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ON THE PROBLEM OF BEST CONVERGENCE RATES OF DENSITY ESTIMATES

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Abstract

Let $X_1, \cdots, X_n$ be iid samples drawn from an m-dimensional population with a probability density f, belonging to the family $C_{k\alpha}$, i. e. the family of all densities whose partial derivatives of order k are bounded by α. It is desired to estimate the value of f at some predetermined point a, for example $a=0$. Farrell obtained some results concerning the best possible convergence rates for all estimator sequence, from which it follows, for example, that there exists no estimator sequence $\{\gamma_n(0)=\gamma_n(X_1, \cdots, X_n, 0)\}$ such that $\sup\limits_{f\in C_{k\alpha}} E_f[\gamma_n(0)-f(0)]^2=o(n^{-2k/(2k+m)})$. This article pursues this problem further and proves that there exists no estimator sequence $\{\gamma_n(0)\}$ such that

$$n^{-k/(2k+m)}(\gamma_n(0)-f(0)) \xrightarrow{P_f} 0, \text{ for each } f\in C_{k\alpha},$$

where $\xrightarrow{P_f}$ denotes convergence in probability.

§ 1. Main Result and Its Applications

Let $X_1, \cdots, X_n$ be iid. samples drawn from an m-dimensional population wtih a probability density function f. The problem is to estimate $f(x)$ by an estimator of the form $\gamma_n(X_1, \cdots, X_n, x)$. In the following we shall simplify $\gamma_n(X_1, \cdots, X_n, 0)$ to $\gamma_n(0)$.

Since the appearance of Farrell's work [1], a number of authors (see [1—6]) discussed the following question: Given a family of m-dimensional densities $\mathscr{F}$, we want to estimate $f(0)$, with $f\in\mathscr{F}$. Then what is the best accuracy in various senses which can be achieved by suitably choosing the estimator $\gamma_n(0)$? Of course, the accuracy referred to should be valid to each $f\in\mathscr{F}$, and not to some members of $\mathscr{F}$ only.

In this respect, the most important work up to now is Farrell's [2]. Owing to its close relation to the present article, we quote the main result of [2] in the most interesting case $C_{k\alpha}$ as follows:

Suppose that k is a positive integer, and $\alpha>0$. Denote by $C_{k\alpha}$ the family of densities f satisfying the following conditions: f possesses all partial derivatives up to

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the order $k-1$, each $(k-1)$-th order partial derivative of f is absolutely continuous with respect to each of the variables $x_1, \cdots, x_m$. Also, for any non-negative integers $k_1, \cdots, k_m$ with $\Sigma k_i=k$, we have

$$\sup_{x\in R_m}\left|\frac{\partial^k f(x_1, \cdots, x_m)}{\partial x_1^{k_1}\cdots\partial x_m^{k_m}}\right|<\alpha.$$

Farrell proved in [2] the following

Theorem. (Farrell) *Let $\{\gamma_n(0), n=1, 2, \cdots\}$ be an arbitrary sequence of estimators of $f(0)$, and $\{a_n\}$ is a sequence of constants. Suppose that*

$$\lim_{n\to\infty}\inf_{f\in C_{k\alpha}} P_f(|\gamma_n(0)-f(0)|\leqslant a_n)=1. \tag{1}$$

Then

$$\lim_{n\to\infty} a_n n^{k/(2k+m)}=\infty. \tag{2}$$

From this theorem it follows easily (see [2]) that there exists no sequence of estimators $\{\gamma_n^0(0)\}$ such that

$$\sup_{f\in C_{k\alpha}} E[\gamma_n^0(0)-f(0)]^2=o(n^{-2k/(2k+m)}). \tag{3}$$

The weakness of this result lies in the fact that condition (1) is too strong. This condition amounts to requiring that

$$\lim_{n\to\infty} P_f(|\gamma_n(0)-f(0)|\leqslant a_n)=1 \tag{4}$$

should hold uniformly for $f\in C_{k\alpha}$. Hence, this result cannot answer the question of whether or not a sequence of estimators $\{\gamma_n(0)\}$ can be found such that

$$E_f[\gamma_n^0(0)-f(0)]^2=o(n^{-2k/(2k+m)}), \text{ for each } f\in C_{k\alpha}. \tag{5}$$

Another question concerning the rate of couvergence is as follows: Let $\{\gamma_n(0)\}$ be a sequence of estimators of $f(0)$ and $A_n\to\infty$. If

$$\lim_{n\to\infty} A_n(\gamma_n(0)-f(0))=0 \quad \text{a.s. for each } f\in C_{k\alpha}$$

or

$$\limsup_{n\to\infty} |A_n(\gamma_n(0)-f(0))|<\infty \quad \text{a.s. for each } f\in C_{k\alpha},$$

we say respectively that $\gamma_n(0)$ converges to $f(0)$ with a rate of $o(A_n^{-1})$ or $(O(A_n^{-1})$ in the family $C_{k\alpha}$. No conclusion concerning this question can be derived from Farrell's theorem.

In this article, we first give a modification to Farrell's theorem as follows:

Theorem 1. *Let $\{\gamma_n(0)\}$ be a sequence of estimators of $f(0)$, and $\{a_n\}$ be a sequence of constants, such that (4) holds for each $f\in C_{k\alpha}$. Then*

$$\liminf_{n\to\infty} a_n n^{k/(2k+m)}>0.$$

In turn, this theorem can be employed in establishing the following result.

Theorem 2. *There exists no sequence of estimators $\{\gamma_n(0)]$ such that*

$$n^{k/(2k+m)}[\gamma_n(0)-f(0)]\xrightarrow{P_f} 0, \text{ for each } f\in C_{k\alpha}. \tag{6}$$

From this theorem it follows that

1. There exists no sequence of estimators $\{\gamma_n(0)\}$ satisfying (5).

On the other hand, by choosing suitably a kernel function K and constant $h_n>0$, defining the kernel estimator

$$\hat{f}_n(x)=(nh_n^m)^{-1}\sum_{i=1}^{n}K\left(\frac{x-X_j}{h_n}\right),$$

we can get

$$\sup_{f\in C_{k\alpha}} E[\hat{f}_n(0)-f(0)]^2=O(n^{-2k/(2k+m)}). \tag{7}$$

Therefore, with respect to the whole family $C_{k\alpha}$, the best convergence rate of the MSE for any estimator of $f(0)$ is the right hand side of (7), both in the uniform and non-uniform senses, and this optimal rate is attainable.

2. There exists no sequence of estimators $\{\gamma_n(0)\}$ such that

$$\gamma_n(0)-f(0)=o(n^{-k/(2k+m)}) \quad \text{a. s. for each } f\in C_{k\alpha}. \tag{8}$$

On the other hand, the present author shows in [7] that the kernel estimator $\hat{f}_n(0)$, defined by suitably chosen K and h_n, satisfies

$$\lim_{n\to\infty} \varepsilon_n n^{k/(2k+m)}(\log n)^{-1/2}\sup_x|\hat{f}_n(x)-f(x)|=0, \quad \text{a. s.} \tag{9}$$

for any constant $\varepsilon_n\to 0$ and $f\in C_{k\alpha}$.

This is equivalent to (see Appendix)

$$\sup_x|\hat{f}_n(x)-f(x)|=O(n^{-k/(2k+m)}(\log n)^{1/2}), \quad \text{a. s. foreach } f\in C_{k\alpha}. \tag{10}$$

Thus for the best convergence rate of $\gamma_n(0)-f(0)$ and $\sup_x|\gamma_n(x)-f(x)|$, the exponential $-k/(2k+m)$ is accurate. The author does not know if there exists estimator sequence $\{\gamma_n\}$ such that

$$\sup_x \gamma_n(x)-f(x)|=O(n^{-k/(2k+m)}) \quad \text{a. s. for each } f\in C_{k\alpha}$$

or even

$$\gamma_n(0)-f(0)=O(n^{-k/(2k+m)}) \quad \text{a. s. for each } f\in C_{k\alpha}.$$

§ 2. Proof of Theorem 1

We employ the idea developed in [2]. The new feature is that we introduce a limiting process in order to avoid the assumption (1). We shall use the notations of [2] whenever possible.

Choose $A>0$ sufficiently large and $a>0$, $0<\varepsilon_0<1$, both sufficiently small, such that a function $\tilde{f}$ can be found satisfying: 1° $\tilde{f}\in C_{k,\alpha/2}$. 2° $\tilde{f}(x)=0$ for $\|x\|\leqslant A$. 3° $\tilde{f}(x)>a$ for $\|x\|\leqslant\varepsilon_0$. Denote by $\mathscr{F}$ the set of all such function $\tilde{f}$.

Let $C(\delta)$ be a continuous increasing function defined in $[0,\infty)$, such that $C(0)=0$ while $C(\delta)>0$ for $\delta>0$. The exact form of $C(\delta)$ will be chosen later. For a constant $\delta>0$ and an m-dimensional density f, define

$$h_{f,\delta}(x)=f(x)+C(\delta)e_{k\delta}(x),$$

where $e_{k\delta}$ is defined by (2.18) of [2]. It is easy to see that if $f\in C_{k,\alpha/2}$ and $\delta>0$ sufficiently small, then $h_{f,\delta}\in C_{k,\alpha/2}$, Write

$$S(g,\ \varepsilon)=\{\tilde{g}:\ \tilde{g}\in\mathscr{F},\ \sup_x|\tilde{g}(x)-g(x)|<\varepsilon\}.$$

Now suppose that $\{\gamma_n(0)\}$ is a sequence of estimators of $f(0)$, $\{a_n\}$ is a sequence of constants, such that (4) holds for each $f\in C_{k\alpha}$ For any $d\in(0,\ 1)$, define

$$C_{k\alpha}^{(N)}(d)=\{g:\ g\in C_{k\alpha},\ P_g(|\gamma_n(0)-f(0)|\leqslant a_n)\geqslant d,\ \text{for any } n\geqslant N\}.$$

Then the condition (4) can be written as

$$\bigcup_{N=1}^{\infty}C_{k\beta}^{(N)}(d)=C_{k\alpha},\ \text{for any } d\in(0,\ 1). \tag{11}$$

We proceed to verify the following assertion: There exists a positive integer N, $f\in C_{k\alpha}$ and $\bar{\delta}>0$, such that

$$h_{f,\delta}\in C_{k\alpha}^{(N)}(d),\ \text{for } 0<\delta<\bar{\delta}. \tag{12}$$

Note the following two trivial fact: If $g\in\mathscr{F}$, $g\overline{\in}C_{k\alpha}^{(n)}(d)$, then

$$S(g,\ \varepsilon)\cap C_{k\alpha}^{(n)}(d)=\varnothing$$

for $\varepsilon>0$ small enough. If $g\in\mathscr{F}$, $g\in C_{k\alpha}^{(n)}(d')$, $d<d'<1$, then $S(g,\ \varepsilon)\subset C_{k\alpha}^{(n)}(d)$ for $\varepsilon>0$ small enough. Further, if $g\in\mathscr{F}$ and $\varepsilon>0$, then $h_{g,\delta}\in S(g,\ \varepsilon)$ for $\delta>0$ small enough.

Now suppose that the assertion is false. Then there exists $\delta_1\in(0,\ \varepsilon_0)$, such that $h_{\bar{f},\delta_1}\overline{\in}C_{k\alpha}^{(1)}(d)$. Write $g_1=h_{\bar{f},\delta_1}$. Find $\varepsilon_1>0$ small enough, such that

$$S(g_1,\ \varepsilon_1)\cap C_{k\alpha}^{(1)}=\phi.$$

Using once again the supposition that the assertion is false, we can find $\delta_2\in(0.\ \min\{\delta_1/2,\ 2^{-2/(2k+m)}\})$, such that

$$h_{g_1,\delta_2}\in S(g_1,\ \varepsilon_1/2),\ h_{g_1,\delta_2}\overline{\in}C_{k\alpha}^{(2)}(d).$$

Write $g_2=h_{g_1,\delta_2}$. Find $\varepsilon_2>0$ small enough such that

$$S(g_2,\ \varepsilon_2)\subset S(g_1,\ \varepsilon_1),\ S(g_2,\ \varepsilon_2)\cap C_{k\alpha}^{(2)}(d)=\varnothing.$$

Repeating this process, one can determine three sequences, $\{g_i\}$, $\{\varepsilon_i\}$ and $\{\delta_i\}$, satisfying

1° $\delta_i\in(0,\ \min\{\delta_{i-1}/2,\ i^{-2/(2k-m)}\})$, $\sum_{i=1}^{\infty}C(\delta_i)<\alpha/2$ (see later),

2° $g_0=\bar{f}$, $g_{i+1}=h_{g_i,\delta_{i+1}}$, $i=0,\ 1,\ \cdots$,

3° $S(g_i,\ \varepsilon_i)\cap C_{k\alpha}^{(i)}(d)=\phi$, $S(g_{i+1},\ \varepsilon_{i+1})\subset S(g_i,\ \varepsilon_i)$.

It is not difficult to prove that not only g_i itself, but also all its partial derivatives with an order not exceeding k, converges uniformly in the whole space R^m to a certain function g^*, the derivatives being with the corresponding order, and the function g^* having the properties: 1° $g^*\in C_{k\alpha}$. 2° $g^*(x)=0$ for $\|x\|\geqslant A$. 3° $g^*(x)\geqslant a$ for $\|x\|\leqslant\varepsilon_0$. In fact, from the property of $g_{k\delta}$ appearing in the definition of $e_{k\delta}$, as pointed out in [2], it follows that there exists a constant M, such that

$$|g_j - g_i| \leqslant |g_{i+1} - g_i| + |g_{i+2} - g_{i+1}| + \cdots - |g_j - g_{j-1}|$$
$$\leqslant M[C(\delta_{i+1}) + \cdots + C(\delta_j)] \to 0, \text{ when } j > i \to \infty. \tag{13}$$

Similar estimation holds true also for

$$|\partial^u g_i / \partial x_1^{u_1} \cdots \partial x_m^{u_m} - \partial^u g_j / \partial x_1^{u_1} \cdots \partial x_m^{u_m}| \quad (1 \leqslant u \leqslant k)$$

with the only difference of changing $k+1$ to $k+1-u$. This proves the convergence property of g_i mentioned above.

Take $d' \in (d, 1)$. Since $g^* \in C_{k\alpha}$, there exists N such that $g^* \in C_{k\alpha}^{(N)}(d')$. Also, $g^*(x) = 0$ for $\|x\| \geqslant A$ and $d' > d$, so $S(g^*, \varepsilon) \subset C_{k\alpha}^{(N)}(d)$ for $\varepsilon > 0$ small enough. Since $g_i \in \mathscr{F}$ for each i and $g_i \in S(g^*, \varepsilon)$ for i large enough, we have $g_i \in C_{k\alpha}^{(N)}(d)$. On the other hand, we have $g_i \overline{\in} C_{k\alpha}^{(i)}(d)$ and $C_{k\alpha}^{(i)}(d) \supset C_{k\alpha}^{(N)}(d)$ for $i \geqslant N$. Thus we reach a contradiction, which proves the assertion mentioned earlier: One can find positive integer N, $f \in \mathscr{F}$ and $\delta > 0$, such that (12) holds. Considering (4), one can assume $f \in C_{k\alpha}^{(N)}(d)$ by increasing N, if necessary.

Now choose arbitrarily $b > 0$. Write $b' = C_3 b/4$, where C_3 is determined by

$$\int_{-\infty}^{\infty} e_{k\delta}^2(t)\, dt = a C_3 \delta^{2k+m}/4$$

(see [2], (2.21), replacing $\eta(\delta/2)$ in it by $\delta/2$). Use the relationship

$$\delta_n^{2k+m} C^2(\delta_n) = 4b'/(c_3 n) = b/n \tag{14}$$

to define δ_n. We have $\delta_n \downarrow 0$ for $n \to \infty$. From these, and using an argument similar to that used in deducing (3.7) of [2], we get

$$P h_{\tilde{f}, \delta_n}(|\gamma_n(0) - h_{\tilde{f}, \delta_n}(0)| \leqslant a_n) \leqslant [P_{\tilde{f}}(|\gamma_n(0) - h_{\tilde{f}, \delta_n}(0)| \leqslant a_n)]^{1/2} (1 + b'/n)^{n/2}$$
$$\leqslant [P_{\tilde{f}}(|\gamma_n(0) - h_{\tilde{f}, \delta_n}(0)| \leqslant a_n)]^{1/2} e^{b'/2}. \tag{15}$$

The first expression in (15) is no smaller than d when $n \geqslant N$. Hence

$$P_{\tilde{f}}(|\gamma_n(0) - h_{\tilde{f}, \delta_n}(0)| \leqslant a_n) \geqslant d^2 e^{-b'}. \tag{16}$$

On the other hand, when $n \geqslant N$ we also have

$$P_{\tilde{f}}(|\gamma_n(0) - \tilde{f}(0)| \leqslant a_n) \geqslant d. \tag{17}$$

Now we choose d in the definition of $C_{k\alpha}^{(n)}(d)$ to satisfy

$$0 < d < 1, \quad d + e^{-b'/2} d^2 > 1. \tag{18}$$

From (16)—(18), it follows that for $n \geqslant N$

$$2a_n \geqslant |\tilde{f}(0) - h_{\tilde{f}, \delta_n}(0)| = C(\delta_n) |e_{k\delta_n}(0)|.$$

Write $T = (k-1)(k-2)/2$. By (2.20) of [2] (changing $\eta(\delta/2)$ to $\delta/2$), we have

$$2a_n \geqslant C(\delta_n) 2^T \delta_n^k. \tag{19}$$

From (14) we have

$$\delta_n = b^{1/(2k+m)} n^{-1/(2k+m)} C^{-2/(2k+m)}(\delta_n). \tag{20}$$

Since $\delta_n \to 0$, $C(\delta_n) \to 0$ when $n \to \infty$, from (20) we see that $\delta_n \geqslant n^{-1/(2k+m)}$ for n sufficiently large. By (19) and the fact that $C(\delta)$ is a strictly increasing function of δ, we have on writing $R = 2^{T-1}$

$$a_n \geqslant RC(n^{-1/(2k+m)}) n^{-k/(2k+m)} b^{k/(2k+m)}. \tag{21}$$

Now take arbitarily positive integer sequence $\{B_n\}$, with $B_1 \geqslant A$ and $B_n \uparrow \infty$ where A is chosen such that $\sum_{n=A}^{\infty}(R_n^2)^{-1}<\alpha/2$. Choose a strictly increasing function $C(\delta)$, satisfying $C(0)=0$ and

$$C(n^{-1/(2k+m)})=(RB_n)^{-1}, \quad n=1, 2, \cdots. \tag{22}$$

Then $\{\delta_i\}$ satisfies the conditions 1°—3° mentioned earlier, and by (21), (22)

$$\liminf_{n\to\infty} B_n a_n n^{k/(2k+m)} \geqslant b^{k/(2k+m)}. \tag{23}$$

Since $b>0$ is arbitrarily given, (23) implies

$$\lim_{n\to\infty} B_n a_n n^{k/(2k+m)}=\infty. \tag{24}$$

Since (24) holds true for any $B_n \uparrow \infty$, We finally get

$$\liminf_{n\to\infty} a_n n^{k/(2k+m)}>0,$$

which concludes the proof of Theorem 1.

§ 3. Proof of Theorem 2

The following elementary lemma is needed.

Lemma 1. *Suppose that* $(b_{mn}, m=1, 2, \cdots, n=1, 2, \cdots)$ *is a double array of constants satisfying*

$$\lim_{n\to\infty} b_{mn}=0, \text{ for } m=1, 2, \cdots. \tag{25}$$

Then there exists a positive sequence $\{b_n\}$, *such that* $\lim_{n\to\infty} b_n=0$ *and* $\lim_{n\to\infty} b_{mn}/b_n=0$ *for* $m=1, 2, \cdots$.

Proof For any sequence $\{a_n\}$ tending to zero, there exists a sequence $\{a'_n\}$ tending to zero such that $a'_n \geqslant |a_n|$, and $a'_1 \geqslant a'_2 \geqslant \cdots$. Hence without losing generality we can assume $b_{m1} \geqslant b_{m2} \geqslant \cdots \geqslant 0$ for $m=1, 2, \cdots$. Also, we can assume

$$b_{1n} \leqslant b_{2n} \leqslant b_{3n} \leqslant \cdots \quad \text{for } n=1, 2, \cdots$$

by replacing b_{mn} with $b'_{mn}=\max\{b_{1n}, \cdots, b_{mn}\}$, if necessary.

Define a subsequence $r_1<r_2<\cdots$ of positive integers as follows: Choose r_1 such that $\sqrt{b_{2n}}<1$ for $n>r_1$. Choose $r_2>r_1$ such that $\sqrt{b_{3n}}<1/2$ for $n>r_2$. In general, after $r_1, \cdots r_i$ are determined, we choose $r_{i+1}>r_i$ such that $\sqrt{b_{i+1,n}}<1/(i+1)$ for $n>r_{i+1}$. Such an r_{i+1} exists in view of (25). Now define

$$b_n=\sqrt{b_{1n}} \text{ for } n\leqslant r_1; \quad =\sqrt{b_{in}} \text{ for } r_{i-1}<n\leqslant r_i, \; i=2, 3, \cdots.$$

Then $b_n<1/i$ for $n>r_i$, hence $b_n \to 0$. Fix m. For $n>r_{m-1}$, there exists i such that $i \geqslant m$ and $r_{i-1}<n\leqslant r_i$. Since $b_{in} \geqslant b_{mn}$ for $i \geqslant m$ and $b_{in}<1$ for $r_{i-1}<n\leqslant r_i$, we have

$$b_{mn}/b_n=b_{mn}/\sqrt{b_{in}} \leqslant b_{mn}/\sqrt{b_{mn}}=\sqrt{b_{mn}},$$

which tends to zero when $n\to\infty$. We can assume that $b_n>0$, for we can replace b_n by b_n+n^{-1} if necessary. The Lemma is proved.

Turning to the proof of Theorem 2, for any positive integer sequence $\{B_n\}$, with $B_1 \geqslant A$ and $B_1 < B_1 < \cdots$ (A was defined in the proof of Theorem 1), we choose the function $C(\delta)$ as follows: For $\delta = n^{-1/(2k+m)}$, $n=1, 2, \cdots$, define $C(\delta)$ as in (22), and let $C(\delta)$ be linear within the interval $[(n+1)^{-1/(2k+m)}, n^{-1/(2k+m)}]$, $n=1, 2, \cdots$. The set U of all possible values of $C(\delta_i)$, with $i=0, 1, 2, \cdots$ ($\delta_0=0$, $C(\delta_0)=0$) and $\{B_i\}$ arbitrarily chosen with the above-mentioned property, is denumerable. Write $U=\{u_0, u_1, \cdots\}$, and $f_i = h_{\bar{f}, u_i}$, $i=0, 1, 2, \cdots$.

Now suppose in the contrary that there exist a sequence $\{\gamma_n(0)\}$ satisfying (6). In particular, (6) holds for $f=f_i$, $i=0, 1, 2, \cdots$. Therefore, for each i, $i=0, 1, 2, \cdots$, a positive constant sequence $\{a_{i1}, a_{i2}, \cdots\}$ can be found such that $\lim_{n\to\infty} a_{in}=0$, and

$$\lim_{n\to\infty} P_{f_i}(|\gamma_n(0)-f(0)| \geqslant a_{in} n^{-k/(2k+m)}) = 1, \quad i=0, 1, 2, \cdots. \tag{26}$$

By Lemma 1, we can find a positive constant sequence $\{a_n\}$ such that $\lim_{n\to 8} a_n=0$ and $\lim_{n\to\infty} a_{in}/a_n=0$ for $i=0, 1, 2, \cdots$. Considering (26), for this $\{a_n\}$ we have

$$\lim_{n\to\infty} P_{f_i}(|\gamma_n(0)-f(0)| \geqslant a_n n^{-k/(2k+m)}) = 1, \quad i=0, 1, 2, \cdots. \tag{27}$$

From (27) and a glance at the final part of the proof of Theorem 1, it follows that

$$\lim_{n\to\infty} B_n a_n n^{-k/(2k+m)} n^{k/(2k+m)} = \lim_{n\to\infty} B_n a_n = \infty$$

for any sequence $\{B_n\}$ with the property mentioned earlier. This in turn implies

$$\liminf_{n\to\infty} a_n > 0,$$

which contradicts to the fact that $\lim_{n\to\infty} a_n = 0$, concluding the proof of Theorem 2.

Appendix

To deduce (10) from (9), we need the following lemma.

Lemma 2. *Let $\{X_n\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathscr{B}, P)$, satisfying the condition*

$$\lim_{n\to\infty} \varepsilon_n X_n = 0 \text{ a. s., for any constant } \varepsilon_n \downarrow 0. \tag{28}$$

Then

$$\limsup_{n\to\infty} |X_n| < \infty, \text{ a. s.} \tag{29}$$

Proof Put

$$A = \{\omega: \omega \in \Omega, \limsup_{n\to\infty} |X_n(\omega)| = \infty\}.$$

We want to prove $P(A)=0$. Suppose in the contrary that $P(A)>0$. Transfer to the new probability space $(A\cap\Omega, A\cap\mathscr{B}, P(\cdot)/P(A))$, we can assume without losing generality that $A=\Omega$. Find a sequence of positive integers $n_1 < n_2 < \cdots$ such that

$$P(\max_{1\leqslant j\leqslant n_i} |X_j| \geqslant i) \geqslant 1-2^{-(i+1)}, \quad i=1, 2\cdots. \tag{30}$$

Such a sequence exists, since $A=\Omega$. Write

$$D=\{\omega;\ \omega\in\Omega,\ \max_{1\leqslant j\leqslant n_i}|X_j(\omega)|\geqslant i,\quad i=1,\ 2,\ \cdots\}.$$

By (30) we have $P(D)\geqslant 1/2$. Now define $\varepsilon_n\downarrow 0$ as follows: Put $\varepsilon_1=\cdots=\varepsilon_{n1}=1$, and

$$\varepsilon_j=i^{-1/2},\ \text{for } n_{i-1}<j\leqslant n_i,\quad i=2,\ 3,\ \cdots.$$

Then for $\omega\in D$ we have

$$\max_{1\leqslant j\leqslant n_i}|\varepsilon_j X_j(\omega)|\geqslant i^{1/2},\quad i=1,\ 2,\ \cdots.$$

Therefore we cannot have $\lim\limits_{n\to\infty}\varepsilon_n X_n(\omega)=0$ for $\omega\in D$. Since $P(D)\neq 0$, Lemma 2 is proved.

References

[1] Farrell, R. H., *Ann. Math. Statist.*, **38** (1967), 471—474,
[2] Farrell, R. H., *Ann, Math. Statist.*, **43** (1972), 170—180.
[3] Meyer, T. G., *Ann. Statist.*, **5** (1977), 136—142.
[4] Sacks, J. & Ylvisaker, D. , *Ann. Statist.*, **9** (1981), 334—346.
[5] Krieger, A. M. & Pickands J. III, *Ann. Statist.* **9** (1981), 1066—1078.
[6] 陈希孺,科学探索, **2**(1982), 73—78.
[7] Chen Xiru, *J., Systems Sciences & Math.*, **3**(1983), 263—272.