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ON THE L^p-BOUNDEDNESS OF SEVERAL CLASSES OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract

To indicate precisely the requirements for smoothness of symbols, generalizations of Hörmander's classes of symbols, $S^m_{\rho,k;\delta,\nu}$ and $S^m_{\rho,k;\delta,\nu,\varepsilon,\varkappa}$ are introduced. The main results are as follows: (1) An optimal L^2-boundedness result is obtained for the pseudo-differential operators with double symbols (amplitude) $a(x, \xi, y)$; (2) By means of the interpolation theorem due to Fefferman and Stein [5], new L^p-boundedness results are established. These results are not only sharp with respect to upper index, but also sharp($p\geqslant 2$)or almost sharp ($1<p<2$) with respect to lower indices.

Since L. Hörmander established the calculus of pseudo-differential operators with symbols in his classes $S^m_{\rho,\delta}$ in [7], many boundedness results about this kind of ψ. d. o's and their various generalizations appeared (see, e. g., [3, 8, 14] and the literatures cited there). Some of these Strove for Sharpening the requirements imposed on the symbols to certain extent. For instance, in [3], the usual L^2 boundedness of the ψ. d. o's with symbols in the class $S^0_{\rho,\delta}$ $(0\leqslant\delta\leqslant\rho\leqslant 0,\ \delta<1)$ was sharpened to the requirement that symbols need only belong to $S^0_{\rho,k;\delta,k}$(a slight generalization of $S^0_{\rho,\delta}$ which we shall define in § 1), where $k=\left[\frac{n}{2}\right]+1$ and n denotes the dimension of the independent variable $x=(x_1, \cdots, x_n)$. In this paper we shall present several similar L^p-boundedness results, all of which are related to a problem raised by Hörmander in [7], which will be recalled presently.

Concerning the L^p-boundedness of ψ. d. o's, in his paper [7], Hörmander pointed out: if $a(x, \xi)\in S^m_{\rho,\delta}(0\leqslant\delta<\rho\leqslant 1)$ and $1<p\leqslant r<\infty$ are such that

$$m<-n\left\{\frac{1}{p}-\frac{1}{r}+(1-\rho)\max\left(\frac{1}{2}-\frac{1}{p},\ \frac{1}{r}-\frac{1}{2},\ 0\right)\right\}, \tag{0.1}$$

then the associated operator $a(x, D)\in\mathscr{L}(L^p, L^r)$; while if the inequality (0, 1) is reversed, then generally speaking, this will be false. He also raised the problem concerning the critical case, that is: when (0, 1) is replaced by the corresponding

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equality, and $1<p\leqslant r<2$ or $2<p\leqslant r<\infty$, does one still have $a(x, 0)\in\mathscr{L}(L^p, L^r)$ or not? It is easy to see that $p=r$ is the crucial case. Since then there were many works about the L^p-boundedness of ψ. d. o's (cf. [1—4], [9—14], [17]). In particular, Fefferman [4] and Chang [2] answered the above Hörmander's question affirmatively in the cases $0<\rho<1$ and $\rho=1$, $\delta=0$, respectively.

Fefferman's short paper [4] is highly interesting. One of his results, judged by our present knowledge on L^2-boundedness of ψ. d. o's, is that the operators $a(x, D)$ with symbols in $S^{-m}_{\rho,k+1;\delta,k}$ are L^p-bounded, provided that $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$, $\rho>0$, $2\leqslant p<+\infty$, $m\geqslant n(1-\rho)\left(\frac{1}{2}-\frac{1}{p}\right)$, $k=\left[\frac{n}{2}\right]+1$. In this paper we shall prove that the condition $a(x, \xi)\in S^{-m}_{\rho,k+1;\delta,k}$ can be sharpened to $a(x, \xi)\in S^{-m}_{\rho,k;\delta,k}$, where the lower indices are in complete agreement with those in the optimal L^2-boundness theorem in [3] as cited above.

The content of this paper consists of two parts. On the one hand, we shall push forward the results of [4] futher to its best (or almost best) possible form. And in addition, we shall prove an optimal L^2-boundedness result about certain class of ψ. d. o's $a(x, D, y)$ with "double symbols", and tnen using the ideas and techniques developed in [4, 5], we are able to establish some optimal L^p-boundedness results about these operators. It seems that applications of such results to ψ. d. o's in Weyl's symmetrical form (cf. [8]) would be interesting.

§ 1. Notations and Definitions

Let R^n be an n-dimensional Euclidean space. For $(x_1, \cdots, x_n)=x\in R^n$ we will use the following usual notations

$$|x|=\left(\sum_{j=1}^n x_j^2\right)^{1/2},\quad \langle x\rangle=(1+|x|^2)^{1/2};$$

$$x\cdot\xi=\sum_{1=j}^n x_j\xi_j,\ \text{where } x\in R^n,\ \xi\in R^n.$$

And when $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multiindex, where α_j $(j=1, \cdots, n)$ are nonnegative integers, we will also use the following usual notations

$$|\alpha|=\sum_{1=j}^n\alpha_j,\quad \alpha!=\alpha_1!\cdots\alpha_n!,\quad \alpha+\beta=(\alpha_1+\beta_1, \cdots, \alpha_n+\beta_n);$$

$$\partial_x^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},\quad D_x^\alpha=(-i)^{|\alpha|}\partial_x^\alpha.$$

To fix the terminology, we state the definitions of the classes of symbols used in this paper.

Definition 1. *Let m be any real number, $0\leqslant\delta$, $\rho\leqslant 1$. $a(x, \xi)\in S^m_{\rho,\delta}$ iff $a(x, \xi)\in C^\infty$ and for any two multiindices α and β, there is a constant $C_{\alpha\beta}$, such that*

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{1.1}$$

Definition 2. *Let k and ν be nonnegative integers. If* (1.1) *holds for all α, β with $|\alpha| \leqslant k$, $|\beta| \leqslant \nu$, then we say that $a(x, \xi)$ is a symbol in $S^m_{\rho,k;\delta,\nu}$ and denote $a(x, \xi) \in S^m_{\rho,k;\delta,\nu}$.*

Definition 3. *Let k, ν and $\varkappa$ be nonnegative integers and let m be any real number, $0 \leqslant \delta$, ε, $\rho \leqslant 1$, $a(x, \xi, y) \in S^m_{\rho,k;\delta,\nu;\varepsilon,\varkappa}$ iff for any multiindices α, β and γ with $|\alpha| \leqslant k$, $|\beta| \leqslant \nu$ and $|\gamma| \leqslant \varkappa$ there is a constant $C_{\alpha,\beta,\gamma}$, such that*

$$|D_x^\beta D_\xi^\alpha D_y^\gamma a(x, \xi, y)| \leqslant C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|+\varepsilon|\gamma|}. \tag{1.2}$$

It follows from Definitions 1, 2 and 3 that the inclusions

$$S^{m_1}_{\rho_1,\delta_1} \subset S^{m_1}_{\rho_1,k;\delta_1,\nu_1} \subset S^{m_2}_{\rho_2,k_2;\delta_2,\nu_2},$$

$$S^{m_1}_{\rho_1,k;\delta_1,\nu_1;\varepsilon_1,\varkappa_1} \subset S^{m_2}_{\rho_2,k_2;\delta_2,\nu_2;\varkappa_2}$$

hold for $\rho_1 \geqslant \rho_2$, $\delta_1 \leqslant \delta_2$, $\varepsilon_1 \leqslant \varepsilon_2$, $k_1 \geqslant k_2$, $\nu_1 \geqslant \nu_2$; $\varkappa_1 \geqslant \varkappa_2$, $m_1 \leqslant m_2$.

For $a(x, \xi) \in S^m_{\rho,k;\delta,\nu}$, set

$$|a|^{(m)}_{k_1,\nu_1} = \max_{|\alpha| \leqslant k_1, |\beta| \leqslant \nu_1} \{\mathring{C}_{\alpha,\beta}\}, \ k_1 \leqslant k, \ \nu_1 \leqslant \nu,$$

where $\mathring{C}_{\alpha\beta}$ is the least constant such that (1.1) holds. Obviously, for $m_1 \leqslant m_2$, $k_1 \geqslant k_2$, $\nu_1 \geqslant \nu_2$ we have

$$|a|^{(m_1)}_{k_1,\nu_1} \geqslant |a|^{(m_2)}_{k_2,\nu_2}. \tag{1.3}$$

Similarly, for $a(x, \xi, y) \in S^m_{\rho,k;\delta,\nu;\varepsilon,\varkappa}$, set

$$|a|^{(m)}_{k_1,\nu_1\varkappa_1} = \max_{|\alpha| \leqslant k, |\beta| \leqslant \nu_1, |\nu| \leqslant \varkappa_1} \{\mathring{C}_{\alpha\beta\gamma}\}, \ k_1 \leqslant k, \ \nu_1 \leqslant \nu, \ \varkappa_1 \leqslant \varkappa.$$

The analogue of (1.3) is true.

Next, let us give the definitions of pseudo-differential operators. A pseudo-differential operator with (left-) symbol $a(x, \xi)$ and acting on the Schwartz space $\mathscr{S}$ is defined as follows

$$a(x, D)u(x) = \iint e^{i(x-y)\xi} a(x, \xi) u(y)\, dy\, đ\xi = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi)\, đ\xi \tag{1.4}$$

and one with right-symbol $a(x, \xi)$ is defined as

$$\tilde{a}(x, D)u(x) = \iint e^{i(x-y)\xi} a(y, \xi) u(y)\, dy\, đ\xi, \tag{1.5}$$

where $đ\xi = (2\pi)^{-n} d\xi$ and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, i. e.

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx.$$

In addition, a pseudo-differential operator with symbol $a(x, \xi, y)$ is defined as

$$a(x, D, y)u(x) = \iint e^{i(x-y)\xi} a(x, \xi, y) u(y)\, dy\, đ\xi. \tag{1.6}$$

Sometimes, these pseudo-differential operators need to be written in the forms

$$a(x, D)u(x) = \int k(x, x-y) u(y)\, dy, \tag{1.4'}$$

$$\tilde{a}(x, D)u(x)=\int k(y, x-y)u(y)dy, \tag{1.5)'$$

$$a(x, D, y)u(x)=\int k(x, x-y, y)u(y)dy, \tag{1.6)'$$

where

$$k(x, y)=\int e^{iy\xi}a(x, \xi)d\!\!^-\xi \tag{1.7}$$

and

$$k(x, z, y)=\int e^{iz\xi}a(x, \xi, y)d\!\!^-\xi \tag{1.8}$$

are called (the distributional) kernel of the pseudo-differential operators (1.4), (1.5) and (1.6) respectively. The integrals in (1.7) and (1.8) are understood to be oscillatory integrals, etc.

We also mention briefly the definitions of the several function spaces used in this paper.

Definition 4. *For $-\infty<s<\infty$, $u(x)\in H^s$ iff $u(x)\in\mathscr{S}'$ and*

$$\|u\|^2_{H^s}=\int|\hat{u}(\xi)|^2\langle\xi\rangle^{2s}d\xi<+\infty.$$

Definition 5. *For a noninteger $m>0$,*

i) *If $0<m<1$, then $u(x)\in C^m$ iff there exists a constant C such that $|u|\leqslant C$ and*

$$|u(x+y)-u(x)|\leqslant C|y|^m$$

holds for every pair $x, y\in\mathbf{R}^n$.

ii) *If $m>1$, we write $m=q+r$, where $q=[m]$, $r=m-[m]>0$, then $u\in C^m$ iff $u\in C^q$ and $D^\alpha u\in C^r$ for all α with $|\alpha|=q$.*

The norm of $u\in C^m$ is defined as

$$\|u\|_{C^m}=\sum_{|\alpha|\leqslant q}\sup_x|D^\alpha u(x)|+\sum_{|\alpha|=q}\sup_{x\neq y}\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^r}.$$

As for the notion of BMO space, we refer to the paper [5]. In what follows we denote the norm on BMO space by $\|\cdot\|_*$.

$$\|u\|_*=\sup_Q\frac{1}{|Q|}\int_Q|u(x)-u_Q|dx<+\infty,$$

where the supremum is taken over all finite cubes Q in $\mathbf{R}^n$, $|Q|$ is the Lebesgue measure of Q and U_Q denotes the mean value of u over Q, namely

$$u_Q=\frac{1}{|Q|}\int_Q u(x)dx.$$

§ 2. L^2-boundedness

For the pseudo-differential operator defined by (1.4), we recall the following remarkable result:

Theorem 1 (Coifman and Meyer). *Let $0\leqslant\delta\leqslant\rho\leqslant1$, $\delta<1$, $k=\left[\frac{n}{2}\right]+1$. Suppose*

$a(x, \xi) \in S^0_{\rho, k; \delta, k}$. *Then the pseudo-differential operator* $a(x, D)$ *defined by (1.4) is* L^2-*bounded.* *) (cf. [3]).

The main purpose of this section is to establish a similar result on the L^2-boundedness of the pseudo-differential operator $a(x, D, y)$ defined by (1.6). Here and hereafter we always assume that the integer $k=\left[\frac{n}{2}\right]+1$.

Let $C>1$ be a constant. Set

$$E_{-1}=\{\xi \mid |\xi| \leqslant 2C\}, \quad E_j=\{\xi \mid C^{-1}2^j \leqslant |\xi| \leqslant C2^{j+1}\}, \ j=0, 1, 2, \cdots \tag{2.1}$$

Lemma 1. *There exist* $\varphi_{-1}(\xi) \in C_0^\infty$ *and* $\varphi(\xi) \in C_0^\infty$, *such that*

i) $\operatorname{supp} \varphi \subset E_0$, $\operatorname{supp} \varphi_{-1} \subset E_{-1}$;

ii) $0 \leqslant \varphi \leqslant 1$, $0 \leqslant \varphi_{-1} \leqslant 1$;

iii) $\varphi_{-1}(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) \equiv 1$.

Lemma 2. *Let* $m>0$ *be not an integer. Suppose* $u(x) \in C^m$. *Then there exists the decomposition*

$$u(x) = \sum_{j=-1}^{\infty} u_j(x),$$

such that

i) *the spectrum of* u_j *is contained in* E_j, *i. e.*

$$\operatorname{supp} \hat{u}_j(\xi) \subset E_j, \ j=-1, 0, 1, \cdots;$$

ii) *there is a constant* C *independent of* u *and* j, *such that*

$$\|u_j\|_\infty \leqslant C 2^{-mj} \|u\|_{C^m}.$$

For the proof, see, e. g., Chapt. II of [3].

Corollary. *Let* $m_2>m_1>0$ *and both be not integers. Suppose* $u(x) \in C^{m_2}$. *Then there is a constant* $C>0$ *independent of* u, *such that for any* $R>1$, *we may decompose*

$$u=v+w$$

satisfying

i) $\operatorname{supp} \hat{v}(\xi) \subset \{\xi \mid |\xi| \leqslant R\}$;

ii) $\|v\|_{C^{m_2}} \leqslant C\|u\|_{C^{m_2}}$; $\|w\|_{C^{m_1}} \leqslant C R^{m_1-m_2} \|u\|_{C^{m_2}}$.

Lemma 3. *If a sequence of functions* $\{u_j\}_{-1}^{\infty}$ *satisfies the following conditions:*

i) $\operatorname{supp} \hat{u}_j \subset E_j$, $j=-1, 0, 1, \cdots$,

ii) $\|u_j\|_2 \leqslant C_j 2^{-js}$, $j=-1, 0, 1, \cdots$, *and* $\|\{C_j\}\|_{l^2} \leqslant K$,

then $\sum_{j=-1}^{\infty} u_j = u \in H^s$ *and*

$$\|u\|_{H^s} \leqslant CK,$$

where C *is a constant independent of* u *and* K.

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*) That a pseudo-differential operator is L^2 (or L^p)-bounded means that it can be extended to being L^2 (or L^p)-bounded.

Proof By the hypothesis ii) we have

$$\|u_j\|^2_{H^s}=|\hat{u}_j(\xi)|^2\langle\xi\rangle^{2s}d\xi\leqslant B2^{j2s}\|u\|^2_2\leqslant BC_j^2,\ j=-1,\ 0,\ 1,\ \cdots,$$

where B is a constant independent of j and u_j. Using the triangle inequality and the hypotheses i), ii), we can obtain

$$\|u\|^2_{H^s}\leqslant B'\sum_{j=-1}^{\infty}\|u_j\|^2_{H^s}\leqslant B''\sum_{j=-1}^{\infty}C_j^2.$$

It follows from this that the required conclusion is valid.

Lemma 4. *Let $m>m'>0$ be two real numbers and m be not an integer. Suppose $a(x,\ \xi,\ y)=0$ whenever $|\xi|\geqslant1$ and $x\to\partial_\xi^\beta\partial_y^\gamma a(x,\ \xi,\ y)\in C^m$ for each fixed ξ, y and β, γ with $|\beta|\leqslant2k$, $|\gamma|\leqslant k$ and with norms not greater than 1 Then the operator $a(x,\ D,\ y)$ defined by (1.6) belongs to $\mathscr{L}(H^{-k},\ H^{m'})$ with norm depending only on n, m and m'.*

Proof Since $x\to\partial_\xi^\beta\partial_y^\gamma a(x,\ \xi,\ y)\in C^m$, according to Lemma 2 we have the decomposition

$$a(x,\ \xi,\ y)=\sum_{j=-1}^{\infty}b_j(x,\ \xi,\ y),$$

where the spectrum in x of b_j is contained in E_j and

$$|\partial_\xi^\beta\partial_y^\gamma b_j(x,\ \xi,\ y)|\leqslant C2^{-mj},\ j=-1,\ 0,\ 1,\ \cdots,\ |\beta|\leqslant2k,\ |\gamma|\leqslant k,\tag{2.2}$$

where C is a constant independent of a, j, α, β, ξ, y.

Now, we write

$$g(x)=a(x,\ D,\ y)u(x)=\iint e^{i(x-y)\xi}a(x,\ \xi,\ y)u(y)\,dy\,đ\xi=\sum_{-1}^{\infty}\iint e^{i(x-y)\xi}b_j(x,\ \xi,\ y)u(y)\,dy\,đ\xi$$

$$\triangleq\sum_{-1}^{\infty}g_j(x).$$

For $u\in H^{-k}$, we set

$$\hat{h}(\xi)=\hat{u}(\xi)\Big(1+\sum_{\nu=1}^{n}|\xi_\nu|^k\Big)^{-1},\tag{2.4}$$

then $h\in L^2$. It follows from (2.3) that

$$\hat{u}(\xi)=\hat{h}(\xi)\Big(1+\sum_{\nu=1}^{n}|\xi_\nu|^k\Big)=\hat{h}(\xi)+\sum_{\nu=1}^{n}\xi_\nu^k(\text{sign }\xi_\nu)^k\hat{h}(\xi)\triangleq\hat{h}(\xi)+\sum_{\nu=1}^{n}\xi_\nu^k\hat{h}_\nu(\xi).$$

Hence

$$u(x)=h(x)+\sum_{\nu=1}^{n}D_{x_\nu}^k h_\nu(x),\tag{2.5}$$

where $h_\nu\in L^2$ $(\nu=1,\ \cdots,\ n)$. Substituting (2.5) into (2.3), we have

$$g_j(x)=\iint e^{i(x-y)\xi}b_j(x,\ \xi,\ y)\Big[h(y)+\sum_{\nu=1}^{n}D_{y_\nu}^k h_\nu(y)\Big]dy\,đ\xi,$$

$$=\iint e^{i(x-y)\xi}(1+|x-y|^2)^{-k}(1-\Delta_\xi)^k b_j(x,\ \xi,\ y)\Big[h(y)+\sum_{\nu=1}^{n}D_{y_\nu}^k h_\nu(y)\Big]dy\,đ\xi.$$

In view of the estimate (2.2) we obtain

$$|g_j(x)|\leqslant C2^{-mj}\int(1+|x-y|^2)^{-k}\Big[|h(y)|+\sum_{\nu=1}^{n}|h_\nu(y)|\Big]dy$$

$$\leqslant C'2^{-mj}\sum_{\nu=0}^{n}\Big\{\int(1+|x-y|^2)^{-k}|h_\nu(y)|^2dy\Big\}^{1/2},$$

where $h_0(y)=h(y)$ and C' is independent of a, j. Hence

$$\|g_j\|_2 \leqslant C 2^{-mj} \sum_{\nu=0}^{n} \|h_\nu\|_2 \leqslant \widetilde{C} 2^{-mj} \|u\|_{H^{-k}}, \quad j=-1,\ 0,\ 1,\ 2,\ \cdots. \tag{2.6}$$

On the other hand, taking the Fourier transform of g_j, we have

$$\hat{g}_j(\eta) = \iint \hat{b}_j(\eta-\xi,\ \xi,\ y) e^{-iy\xi} u(y)\, dy\, đ\xi,$$

where $\hat{b}_j(\eta,\ \xi,\ y)$ denotes the Fourier transform of $b_j(x,\ \xi,\ y)$ with respect to x. By the hypothesis and (2.2)

$$\hat{b}_j(\eta-\xi,\ \xi,\ y)=0, \text{ if } |\xi| \geqslant 1 \text{ or } \eta-\xi \notin E_j.$$

Therefore

$$\hat{g}_j(\eta)=0, \text{ if } \eta \notin \{\eta \mid C^{-1}2^j-1 \leqslant |\eta| \leqslant C 2^{j+1}+1\}.$$

It follows that

$$\operatorname{supp} \hat{g}_j(\eta) \subset E'_j, \text{ if } j \text{ is sufficiently large}, \tag{2.7}$$

where E'_j is the set E_j with another constant C. Thus from (2.6) and (2.7), we can apply Lemma 3 to the sequence $\{g_j\}$ to conclude that $g \in H^{m'}$ and

$$\|g\|_{H^{m'}} \leqslant C \|u\|_{H^{-k}}.$$

The proof is completed.

Now, we shall give the theorem on L^2-boundedness of the pseudo-differential operators with symbol in the class $S^0_{\rho,2k;\delta,k;\varepsilon,k}$.

Proposition 1. *Let $m>\frac{n}{2}$ be not an integer. Suppose that $x \to \partial_y^\gamma \partial_\xi^\beta a(x,\ \xi,\ y) \in C^m$ for each fixed ξ, y and β, γ with $|\beta| \leqslant 2k$, $|\gamma| \leqslant k$ and with their norms not greater than 1. Then the operator $a(x,\ D,\ y) \in \mathscr{L}(L^2,\ L^2)$ and its norm depends only on n and m.*

Proof Take a function $\psi \in C_0^\infty$ satisfying

i) $\psi(x)=1,\ \ |x| \leqslant \frac{1}{3};\ \psi(x)=0\ \ |x| \geqslant 1;$

ii) $\sum_{\nu \in \mathbf{Z}^n} \psi(\xi-\nu) \equiv 1$, where $\mathbf{Z}$ denotes the set of all integers and $\mathbf{Z}^n = \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n \text{ times}}$.

Then we can write

$$g(x) = \iint e^{i(x-y)\xi} a(x,\ \xi,\ y) u(y)\, dy\, đ\xi = \sum_\nu \iint e^{i(x-y)\xi} a(x,\ \xi,\ y) \psi(\xi-\nu) u(y)\, dy\, đ\xi.$$

$$= \sum_\nu e^{i\nu x} \iint e^{i(x-y)\xi} a(x,\ \xi+\nu,\ y) \psi(\xi) e^{-i\nu y} u(y)\, dy\ đ\xi \quad \sum_\nu e^{i\nu x} g_\nu(x). \tag{2.8}$$

If we denote

$$u_\nu(y) = e^{-i\nu y} u(y),\ a_\nu(x,\ \xi,\ y) = a(x,\ \xi+\nu,\ y) \psi(\xi),$$

then

$$g_\nu(x) = a_\nu(x,\ D,\ y) u_\nu(y).$$

It is easy to see that the symbol a_ν satisfies the conditions of Lemma 4. Consequently we have

$$\|g_\nu\|_{H^{m'}} \leqslant \|u_\nu\|_{H^{-k}}, \tag{2.9}$$

where $m>m'>\frac{n}{2}$. By means of Schwartz's inequality, from (2.8) we have

$$\|\hat{g}\|_2^2=\int|\sum_\nu \hat{g}_\nu(\xi-\nu)|^2 d\xi\leqslant\int(\sum_\nu|\hat{g}_\nu(\xi-\nu)|^2\langle\xi-\nu\rangle^{2m'})\cdot\sum_\nu\langle\xi-\nu\rangle^{-2m'}d\xi$$
$$\leqslant C(n,\ m')\sum_\nu\|g_\nu\|_{H^{m'}}^2. \tag{2.10}$$

Here in the last step we have used the fact that the series

$$\sum_\nu\langle\xi-\nu\rangle^{-2m'}\leqslant C(n,\ m')<\infty,$$

since $2m'>n$. From (2.10) and (2.9) we obtain

$$\|g\|_2^2\leqslant C(n,\ m')\sum_\nu\|u_\nu\|_{H^{-k}}^2. \tag{2.11}$$

By definition

$$\|u_\nu\|_{H^{-k}}^2=\int|\hat{u}_\nu(\xi)|^2\langle\xi\rangle^{-2k}d\xi=\int|\hat{u}(\xi)|^2\langle\xi-\nu\rangle^{-2k}d\xi.$$

Substituting this into (2.11), one gets

$$\|g\|_2^2\leqslant C(n,\ m')\sum_\nu\int|\hat{u}(\xi)|^2\langle\xi-\nu\rangle^{-2k}d\xi\leqslant C'(n,\ m')\|u\|_2^2.$$

This completes the proof of Proposition 1.

Corollary. *If* $a(x,\ \xi,\ y)\in S^0_{0,2k;0,k\cdot 0,k}$ and $|a|^{(0)}_{2k,k,k}\leqslant 1$, *then the associated pseudo-differential operator* $a(x,\ D,\ y)\in\mathscr{L}(L^2,\ L^2)$ *and*

$$\|a(x,\ D,\ y)\|\leqslant C(n).$$

Proposition 2. *Let* $0<\rho<1$. *If*

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma a(x,\ \xi,\ y)|\leqslant\langle\xi\rangle^{-\rho(|\beta|-|\alpha|-|\gamma|)}$$

for $|\alpha|\leqslant k$, $|\beta|\leqslant 2k$ *and* $|\gamma|\leqslant k$, *then the associated operator* $a(x,\ D,\ y)\in\mathscr{L}(L^2,\ L^2)$ *and*

$$\|a(x,\ D,\ y)\|\leqslant C(n,\ \rho).$$

Proof Let φ_{-1} and φ be the functions given in Lemma 1, supported in E_{-1} and E_0 with $C=2$ respectively. Then we have

$$g(x)=a(x,\ D,\ y)u(x)=\sum_{j=0}^\infty\iint e^{i(x-y)\xi}a(x,\ \xi,\ y)\varphi(2^{-j}\xi)u(y)dy\,d\!\!{}^-\xi$$
$$+\iint e^{i(x-y)\xi}a(x,\ \xi,\ y)\varphi_{-1}(\xi)u(y)dy\,d\!\!{}^-\xi$$
$$=\sum_{j=0}^\infty\iint e^{i(2^{j\rho}x-y)\xi}a(x,\ 2^{j\rho}\xi,\ 2^{-j\rho}y)\varphi(2^{-j(1-\rho)}\xi)u(2^{-j\rho}y)dy\,d\!\!{}^-\xi$$
$$+\iint e^{i(x-y)\xi}a(x,\ \xi,\ y)\varphi_{-1}(\xi)u(y)dy\,d\!\!{}^-\xi\triangleq\sum_{j=0}^\infty g_j(2^{j\rho}x)+g_{-1}(x), \tag{2.12}$$

where

$$g_j(x)=\iint e^{i(x-y)\xi}a(2^{-j\rho}x,\ 2^{j\rho}\xi,\ 2^{-j\rho}y)\varphi(2^{-j(1-\rho)}\xi)u(2^{-j\rho}y)dy\,d\!\!{}^-\xi. \tag{2.13}$$

Set

$$a_j(x,\ \xi,\ y)=a(2^{-j\rho}x\cdot 2^{j\rho}\xi\cdot 2^{-j\rho}y)\varphi(2^{-j(1-\rho)}\xi)j=0,\ 1,\ 2,\ \cdots.$$

Clearly $a_j \in S^0_{0,2k;0,k;0,k}$. Choose m and m' so that $k>m>m'>\left[\frac{n}{2}\right]$. By the Corollary of Lemma 2 we may decompose

$$a_j(x, \xi, y)=a_{1j}(x, \xi, y)+a_{2j}(x, \xi, y)$$

satisfying

i) the spectrum of $x \to \partial^\beta_\xi \partial^\gamma_y a_{1j}(x, \xi, y)$ is contained in the ball $|\eta| \leqslant \frac{1}{10} 2^{j(1-\rho)}$ and

$$\|\partial^\beta_\xi \partial^\gamma_y a_{1j}(x, \xi, y)\|_{C^m(\mathbf{R}^n_x)} \leqslant C_0, \ |\beta| \leqslant 2k, \ |\gamma| \leqslant k;$$

ii)

$$\|\partial^\beta_\xi \partial^\gamma_y a_{2j}(x, \xi, y)\|_{C^{m'}(\mathbf{R}^n_x)} \leqslant C_0 2^{-(m-m')j(1-\rho)},$$

C_0 is independent of j.

Then we have

$$g_j(x)=g_{1j}(x)+g_{2j}(x), \ j=0, 1, 2, \cdots, \tag{2.14}$$

where

$$g_{lj}(x)=\iint e^{i(x-y)\xi} a_{lj}(x, \xi, y) u(2^{-j\rho}y)\, dy\, đ\xi, \ l=1, 2.$$

Applying Proposition 1 to a_{2j}, we obtain

$$\|g_{2j}\|_2 \leqslant C_1 2^{j\rho n/2} \cdot 2^{-(m-m')j(1-\rho)} \|u\|_2, \ j=0, 1, 2, \cdots. \tag{2.15}$$

To estimate g_{1j}, we write

$$g_{1j}(x)=\sum_\nu \iint e^{-i(x-y)\xi} a_{1j}(x, \xi, y) \psi(\xi-\nu) u(2^{-j\rho}y)\, dy\, đ\xi = \sum_\nu e^{i\nu x} g_{1j,\nu}(x). \tag{2.16}$$

where $\sum_\nu \psi(\xi-\nu) \equiv 1$ is the partition of unity used in the proof of Proposition 1 and

$$g_{1j,\nu}(x)=\iint e^{i(x-y)\xi} a_{1j}(x, \xi+\nu, y)\ \psi(\xi)\ f_{j,\nu}(y)\, dy\, đ\xi,$$

$$f_{j,\nu}(y)=e^{-i\nu y} u(2^{-j\rho}y).$$

As in the proof of Proposition 1, from (2.16) we have

$$\|g_{1j}\|^2_2 \leqslant C\ 2^{2j\rho n} \sum_\nu \int |\hat{u}(2^{j\rho}(\xi+\nu))|^2 \langle\xi\rangle^{-2k} d\xi = C_2 2^{2j\rho n} \int |\hat{u}(2^{j\rho}\xi)|^2 \sum_\nu \langle\xi-\nu\rangle^{-2k} d\xi. \tag{2.17}$$

Here the range of ν in the summation (2.16) and (2.17) is $\nu \approx 2^{j(1-\rho)}$; this is crucial for the following arguments. In fact, inclusions

$$\operatorname*{supp}_\xi a_{1j}(x, \xi+\nu, y) \subset \{\xi \mid 2^{j(1-\rho)-1} \leqslant |\xi+\nu| \leqslant 2^{j(1-\rho)+2}\}$$

$$\operatorname{supp} \psi(\xi) \subset \{\xi \mid |\xi| \leqslant 1\}$$

mean that we need only consider those ν which satisfy

$$2^{j(1-\rho)-1}-1 \leqslant |\nu| \leqslant 2^{j(1-\rho)+2}+1.$$

It follows that

$$\frac{1}{8} 2^{j(1-\rho)} \leqslant |\nu| \leqslant 8 \cdot 2^{j(1-\rho)}, \ \text{ if } j \geqslant j_0, \ j_0 \text{ is large enough.}$$

Set

$$S_{1j}=\left\{\xi\left|\frac{1}{16}2^{j(1-\rho)}\leqslant|\xi|\leqslant 16\cdot 2^{j(1-\rho)}\right.\right\},$$

$$S_{2j}=\left\{\xi\left|\frac{1}{16}2^{j}\leqslant|\xi|\leqslant 16\cdot 2^{j}\right.\right\}.$$

Then we have

$$\int|a(2^{j\rho}\xi)|^2\sum_\nu\langle\xi-\nu\rangle^{-2k}d\xi=\int_{S_{1j}}+\int_{\mathscr{C}S_{1j}}$$

$$\leqslant C\int_{S_{1j}}|\hat{u}(2^{j\rho}\xi)^2d\xi+\int_{\mathscr{C}S_{1j}}|\hat{u}(2^{j\rho}\xi|^2\sum_\nu\langle\xi-\nu\rangle^{-2k}d\xi,\tag{2.18}$$

where $\mathscr{C}S$ denotes the complement of S. If $\xi\in\mathscr{C}S_{1j}$, then

$$\sum_\nu\langle\xi-\nu\rangle^{-2k}\sim\int_{\frac{1}{8}2^{j(1-\rho)}\leqslant|\eta|\leqslant 8\cdot 2^{j(1-\rho)}}\langle\xi-\eta\rangle^{-2k}d\eta=C\cdot 2^{j(1-\rho)(n-2k)}.$$

Substituting this into (2.18), one gets

$$\int|\hat{u}(2^{j\rho}\xi)|^2\sum_\nu\langle\xi-\nu\rangle^{-2k}d\xi\leqslant C_3\cdot 2^{-j\rho n}\left\{\int_{S_{2j}}|\hat{u}(\xi)|^2d\xi+2^{j(1-\rho)(n-2k)}\|u\|_2^2\right\}.\tag{2.19}$$

Using the argument on the spectrum of g_j similar to that used in the proof of Lemma 4, we can show that

$$\operatorname{supp}\hat{g}_{1j}(\eta)\subset\left\{\eta\left|\frac{1}{8}2^{j(1-\rho)}\leqslant|\eta|\leqslant 8\cdot 2^{j(1-\rho)}\right.\right\}.$$

Since $1-\rho>0$, there is a positive integer N, such that the spectra of g_j and g_ν are disjoint, if $|j-\nu|\geqslant N$. Consequently we have

$$\left\|\sum_{j=j_0}^\infty g_{1j}(2^{j\rho})\right\|_2^2\leqslant C\sum_{j=j_0}^\infty 2^{-j\rho n}\|g_{1j}\|_2^2,\tag{2.20}$$

where C is a constant depending only on N. Moreover, since the associated symbol has compact support in ξ, we can obtain

$$\left\|\sum_{j=0}^{j_0-1}g_j(2^{j\rho})+g_{-1}\right\|_2\leqslant C_4\|u\|_2.\tag{2.21}$$

From the estimates (2.12), (2.14). (2.15), (2.17), (2.19)—(2.21), we obtain finally

$$\|g\|_2\leqslant\left\|\sum_{j=j_0}^\infty g_j(2^{j\rho})\right\|_2+\left\|\sum_{j=0}^{j_0-1}g_j(2^{j\rho}\cdot)+g_{-1}\right\|_2$$

$$\leqslant\sum_{j=j_0}^n g_{1j}(2^{j\rho}\cdot)\Big\|_2+\sum_{j=j_0}^n\|g_{2j}(2^{j\rho}\cdot)\|_2+C_4\|u\|_2\leqslant C_5\|u\|_2,$$

which completes the proof of Proposition 2.

Combining Propositions 1 and 2, we get the following

Theorem 2. *Let* $0\leqslant\delta\leqslant\rho\leqslant 1$, $0\leqslant s\leqslant\rho\leqslant 1$, $\delta<1$, $s<1$, *and* $k=\left[\frac{n}{2}\right]+1$. *Suppose* $a(x,\xi,y)\in S^0_{\rho,2k;\delta,k;s,k}$, *then the pseudo-differential operator* $a(x,D,y)$ *difined by* (1.6) *is* L^2*-bounded.*

As the end of this section, we go to illustrate the sharpness of the above theorem.

Example 1 Let $a_1(x,\xi,y)=e^{-(x-y)\cdot\xi}\langle x-y\rangle^{-\nu}\langle\xi\rangle^{-2\mu}$, where ν and μ are positive integers. Obviously, $a_1\in S^0_{0,\nu;0,\mu;0,\mu}$. By definition (1.6),

$$a_1(x,\ D,\ y)u(x)=\iint\langle x-y\rangle^{-\nu}\langle\xi\rangle^{-2\mu}u(y)\,dy\,d\!\!\bar{}\,\xi=\int\langle\xi\rangle^{-2\mu}\,d\!\!\bar{}\,\xi\int\langle x-y\rangle^{-\nu}u(y)\,dy.$$

The Fourier transform of the later convolution is $G_\nu(\xi)\hat{u}(\xi)$, at least whenever $u\in\mathscr{S}$, where $G_\nu(\xi)$ denotes the Fourier transform of $\langle x\rangle^{-\nu}$ in the sense of distribution, i. e. the kernel of Bessel potential of order ν. To guarantee that $G_\nu(\xi)\hat{u}(\xi)\in L^2$ would be true for any $u\in L^2$, we should have had $G_\nu(\xi)\in L^\infty$. But if $\nu\leqslant n$, then this is false, because the following asymptotic relations hold:

$$G_\nu(\xi)\approx\begin{cases}C_{n,\nu}|\xi|^{\nu-n} & \text{when } 0<\nu<n\\ C_n\cdot\log\dfrac{1}{|\xi|} & \nu=n\end{cases}\qquad(|\xi|\to 0),$$

(see [15], pp. 289—290). So in order that $a_1(x,\ D,\ y)\in\mathscr{L}(L^2,\ L^2)$ we must demand $2\mu\geqslant n+1$ and $\nu\geqslant n+1$. This just shows that, in the case $\rho=\delta=\varepsilon=0$, the result of Theorem 2 is sharp in a reasonable sense. And for the case $0<\rho=\delta=\varepsilon<1$, this is also true, as it can be seen by the same reasoning used below Theoren 7 in [3].

Example 2 Consider a slight modification to the above example

$$a_2(x,\ \xi,\ y)=e^{-i(x-y)\cdot\xi}\langle x-y\rangle^{-\nu}\langle\xi\rangle^{-2(\mu+\nu)}.$$

Obviously, $a_2\in S_{1,\nu;,0,\mu;0,\mu}$. The argument used in the preceding example shows that the operator $a_2(x,\ D,\ y)$ cannot be L^2-bounded, if $\nu\leqslant n$.

§ 3. Some auxiliary results

For the convenience of proving the theorems on L^p-boundedness in § 4, in the present section we prove some useful lemmas and propositions first. We set $\mu=\frac{n}{2}(1-\rho)$ and let

$$E'_0=\left\{\xi\,\middle|\,\frac{1}{2}<|\xi|<2\right\},\ E'_{-1}=\{\xi\,|\,|\xi|\leqslant 1\}.$$

We call Lemma 1 with E_0 and E_{-1} replaced by E'_0 and E'_{-1}, respectively, Lemma 1′.

Lemma 5. *Suppose* $\mathrm{supp}_\xi a(x,\ \xi)$ (*or* $a(x,\ \xi,\ y)$) $\subset\{\xi\,|\,|\xi|\leqslant 1\}$.

(1) *If* $a(x,\ \xi)\in S^0_{\rho,k;\delta,0}$, *then the pseudo-differential operator* $a(x,\ D)$ *defined by* (1.4) *is* L^∞*-bounded, and we have the estimate*

$$\|a(x,\ D)u\|_\infty\leqslant C|a|^{(0)}_{k,0}\|u\|_\infty,$$

where the constant C *does not depend on* a *and* u.

(2) *If* $a(x,\ \xi)\in S^0_{\rho,n+1;\delta,0}$, *then the pseudo-differential operator* $\tilde{a}(x,\ D)$ *defined by* (1.5) *is* L^∞*-bounded, and we have the estimate*

$$\|\tilde{a}(x,\ D)u\|_\infty\leqslant C|a|^{(0)}_{n+1,0}\|u\|_\infty$$

with the constant C *as above.*

(3) *If* $a(x,\ \xi,\ y)\in S^0_{\rho,n+1;\delta,0;\varepsilon,0}$, *then the pseudo-differential operator* $a(x,\ D,\ y)$

defined by (1.6) is L^∞-bounded, and we have the estimate

$$\|a(x, D, y)u\|_\infty \leqslant C|a|^{\{0\}}_{n+1,0,0}\|u\|_\infty$$

with the constant C as above.

Proof By (1.4) we have

$$|a(x, D)u(x)| = \left|\iint e^{i(x-y)\xi} a(x, \xi)u(y)dy\,d\xi\right| \leqslant \|u\|_\infty \int |\hat{a}_x(y)|dy,$$

where $\hat{a}_x(y)$ is the Fourier transform of $a(x, \xi)$ with respect to ξ, x being regarded as a parameter. Clearly, it suffices to prove $\hat{a}_x(y) \in L^1$ and

$$\int |\hat{a}_x(y)|dy \leqslant C|a|^{\{0\}}_{k,0}. \tag{3.1}$$

For this purpose, by means of Schwartz's inequality, we have

$$\int |\hat{a}_x(y)|dy = \int (1+|y|^2)^{-k/2}(1+|y|^2)^{k/2}|\hat{a}_x(y)|dy \leqslant C\left\{\int (1+|y|^2)^k |\hat{a}_x(y)|^2 dy\right\}^{1/2}$$

$$\leqslant C' \sum_{|\alpha|\leqslant k}\left\{\int |y^\alpha \hat{a}_x(y)|^2 dy\right\}^{1/2}.$$

Integrating by parts and using Parseval's identity, we obtain

$$\int |\hat{a}_x(y)|dy \leqslant C' \sum_{|\alpha|\leqslant k}\left\{\int \{|D_\xi^\alpha a(x, \xi)|^2 d\xi\right\}^{1/2} \leqslant C''|a|^{\{0\}}_{k,0},$$

where C, C' and C'' are constants independent of a. This pooves (3.1).

To prove (2), we need only notice that

$$|\tilde{a}(x, D)u(x)| \leqslant \|u\|_\infty \int \left|\int e^{-iy\xi} a(y+x, \xi)d\xi\right| dy$$

$$\leqslant \|u\|_\infty \int (1+|y|)^{-(n+1)} \sum_{|\alpha|\leqslant n+1} C_\alpha \left|\int e^{-iy\xi} D_\xi^\alpha a(y+x, \xi)d\xi\right| dy \leqslant C|a|^{\{0\}}_{n+1,0}\|u\|_\infty.$$

The proof of (3) is parallel to that of (2). So we omit it. Lemma 5 is proved.

Lemma 6. *Suppose $\operatorname{supp}_\xi a(x, \xi)$ (or $a(x, \xi, y)$) $\subset \{\xi \,|\, 1 \leqslant |\xi| \leqslant l^{-1}\}$.*

(1) *If $a(x, \xi) \in S^{-\mu}_{\rho,k;\delta,1}$, then there is a constant C independent of l, a, such that*

$$\int |K(x, x-y) - K(z, z-y)|dy \leqslant C|a|^{\{-\mu\}}_{k,1}$$

holds for $|x| \leqslant l$, $|z| \leqslant l$, where K is defined by (1.7).

(2) *If $a(x, \xi) \in S^{-\mu}_{\rho,2k;\delta,k}$ with $0 \leqslant \delta \leqslant \rho \leqslant 1$, $\delta < 1$, then there is a constant C independent of l, a, such that*

$$\int |K(y, x-y) - K(y, z-y)|dy \leqslant C|a|^{\{\mu\}}_{2k,k}$$

holds for $|x| \leqslant l$, $|z| \leqslant l$.

(3) *If $a(x, \xi, y) \in S^{-\mu}_{\rho,2k;\delta,1;s,k}$ with $0 \leqslant s \leqslant \rho \leqslant 1$, $s < 1$, then there is a constant C independent of l, a, such that*

$$\int |K(x, x-y, y) - K(z, z-y, y)|dy \leqslant C|a|^{\{-\mu\}}_{2k,1,k}$$

holds for $|x| \leqslant l$, $|z| \leqslant l$, where K is defined by (1.8).

Proof By (1.7)

$$I \triangleq \int |K(x, x-y) - K(z, z-y)|\, dy = \int \left| \int e^{-iy\xi}[e^{ix\xi} a(x, \xi) - e^{iz\xi} a(z, \xi)]\, đ\xi \right| dy.$$

Using the theorem of mean, we have

$$I = \int dy \left| \int e^{-iy\xi} \left\{ \int_0^1 [e^{i(z+\theta(x-z))\xi} a(x, \xi) i(x-z)\xi \right.\right.$$

$$\left.\left. + e^{ix\xi}(x-z) \cdot \nabla_x a(z+\theta(x-z), \xi) d\theta \right\} đ\xi \right| \leqslant \int \left| \int e^{-iy\xi} a(x, \xi)(x-z)\xi\, đ\xi \right| dy$$

$$+ \int_0^1 d\theta \int \left| \int e^{-iy\xi}(x-z) \cdot \nabla_x (z+\theta(x-z), \xi)\, đ\xi \right| dy. \tag{3.2}$$

By the hypothesis, $l<1$. Choose a positive integer N, so that $1 \leqslant 2^N l < 2$, and set

$$\Phi(\xi) = \sum_{\nu=0}^{N} \varphi(2^\nu l \xi),$$

where φ is the function given in Lemma 1′. It is obvious that supp $\Phi \subset \{\xi \mid \frac{1}{4} \leqslant |\xi| \leqslant 2l^{-1}$ and $\Phi(\xi) \equiv 1$ on $1 \leqslant |\xi| \leqslant l^{-1}$. Thus, for the first term in the right-hand side of (3.2) we have

$$\int \left| \int |e^{-iy\xi} a(x, \xi)(x-z)\xi\, đ\xi \right| dy \leqslant \sum_{\nu=0}^{N} \int \left| \int e^{-iy\xi} a(x, \xi) \varphi(2^\nu l\xi)(x-z)\xi\, đ\xi \right| dy$$

$$\triangleq \sum_{\nu=0}^{N} I_\nu. \tag{3.3}$$

Put $b_\nu = (2^\nu l)^\rho$ and break each I_ν into two parts

$$I_\nu = \int_{|y| \leqslant b_\nu} + \int_{|y| > b_\nu} \triangleq I'_\nu + I''_\nu. \tag{3.4}$$

For I'_ν, using Schwartz's inequality and Parseval's identity, we get

$$I'_\nu \leqslant C b_\nu^{n/2} \left\{ \int \left| \int e^{-iy\xi} a(x, \xi) \varphi(2^\nu l\xi)(x-z)\xi\, đ\xi \right|^2 dy \right\}^{1/2}$$

$$= C' b_\nu^{n/2} \left\{ \int |a(x, \xi) \varphi(2^\nu l\xi)(x-z)\xi|^2\, đ\xi \right\}^{1/2} \leqslant C_1 |a|_{0,0}^{(-\mu)} \cdot 2^{-\nu}. \tag{3.5}$$

And for I''_ν, by Schwartz's inequality

$$I''_\nu \leqslant C b_\nu^{\frac{n}{2}-k} \left\{ \int |y|^{2k} \left| \int e^{-iy\xi} a(x, \xi) \varphi(2^\nu l\xi)(x-z)\xi\, đ\xi \right|^2 dy \right\}^{1/2}.$$

Integrating by parts and using Parseval's identity, we also have

$$I''_\nu \leqslant C b_\nu^{n/2-k} \sum_{|\alpha|=k} \left\{ |\partial_\xi^\alpha [a(x, \xi) \varphi(2^\nu l\xi)(x-z)\xi]|^2\, đ\xi \right\}^{1/2} \leqslant C_2 |a|_{k,0}^{(-\mu)} \cdot 2^{-\nu}. \tag{3.6}$$

Combining (3.5), (3.6) and (3.4), we obtain

$$I_\nu \leqslant C |a|_{k,0}^{(-\mu)} \cdot 2^{-\nu}, \quad \nu = 0, 1, \cdots, N, \tag{3.7}$$

where C is a constant independent of l, a. Substituting (3.7) into (3.3), it follows that

$$\int \left| \int e^{-iy\xi} a(x, \xi)(x-z)\xi\, đ\xi \right| dy \leqslant C |a|_{k,0}^{(-\mu)}.$$

Similarly we can get the following estimate for the second term in the right-hand side of (3.2):

$$\int_0^1 d\theta \int \left| \int e^{-iy\xi}(x-z)\cdot\nabla_x a(z+\theta(x-z),\ \xi)d\xi \right| dy \leqslant C|a|_{k,1}^{\{\mu\}},$$

where C is a constant independent of l, a. Thus, the proof of Lemma 6-**(1)** is completed.

As to (2), it may be proved parallelly, provided in the argument above we apply the L^2-boundedness of pseudo-differential operators stated in Theorem 1, instead of Parseval's identity.

For (3), from (1.8) we have

$$\begin{aligned}
&\int |K(x,\ x-y,\ y)-K(z,\ z-y,\ y)|dy \\
&\leqslant \int |K(x,\ x-y,\ y)-K(x,\ z-y,\ y)|\,dy+\int |K(x,\ z-y,\ y)-K(z,\ z-y,\ y)|dy \\
&\leqslant \int_0^1 d\theta \int dy \left| \int e^{-iy\xi} a(x,\ \xi,\ y+z+\theta(x-z))(x-z)\xi\, d\xi \right| \\
&\quad + \int_0^1 d\theta \int dy \left| \int e^{-iy\xi} \nabla_x a(z+\theta(x-z),\ \xi,\ y+z)(x-z)d\xi \right|.
\end{aligned}$$

Thus, we reduce (3) to the case of (2), and hence can complete the proof of (3) similarly.

Lemma 7. *Suppose* $\operatorname{supp}_\xi\ a(x,\xi)$ *(or* $a(x,\ \xi,\ y)$*)* $\subset\{\xi\,|\,|\xi|\geqslant l^{-1}\}$, *where* $0<l<1$.

(1) *If* $a(x,\ \xi)\in S^{-\mu}_{\rho,k;\delta,0}$ *with* $\rho>0$, *then there is a constant C independent of l, a, such that the estimate*

$$\int_{|y|>2l^\rho} |K(x,\ x-y)|dy \leqslant C|a|_{k,0}^{\{-\mu\}}$$

holds for $|x|\leqslant l$.

(2) *If* $a(x,\ \xi)\in S^{-\mu}_{\rho,2k;\delta;k}$ *with* $0\leqslant\delta\leqslant\rho\leqslant1$, $\delta<1$ *and* $\rho>0$, *then there is a constant C independent of l, a, such that*

$$\int_{|y|-2l^\rho} |K(y,\ x-y)|dy \leqslant C|a|_{2k,k}^{\{-\mu\}}$$

holds for $|x|\leqslant l$.

(3) *If* $a(x,\ \xi,\ y)\in S^{-\mu}_{\rho,2k;\delta,0;\varepsilon,k}$ *with* $0\leqslant\varepsilon\leqslant\rho\leqslant1$, $\varepsilon<1$, *and* $\rho>0$, *then there is a constant C independent of l, a, such that*

$$\int_{|y|>2l^\rho} |K(x,\ x-y,\ y)|dy \leqslant C|a|_{2k,0,k}^{\{-\mu\}}$$

holds for $|x|\leqslant l$.

The proof of Lemma 7 is similar to the estimate for I''_ν in the proof of Lemma 6, so we omit it.

Lemma 8. *Let* $l\geqslant1$.

(1) *If* $a(x,\ \xi)\in S^{-\mu}_{\rho,k;\delta,0}$ *with* $\rho>0$, *then*

$$\int_{|y|>2l} |K(x,\ x-y)|dy \leqslant C|a|_{k,0}^{\{-\mu\}}$$

holds for $|x|\leqslant l$, *where C is a constant independent of* l, a.

(2) *If* $a(x, \xi)\in S^{-\mu}_{\rho,2k;\delta,k}$ *with* $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$ *and* $\rho>0$, *then*

$$\int_{|y|\geqslant 2l}|K(y, x-y)|dy\leqslant C|a|^{(-\mu)}_{2k,k}$$

holds for $|x|\leqslant l$, *where C is as above.*

(3) *If* $a(x, \xi, y)\in S^{-\mu}_{\rho,2k;\delta,0;s,k}$ *with* $0\leqslant s\leqslant\rho\leqslant 1$, $s<1$ *and* $\rho>0$, *then*

$$\int_{|y|\geqslant 2l}|K(x, x-y, y)|dy\leqslant C|a|^{(-\mu)}_{2k,0,k}$$

holds for $|x|\leqslant l$, *where C is as above.*

Proof As in the proof of Lemma 6, we have

$$\int_{|y|\geqslant 2l}|K(x, x, y)|dy\leqslant\int_{|y|\geqslant l}|K(x, -y)|dy$$
$$\leqslant Cl^{-\left(k-\frac{n}{2}\right)}\left\{\int|y|^{2k}|K(x, -y)|^2 dy\right\}^{1/2}\leqslant C\left\{\int|\nabla^k_\xi a(x, \xi)|^2 d\xi\right\}^{1/2}$$
$$\leqslant C'|a|^{(-\mu)}_{k,0}.$$

This completes the proof of (1). And the proofs of (2) and (3) are completely parallel.

Now, we are prepared to state and prove the following result which is crucial for the next section.

Proposition 3. *Let* $k=\left[\frac{n}{2}\right]+1$ *and* $\mu=\frac{n}{2}(1-\rho)$.

(1) *If* $a(x, \xi)\in S^{-\mu}_{\rho,k;\delta,k}$ *with* $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$ *and* $\rho>0$, *then*

$$\|a(x, D)u\|_*\leqslant C|a|^{(-\mu)}_{k,k}\|u\|_\infty, \text{ for } u\in L^2\cap L^\infty,$$

where C is a constant independent of a and u.

(2) *If* $a(x, \xi)\subset S^{-\mu}_{\rho,2k;\delta,k}$ *with* $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$ *and* $\rho>0$, *then*

$$\|\tilde{a}(x, D)u\|_*\leqslant C|a\{^{(-\mu)}_{2k,k}\|u\|_\infty, \text{ for } u\in L^2\cap L^\infty,$$

where C is as in (1).

(3) *If* $a(x, \xi, y)\in S^{-\mu}_{\rho,2k,;\delta,k,\delta,k}$ *with* $0\leqslant\delta\leqslant\rho\leqslant 1$, $0\leqslant s\leqslant\rho\leqslant 1$, $s<1$, $\delta<1$ *and* $\rho>0$, *then*

$$\|a(x, \xi, y)u\|_*\leqslant C|a|^{(-\mu)}_{2k,k,k}\|u\|_\infty, \text{ for } u\in L^2\cap L^\infty,$$

where C is as in (1).

Proof Let Q be any cube of diameter $2l$. Without loss of generality, we may assume that it is centered at the origin. We are going to prove that

$$\frac{1}{|Q|}\int_Q|a(x, D)u(x)-a_Q|dx\leqslant C|a|^{(-\mu)}_{k,k}\|u\|_\infty, \text{ for } u\in L^2\cap L^\infty, \tag{3.8}$$

where a_Q denotes the mean value of $a(x, D)u(x)$ over Q.

We divide the proof into three parts.

(i) Let $l\geqslant 1$. For $u\in L^2\cap L^\infty$, set

$$u_1(x)=u(x)\chi_l(x),\ u_2(x)=u(x)-u_1(x),$$

where $\chi_l(x)$ is the characteristic function of the ball $\{x||x|\leqslant 2l\}$. Then

$$\frac{1}{|Q|}\int_Q |a(x, D)u(x)|dx \leqslant \frac{1}{|Q|}\int_Q |a(x, D)u_1(x)|dx + \frac{1}{|Q|}\int_Q |a(x, D)u_2(x)|dx. \tag{3.9}$$

For the first term on the right-hand side of (3.9), by Schwartz's inequality and L^2-boundedness of the operator $a(x, D)$, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |a(x, D)u_1(x)|dx &\leqslant \frac{1}{|Q|^{1/2}}\|a(x, D)u_1(x)\|_2 \\ &< \frac{C}{|Q|^{1/2}}|a|_{k,k}^{(0)}\|u_1\|_2 \leqslant C_1|a|_{k,k}^{(-\mu)}\|u\|_\infty,\end{aligned} \tag{3.10}$$

where C_1 does not depend on a, u and l. As to the second term on the right-hand side of (3.9), by (1.4)′ we have

$$\frac{1}{|Q|}\int_Q |a(x, D)u_2(x)|dx \leqslant \frac{\|u\|_\infty}{|Q|}\int_Q\left[\int_{|y|>2l}|K(x, x-y)|dy\right]dx.$$

Thus in virtue of Lemma 8-(1) we obtain

$$\frac{1}{|Q|}\int_Q |a(x, D)u_2(x)|dx \leqslant C_2|a|_{k,0}^{(-\mu)}\|u\|_\infty, \tag{3.11}$$

where C_2 does not depend on a, u and l. Combining (3.10) and (3.11), it follows that

$$\frac{1}{|Q|}\int_Q |a(x, D)u(x)|dx \leqslant C_3|a|_{k,k}^{(-\mu)}\|u\|_\infty, \tag{3.12}$$

where $C_3 = C_1 + C_2$ is, of course, a constant independent of a, u and l.

(ii) Let $l<1$. Set

$$a'(x, \xi) = a(x, \xi)\varphi_{-1}(l\xi), \quad a''(x, \xi) = a(x, \xi) - a'(x, \xi),$$

where φ_{-1} is the function given in Lemma 1′. Obviously

$$|a'|_{k,k}^{(-\mu)} \leqslant C'|a|_{k,k}^{(-\mu)}, \quad |a|_{k,k}^{(-\mu)} \leqslant C''|a|_{k,k}^{(-\mu)}.$$

Now, we are going to prove that

$$\frac{1}{|Q|}\int_Q |a'(x, D)u(x) - a'_Q|dx \leqslant C_4|a|_{k,k}^{(-\mu)}\|u\|_\infty. \tag{3.13}$$

It is easy to see that it suffices to show that

$$|a'(x, D)u(x) - a'(z, D)u(z)| \leqslant C_4|a|_{k,k}^{(-\mu)}\|u\|_\infty \tag{3.13'}$$

holds for $|x|\leqslant l$, $|z|\leqslant l$. From (1.4)′ we have

$$\begin{aligned}|a'(x, D)u(x) - a'(z, D)u(z)| &= \left|\int [K'(x, x-y) - K'(z, z-y)]u(y)dy\right| \\ &\leqslant \|u\|_\infty\int |K'(x, x-y) - K'(z, z-y)|dy,\end{aligned} \tag{3.14}$$

where $K'(x, y)$ is difined by (1.7) with a replaced by a'. Let N be the positive integer used in the proof of Lemma 6. According to Lemma 1′

$$\varphi_{-1}(l\xi) = \varphi_{-1}(2^N l\xi) + \sum_{\nu=1}^{N}\varphi(2^\nu l\xi).$$

By means of this decomposition and using Lemma 5-(1), Lemma 6-(1), we obtain

$$\int |K'(x, x-y) - K'(z, z-y)|dy \leqslant C|a|_{k,k}^{(-\mu)}.$$

Substituting this into (3.14), one gets (3.13)'.

(iii) Let $l<1$. What has to be proved now is that

$$\frac{1}{|Q|}\int_Q |a''(x, D)u(x)|dx \leqslant C_5|a|_{k,k}^{(-\mu)}\|u\|_\infty. \tag{3.15}$$

Set

$$u_3(x)=u(x)\chi_l'(x),\ u_4(x)=u(x)-u_3(x),$$

where $\chi_l'(x)$ is the characteristic function of the ball $\{x||x|\leqslant 2l^\rho\}$. Then

$$\frac{1}{|Q|}\int_Q |a''(x, D)u(x)|dx \leqslant \frac{1}{|Q|}\int_Q |a''(x, D)u_3(x)|dx + \frac{1}{|Q|}\int_Q |a''(x, D)u_4(x)|dx$$
$$\triangleq I_1+I_2. \tag{3.16}$$

Using (1.4)', we write

$$I_2=\frac{1}{|Q|}\int_Q \left|\int K''(x, x-y)u_4(y)dy\right|dx,$$

where $K''(x, y)=\int e^{iy\xi}a''(x, \xi)đ\xi$.

$$I_2 \leqslant \|u\|_\infty \frac{1}{|Q|}\int_Q dx \int_{|y|\geqslant 2l^\rho} |K''(x, x-y)|dy \leqslant C_5'|a|_{k,0}^{(-\mu)}\|u\|_\infty. \tag{3.17}$$

On the other hand, since $\operatorname{supp}_\xi a''(x, \xi)\subset\{\xi||\xi|\geqslant l^{-1}\}$, by definition we have

$$|a|_{k,k}^{(0)} \leqslant |a|_{k,k}^{(-\mu)}\cdot l^\mu.$$

Therefore, using Theorem 1, we can obtain

$$\|a''(x, D)u_3(x)\|_2 \leqslant C|a|_{k,k}^{(0)}\|u_3\|_2 \leqslant C|a|_{k,k}^{(-\mu)}l^\mu\|u_3\|_2 \leqslant C_5''|a|_{k,k}^{(-\mu)}l^{n/2}\|u\|_\infty.$$

As a consequence, we have

$$I_1 \leqslant \frac{1}{|Q|^{1/2}}\|a''(x, D)u_3\|_2 \leqslant C_5'''|a|_{k,k}^{(-\mu)}\|u\|_\infty. \tag{3.18}$$

Combining (3.17), (3.18) with (3.16), we get (3.15).

Combining (3.12), (3.13) and (3.15), (3.8) follows. This completes the proof of (1). The proofs of (2) and (3) is completely parallel.

§ 4. L^p-boundedness

In the present section we are going to establish theorems on L^p-boundedness of the pseudo-differential operators given in § 1. At the same time we give some examples to exhibit that the conditions proposed in these theorems are sharp or almost sharp.

To prove these theorems, we will need a deep and powerful interpolation theorem due to Fefferman and Stein (cf. [5]).

Lemma 9 (Fefferman and Stein). *Suppose that the mapping $z\to T_z$ from the closed strip $0\leqslant \operatorname{Re} z\leqslant 1$ to $\mathscr{L}(L^2, L^2)$ is analytic in the interior of this strip and strongly continuous and uniformly bounded on the closure of the strip. If*

$$\sup_{-\infty<y<\infty}\|T_{iy}u\|_* \leqslant M_0\|u\|_\infty,\ \textit{for } u\in L^2\cap L^\infty, \tag{4.1}$$

and

$$\sup_{-\infty<y<\infty} \|T_{1+iy}u\|_2 \leqslant M_1\|u\|_2, \quad \textit{for } u\in L^2, \tag{4.2}$$

then the estimate

$$\|T_t u\|_p \leqslant C_t M_0^{1-t} M_1^t \|u\|_p$$

holds for $u\in L^2\cap L^p$, *where* $0<t\leqslant 1$, $p=\dfrac{2}{t}$ *and the constant is independent of* M_0, M_1 *and* u.

Note that the condition (4.1) is only imposed on $u\in L^2\cap L^\infty$, and we do not demand that it is valid for all $u\in L^\infty$. This is very important for the following applications.

We are now ready to establish theorems on L^p-boundedness.

Theorem 3. *Let* $0\leqslant\delta\leqslant\rho\leqslant 1$, $\delta<1$, $\rho>0$, $k=\left[\dfrac{n}{2}\right]+1$ *and* $0\leqslant m\leqslant\mu=\dfrac{n}{2}(1-\rho)$. *Suppose* $a(x,\xi)\in S^{-m}_{\rho,k;\delta,k}$. *Then the pseudo-differential operator* $a(x,D)$ *defined by* (1.4) *is* L^p*-bounded, if* $0\leqslant\left(\dfrac{1}{2}-\dfrac{1}{p}\right)n(1-\rho)\leqslant m$ (*when* $\rho=1$, *this inequality is to be replaced by* $p\geqslant 2$).

Proof Set

$$b_z(x,\xi)=e^{z^2}a(x,\xi)\langle\xi\rangle^{m+\mu(z-1)}\ (z=t+is),$$

where $z\in\{z\,|\,0\leqslant\mathrm{Re}\,z\leqslant 1\}$ is a complex parameter. We write

$$B_z u(x)=\iint e^{i(x-y)\xi}b_z(x,\xi)u(y)\,dy\,đ\xi. \tag{4.3}$$

It is easy to see that

$$|b_z|^{(0)}_{k,k}\leqslant e^{t^2-s^2}p(z)\,|a|^{(-m)}_{k,k}\leqslant C\,|a|^{(-m)}_{k,k}, \tag{4.4}$$

where $p(z)$ is a polynomial of degree k and C is a constant independent of z. Therefore, it follows from Theorem 1 and (4.4) that the family of operators $\{B_z\}$ is uniformly bounded in $\mathscr{L}(L^2, L^2)$. It is not hard to verify that the mapping

$$\{z\,|\,0\leqslant\mathrm{Re}\,z\leqslant 1\}\ni z\to B_z\in\mathscr{L}(L^2,L^2)$$

is analytic in the open strip $0<\mathrm{Re}\,z<1$ and strongly continuous on its closure. In particular, we have

$$\sup_{-\infty<s<\infty}\|B_{1+is}u\|_2\leqslant M_1|a|^{(-m)}_{k,k}\|u\|_2. \text{ for } u\in L^2,$$

where M_1 is a constant independent of a, u. On the other hand, by (4.3) we have

$$B_{is}u(x)=\iint e^{i(x-y)\xi}(e^{-s^2}a(x,\xi)\langle\xi\rangle^{m-\mu+is\mu})u(y)\,dy\,đ\xi.$$

It is clear that $b_{is}(x,\xi)\in S^{-\mu}_{\rho,k;\delta,k}$ and

$$|b_{is}|^{(-\mu)}_{k,k}\leqslant C\,|a|^{-m}_{k,k},$$

where C is independent of z. Thus, applying Proposition 3-(1) to the operotor B_{is}, one gets

$$\|B_{is}u\|_*\leqslant C\,|a|^{(-m)}_{k,k}\|u\|_\infty, \text{ for } u\in L^2\cap L^\infty.$$

It follows that the family of operators B_z satisfies all the conditions of Lemma 9. Hence we obtain

$$\|B_t u\|_p \leqslant M_p |a|_{k,k}^{\{-m\}} \|u\|_p, \text{ for } p=\frac{2}{t},\ 0<t\leqslant 1.$$

It is seen from (4.3) that the symbol of B_t is

$$b_t(x, \xi) = e^{t^2} a(x, \xi) \langle \xi \rangle^{m+\mu(t-1)}.$$

Therefore, the pseudo-differential operator $a(x, D)$ is L^p-bounded $\left(p=\frac{2}{t}\right)$ and

$$\|a(x, D)u\|_p \leqslant M_p |a|_{k,k}^{\{-m\}} \|u\|_p,$$

if $m \geqslant \mu(1-t) = \frac{n}{2}(1-\rho)\left(1-\frac{2}{p}\right) = \left(\frac{1}{2}-\frac{1}{p}\right) n(1-\rho) \geqslant 0$. This completes the proof of Theorem 3.

Corollary 1. *Assume the hypotheses of Theorem 3. Then the pseudo-differential operator* $\tilde{a}(x, D)$ *defined by (1.5) is* L^p*-bounded, if* $0 \leqslant \left(\frac{1}{p}-\frac{1}{2}\right) n(1-\rho) \leqslant m$ *(when* $\rho=1$, *this condition is to be replaced by* $1<p\leqslant 2$).

Corollary 2. *Suppose* $a(x, \xi) \in S^{-m}_{0,k;0,k}$. *Then*

(i) *the operator* $a(x, D)$ *is* L^p*-bounded, if*

$$0 \leqslant n\left(\frac{1}{2}-\frac{1}{p}\right) < m;$$

(ii) *the operator* $\tilde{a}(x, D)$ *is* L^p*-bounded, if*

$$0 \leqslant n\left(\frac{1}{p}-\frac{1}{2}\right) < m.$$

Proof For any $\rho>0$, we have

$$S^{-m}_{0,k;0,k} \subset S^{-m+k\rho}_{\rho,k;0,k}.$$

When $0<\rho<\frac{m}{k}$, by Theorem 3, the operator $a(x, D)$ is L^p-bounded, if

$$0 \leqslant n(1-\rho)\left(\frac{1}{2}-\frac{1}{p}\right) \leqslant m-k\rho.$$

This conditions is equivalent to

$$0 \leqslant n\left(\frac{1}{2}-\frac{1}{p}\right) \leqslant \frac{m-k\rho}{1-\rho} \nearrow m \quad (\rho \to 0).$$

It follows that Corollary 2-(1) is true, hence (2) is also true.

Corollary 3. *Suppose* $a(x, \xi) \in S^{-m}_{1,k;1,k}$ *with* $m>0$. *Then*

(i) *the operator* $a(x, D)$ *is* L^p*-bounded, if* $2 \leqslant p < \infty$,

(ii) *the operator* $\tilde{a}(x, D)$ *is* L^p*-bounded, if* $1<p\leqslant 2$.

Theorem 4. *Every pseudo-differential operator* $a(x, D)$ *(or* $\tilde{a}(x, D)$*) with symbol* $a(x, \xi) \in S^{-m}_{\rho,2k;\delta,k}$ $(0 \leqslant \delta \leqslant \rho \leqslant 1,\ \delta<1$ *and* $\rho>0)$ *is* L^p*-bounded, iff*

$$n(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right| \leqslant m.$$

Proof Sufficiency. In virtue of Theorem 3 we see that it suffices to prove the conclusion is true for the case $1<p<2$. As a consequence, it suffices also to prove it for the operator $\tilde{a}(x, D)$ in the case $2\leqslant p<\infty$. However, this can be done readily, for it can be proved in the same manner as in Theorem 3, differences only consisting in that Lemmas 5-(1). 6-(1), 7-(1) and 8-(1) ought to be replaced by Lemmas 5-(2), 6-(2), 7-(2) and 8-(2), respectively.

Necessity. To prove it, it is enough to cite the famous counterexample of Hardy-Littlewood-Hirschman-Wainger. Consider

$$\sigma_{\rho m}(x, \xi)=\varphi(\xi)e^{i|\xi|^{1-\rho}}/(1+|\xi|^{m})\in S^{-m}_{\rho,0},$$

where $0<\rho<1$, $m<\mu$ and $\varphi\in C^{\infty}$, $\varphi=0$ *near* the origin, $\varphi=1$ for $|\xi|\geqslant 2$. If

$$\left|\frac{1}{2}-\frac{1}{p}\right|n(1-\rho)>m,$$

then the pseudo-differential operator with symbol $\sigma_{\rho m}(x, \xi)$ is not bounded on L^p (cf. Hirschman [6] and Wainger [16]).

Corollary 1. *Suppose $a(x, \xi)\in S^{-m}_{0,2k;0,k}$. Then the associated pseudo-differential operators $a(x, D)$ and $\tilde{a}(x, D)$ are L^p-bounded* ($1<p<\infty$), *if* $n\left|\frac{1}{2}-\frac{1}{p}\right|<m$.

Corollary 2. *Suppose $a(x, \xi)\in S^{-m}_{1,2k;1,k}$ with $m>0$. Then the operators $a(x, D)$ and $\tilde{a}(x, D)$ are L^p-bounded for all p, $1<p<\infty$.*

Theorem 5. *Every pseudo-differential operator $a(x, D, y)$ with symbol $a(x, \xi, y)$ $\in S^{-m}_{\rho,2k;\delta,k;s,k}$ with $0\leqslant\delta\leqslant\rho\leqslant 1$, $0\leqslant s\leqslant\rho\leqslant 1$. $\delta<1$, $s<1$, $\rho>0$ is L^p-bounded, iff*

$$n(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right|\leqslant m.$$

The proof is similar to the proofs of Theorems 3 and 4, so we omit it.

Corollary 1. *Suppose $a(x, \xi, y)\in S^{-m}_{0,2k;0,k;0,k}$. Then the associated operator $a(x, D, y)$ is L^p-bounded, if* $n\left|\frac{1}{2}-\frac{1}{p}\right|<m$.

Corollary 2. *Suppose $a(x, \xi, y)\in S^{-m}_{1,2k;1,k;1,k}$ with $m>0$. Then the operator $a(x, D, y)$ is L^p-bounded for all p, $1<p<\infty$.*

Note that in Theorems 3 and 4 the conditions to assure the L^p-boundedness of the operator $a(x, D)$ in the case $2\leqslant p<\infty$ and the case $1<p<2$ are different, and the latter are stronger than the former. It is natural to ask if this difference is essential. To answer this question, let us look at the following

Example 3 Put

$$a_3(x, \xi)=e^{-ix\xi}\langle x\rangle^{-\nu}\langle\xi\rangle^{-2n-\nu}, \tag{4.5}$$

where ν is an integer.

Obviously, $a_3(x, \xi)\in S^{-n}_{1,\nu;0,n}$ and

$$a_3(x, D)u(x)=\langle x\rangle^{-\nu}\int\langle\xi\rangle^{-2n-\nu}\,\hat{u}(\xi)\,đ\xi. \tag{4.6}$$

If $\nu<n$, choose $p>1$ so that $p\nu<n$, then it follows from (4.6) that the $a_3(x, D)$ cannot be L^p-bounded.

This exrmple makes it clear that our result about the L^p-boundedness of the operators $a(x, D)$ in the case $1<p<2$ is also almost optimal.

As to Theorem 5, clearly, the requirements represented by the three lower indices, $2k$, k, k, are the lowest.

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