begin{cases}\dot{X}_l^i(t)=\sum_{p=1}^r a_p^i(X_l(t))\dot{W}_l^p(t)+b^i(X_l(t))\\ X_l^i(0)=x_0^i \quad i=1, 2, \cdots, d\end{cases} \quad (4.18)$$

*converges to $X$ defined by (3.2) in law, providod $\alpha$, $\{a_p^i\}$ and $\{b^i\}$ have same smoothness as above.*

Now let's set for every $l=1, 2, \cdots$

$$\sigma_0^{l,i}\equiv 0, \quad \sigma_1^{l,i}(w)=\inf\left\{t: |w^i(t)|=\sqrt{\frac{1}{l}}\right\}$$

and

$$\sigma_k^{l,i}(w)=\sigma_{k-1}^{l,i}(w)+\sigma_1^{l,i}(\theta_{\sigma_{k-1}^{l,i}}w) \text{ for } k>1, i=1, \cdots, r.$$

Then it is obvious that $\sigma_k^{l,i}$ satisfies the conditions(c.1)′—(c.5)′. Hence it is easy to know the processes $\{\overline{W}_l(t)\}_{l\geq 1}$ defined by

$$\overline{W}_l^i(t)=\begin{cases} w^i(\sigma_k^{l,i}) & \text{if } t=\frac{k}{l} \\ \text{linear} & \text{if } \frac{k}{l}\leqslant t\leqslant\frac{k+1}{l} \end{cases} \quad (i=1,\ 2,\ \cdots,\ r;\ l=1,\ 2,\ \cdots)$$

are an approximation of $D$-class of $W$. Therefore, by Theorems 3 and 4 we may obtain following conclusion:

**Theorem 8.** *If for every* $l=1,\ 2,\ \cdots$ on any $(\Omega_l,\ \mathscr{B}_l,\ P_l)$ *a sequence of random variables* $\{\xi_k^l=(\xi_k^{l,1},\ \cdots,\ \xi_k^{l,r})\}$ *is given such that*

(i) *for every* $l$, $\xi_k^{l,1},\ \cdots,\ \xi_k^{l,r}$ *are mutually independent, and*

(ii) $P\left\{\xi_k^{l,i}=\frac{1}{\sqrt{l}}\right\}=P\left\{\xi_k^{l,i}=-\frac{1}{\sqrt{l}}\right\}=\frac{1}{2}$.

*then random walk processes* $\{W_l(t)\}$ *defined by* (4.17) *have the same distributions as* $[\overline{W}_l(t)\}_{l\geq 1}$. *Moreover*.

(I) $\{A(t,\ \alpha;\ W_l)\}_{l\geq 1}$ *converges to* $A(t,\ \alpha;\ w)$ *in law, and*

(II) *the solutions* $\{X_l\}$ *of SDE* (4.18) *converges to* $X$ *defined by* (3.2) *in law provided* $\alpha$, $\{a_p^i\}$ *and* $\{b^i\}$ *have same smoothness as above.*

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